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МАТЕМАТИЧЕСКАЯ ЖИЗНЬ

ДЖУМАДИЛЬДАЕВ АСКАР СЕРКУЛОВИЧ
(к 60-летию со дня рождения)

Джумадильдаеву Аскару Серкуловичу 25 февраля 2016 года исполнилось 60 лет со дня рождения. Джумадильдаев Аскар Серкулович, академик Национальной Академии наук Республики Казахстан, доктор физико-математических наук, профессор, на протяжении многих лет успешно занимается педагогической и научной деятельностью. Окончил в 1977 году механико-математический факультет Московского государственного университета с красным дипломом. Был Ленинским стипендиатом. В 1981 году в Математическом институте АН СССР защитил диссертацию на соискание ученой степени кандидата физико-математических наук. В 1988 году в Ленинградском отделении Математического института АН СССР защитил диссертацию на соискание ученой степени доктора физико-математических наук. С 1980 года работает в Институте Математики НАН РК.

В 1995-2003 годах работал в различных научно-исследовательских центрах и университетах Европы, в частности, в Институте высших исследований (Париж), Институте Макса Планка (Бонн), Международном центре теоретической физики (Триест), институте Филдса (Торонто), Королевском Институте математики Миттаг-Лефлер (Стокгольм), Институте

Ньютона (Кэмбридж), а также в университетах Мюнхена, Билефельда, Киото, Стокгольма и других. Получал гранты Фонда Сороса (дважды), Фонда ИНТАС (трижды), Японского Фонда поддержки науки и технологии, Королевской Академии Наук Швеции. Получал стипендии Фонда Гумбольдта (Германия) и стипендию Американского математического общества. Неоднократно получал государственные научные стипендии для ученых и специалистов, внесших выдающийся вклад в развитие науки и техники. Он – лауреат Государственной премии Республики Казахстан в области науки и техники 2011 года, лауреат 25-ой международной премии им. аль-Хорезми Исламской Республики Иран в 2012 году.

На протяжении многих лет Джумадильдаев А.С. читает общие курсы по дискретной математике, дифференциальной геометрии и дифференциальным уравнениям, по линейной алгебре и аналитической геометрии. Его методы по привлечению студентов к научной работе вызвали большой интерес у студентов. Так, в 2004 г. и 2006 г. по опросам студентов он был избран самым лучшим профессором КБТУ. В 2007 году получил Государственный грант "Лучший преподователь вуза".

Джумадильдаев А.С. является председателем жюри Республиканской математической олимпиады, неизменным председателем жюри Жаутыковских олимпиад по математике и членом жюри конкурса научных проектов корпорации Интел. Под его руководством были защищены 8 кандидатских диссертаций, 1 ученик защитил докторскую диссертацию, 2 ученика – доктора PhD. Среди его учеников есть лауреаты молодежной премии "Дарын"и независимой премии "Тарлан". В 1993 году он и его коллегии и ученики были награждены стипендией фонда Сороса, как лучший коллектив ученых, работающих в кризисное время в области науки. Он – автор учебника по дискретной математике, член редколлегии ряда научных журналов.

Джумадильдаев А.С. занимает активную общественную позицию. Его выступления на телевидении и в печати пользуются большой популярностью. Особый интерес среди молодежи вызвали его выступления на дискуссионных передачах клуба "Култобе", "Біз айтсақ", "Көзқарас", теле и радио компаний Казахстан и Хабар. Он написал четыре научно-популярные книжки для детей.

Джумадильдаев А.С. дважды избирался депутатом Верховного Совета Республики Казахстан (1990-95 гг). Будучи самым молодым доктором наук, первый раз он прошел в депутаты, как представитель совета молодых ученых. Второй раз он успешно провел выборную компанию по Кзыл-Ординскому городскому округу и выиграл конкурс из 13 претендентов. Он был одним из 360 депутатов, принимавших судьбоносные законы Республики Казахстан. Он принимал Декларацию Независимости, Закон о Независимости, первую конституцию и Государственные символы: герб, флаг и гимны Республики Казахстан. Он принимал участие в выборе первого президента Республики Казахстан, он был первым председателем счетной комиссии Верховного Совета независимого Казахстана и заместителем комитета по науке и образованию Верховного Совета РК. Как член комитета по науке и образованию, он принимал активное участие при принятии законов РК по науке и образованию. Создание фонда Болашак было пунктом номер один в его предвыборной программе и он был инициатором создания этого фонда.

Аскар Серкулович владеет четырьмя языками: казахским, русским, английским и немецким. Он – отец четверых детей.

Основное поле научной деятельности Джумадильдаева Аскара Серкуловича – теория неассоциативных алгебр. В этой области им опубликовано более 80 работ в известных реферируемых западных журналах. По данным Американского Математического общества его работы цитируются 160 раз 100 авторами. Его научные работы в этой области получили широкое признание математической общественности.

Коллектив Института математики и математического моделирования Комитета науки МОН РК желает Джумадильдаеву Аскар Серкуловичу крепкого здоровья, долгих и счастливых лет жизни, полных творчества и созидания, покорения всех вершин и в науке и в жизни, счастья и успехов.

Редакционная коллегия "Математического журнала"

Основные научные результаты и публикации А.С. Джумадильдаева

Научные интересы Джумадильдаева Аскара Серкуловича разнообразны: теория алгебр Ли, теория неассоциативных алгебр и комбинаторика. Им получены фундаментальные результаты в следующих направлениях:

1) когомологические результаты (в частности, решение гипотезы Селигмана. Отметим также вычисления, связанные с проблемой Картана, и вычисления третьей группы когомологий с тривиальными коэффициентами);

2) полиномиальные тождества дифференциальных операторов (алгебры Винберга-Кошуля, алгебра Витта);

3) обобщенные коммутаторы для дифференциальных операторов;

4) правосимметрические алгебры (базисы, тождества, алгебры Новикова);

5) алгебры, построенные с помощью дифференциальных операторов (аналог теоремы Энгеля для алгебр Новикова);

6) алгебры, построенные с помощью операторов интегрирования (тождества алгебры Цинбиела);

7) алгебры Лейбница (тождества);

8) алгебры с кососимметрическим тождеством степени 3;

9) коммутативные циклы и Алиа-алгебры;

10) алгебры с кососимметрическим тождеством степени 4;

11) n-лиевские алгебры (установлен ряд свойств алгебр Филиппова);

12) n-коммутаторы (исследованы свойства n-коммутаторов);

13) лиевы элементы и комбинаторика перестановок (отмечены связи различных подходов);

14) бикоммутативные алгебры;

15) q-ассосимметрические алгебры;

16) статистика на перестановках (в русле развития теорем МакМагона и Фоата-Шутценберже установлены свойства распределений главных кодов и кодов инверсии).

Отметим наиболее важные результаты. В основе теории алгебр Ли над полем нулевой характеристики лежит теорема Леви-Мальцева об отщепляемости радикала. Она, в свою очередь, следует из леммы Уайтхеда, которая гласит, что когомологии полупростых алгебр Ли с коэффициентами в неприводимом модуле тривиальны, если модуль нетривиален. В случае положительной характеристики лемма Уайтхеда неверна. Джекобсон доказал, что любая модулярная алгебра Ли имеет неразложимый модуль. Селигман высказал гипотезу о том, что любая модулярная конечномерная алгебра Ли имеет модуль с нетривиальной когомологией. Джумадильдаев доказал гипотезу Селигмана. Он показал, что когомологическая картина в модулярном случае богаче и интереснее, чем в случае характеристики нуль. Софус Ли ввел в рассмотрение коммутаторы более ста лет назад и нахождение в нем новых нетривиальных операций кажется особенно удивительным. Последние работы связаны с тождествами алгебр, построенных с помощью дифференцирования и интегрирования. Эти алгебры – новые и обладают интересными свойствами, связанными с интегрируемыми системами. Когомологии и деформации правосимметрических алгебр – активно развиваемая современная область математики. Работа Джумадильдаева о когомологиях правосимметрических алгебр является особенно цитируемой. Более ста лет назад Софус Ли заметил, что композиция двух векторных полей не обязательно является векторным полем, но их коммутатор – обязательно таковой. Это наблюдение лежит в основе теории алгебр и групп Ли. Аскар Джумадильдаев заметил, что конструкция Ли обобщается и на многомерный случай. Этот удивительный факт, открытый через 150 лет после Ли, показывает, что многомерные версии алгебр Ли допустимы в геометрии и физике и что они могут служить источником новых открытий.

Алгебры Новикова определяются тождеством правосимметричности ассоциатора и тождеством левокоммутативности. Джумадильдаев начал изучать тождества алгебр многочленов относительно умножений, построенных с помощью операций дифференцирования и интегрирования. Он построил ряд новых классов алгебр, обобщающих алгебры Новикова. Базис Джумадильдаева-Лофволла алгебры Новикова имеют такую же роль, что и базисы Холла и Ширшова в алгебрах Ли.

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МАТЕМАТИЧЕСКАЯ ЖИЗНЬ

КУДАЙБЕРГЕНОВ КАНАТ ЖАНЗАКОВИЧ
(к 60-летию со дня рождения)

Исполнилось 60 лет со дня рождения Кудайбергенова Каната Жанзаковича, известного казахстанского математика, доктора физико-математических наук, профессора. Кудайбергенов Канат Жанзакович родился 5 ноября 1955 года в городе Чимкент КазССР. В 1973 году поступил на механико-математический факультет Казахского государственного университета им. С.М. Кирова (КазГУ) и в 1978 году окончил его с красным дипломом. Был Ленинским стипендиатом. В 1982 году в Новосибирском государственном университете защитил диссертацию на соискание ученой степени кандидата физико-математических наук, а в 1992 году в Институте математики СО РАН – диссертацию на соискание ученой степени доктора физико-математических наук. С 1981 года работает в Институте математики НАН РК, а с 2002 года – в Казахстанском институте менеджмента, экономики и прогнозирования (КИМЭП). Кудайбергенов Канат Жанзакович на протяжении многих лет занимается научной и педагогической деятельностью. В 1990-2004 годах проводил научные исследования в различных университетах Европы, в частности, в Университете Париж-7, Оксфордском университете, Университете Лидса. Он является лауреатом Независимой премии "Тарлан" за 2001 год. В 2011 году награжден серебряным значком семинара "Алгебра и Логика" (объединенный семинар Новосибирского госу-

дарственного университета и Института математики имени С.Л. Соболева СО РАН).

На протяжении многих лет К.Ж. Кудайбергенов читал спецкурсы по теории моделей и теории групп в КазНУ и Алматинском государственном университете. В настоящее время в КИМЭП он читает такие курсы, как Математика для бизнеса и экономики, Математика для юристов.

Научная деятельность К.Ж. Кудайбергенова связана, в основном, с теорией моделей. Он опубликовал 50 работ в реферируемых научных журналах. Полученные им результаты получили признание специалистов.

Коллектив Института математики и математического моделирования Комитета науки МОН РК желает Кудайбергенову Канату Жанзаковичу крепкого здоровья, долгих и счастливых лет жизни, полных творчества и созидания, неугасаемого интереса к новым задачам, которые ставит перед нами жизнь.

Редакционная коллегия "Математического журнала"

Основные научные результаты и публикации К.Ж. Кудайбергенова

Изучались конструктивные модели полных теорий. Построена полная разрешимая теория, имеющая ровно две сильно конструктивизируемые модели [1]; это контрастирует с известной теоремой Воота о том, что полная счетная теория не может иметь ровно две счетные модели. Доказано, что для любого натурального числа n существует ω_1-категоричная теория, имеющая ровно n конструктивизируемых моделей [2]. Построена сильно конструктивизируемая однородная модель, не являющаяся эффективно однородной ни в какой конструктивизации [3].

Изучались однородные модели. Решена проблема Кейслера и Морли о числе однородных моделей в различных мощностях [4]. Опровергнуты гипотеза Шелаха о мощности абсолютно однородных моделей и некоторым образом связанная с ней гипотеза Кейслера и Морли [5].

Всесторонне исследовались однородные модели стабильных теорий [6], в том числе однородные модели одноразмерностных теорий [7], локально модулярных теорий конечного ранга [8], слабо минимальных теорий [9]. Описан спектр однородных моделей тотально трансцендентных немультиразмерностных теорий [10]. Исследованы однородные модели при более слабом, чем стабильность теории, условии стабильности диаграммы [11].

Изучались генерические автоморфизмы моделей. Доказано, что неподвижное поле генерического автоморфизма сепарабельно замкнутого поля является регулярно замкнутым и вычислена его абсолютная группа Галуа [12]; этот результат обобщает соответствующие результаты Макинтайра для алгебраически замкнутых полей. Исследованы генерические последовательности автоморфизмов [13]; они были применены для доказательства существования плотных свободных подгрупп групп автоморфизмов однородных моделей [14], что дополняет результаты Меллеса и Шелаха, полученные для насыщенных моделей.

Исследовалось понятие o-минимальности и различные его обобщения. Усилены результаты Маркера о малых расширениях моделей o-минимальных теорий [15]. Введено и изучено понятие слабо квази-o-минимальной модели и теории [16]. Введены и изучены различные обобщения o-минимальности на частичные порядки [17].

Изучалось классическое понятие модельного компаньона. Получены результаты общего вида о существовании и несуществовании модельного компаньона для теорий структур с выделенным автоморфизмом, а также соответствующие результаты для некоторых конкретных структур [18]. Дан ответ на вопрос Кикио и Цубои о существовании модельного компаньона для теории C-минимальных моделей с выделенным автоморфизмом специального вида [19].

Изучались вопросы однородности для конкретных алгебраических систем. Изучалась однородность для модулей [20] и линейных порядков [21]. Построена счетная строго 2-однородная дистрибутивная решетка [22], вопрос о существовании которой был поставлен Макферсоном и Дростом.

Понятие однородности изучалось с самых разных сторон. В ответ на вопрос Перетятькина доказано, что понятие однородности не является абсолютным в смысле теории множеств [23]. Исследовался вопрос о сохранении и несохранении однородности при специальном обогащении модели; получены условия, при которых однородность модели M влечет однородность модели M^{eq}, а также показано, что в общем случае однородность модели M не влечет однородность модели M^{eq} [24].

Введено общее понятие надстройки над моделью, обобщающее понятие наследственно конечной надстройки, и получены результаты о мощности множеств, интерпретируемых в надстройке над моделью [25], аналогичные результатам Ю.Л. Ершова для наследственно конечной надстройки.

Изучалось свойство независимости для теорий первого порядка. Опровергнута сильная форма гипотезы Шелаха о существовании бесконечных неразличимых последовательностей в моделях большой мощности теорий без свойства независимости [26]. Построена теория со свойством независимости, атомные формулы которой не имеют свойства независимости [27], что опровергает утверждение Адлера.

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ЕРКІН КОММУТАТИВТІ n-АЛГЕБРАНЫҢ КЕЙБІР АҒАШТАРЫНДАҒЫ S_n ТОБЫНЫҢ КӨРСЕТІЛІМДЕРІ

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Аннотация: Жұмыста еркін коммутативті n-алгебра S_n-модуль ретінде қарастырылған. Еркін коммутативті n-алгебраның базистік элементтері түбірлі нонпланар n-арлы ағаштармен туындатылады. Еркін коммутативті n-алгебраның кейбір n-арлы ағаштарының S_n-модульге жіктелуі толығымен сипатталған.

Кілттік сөздер: n-арлы алгебра, n-арлы ағаштар, еркін алгебра, S_n-модуль, Костка сандары, Литтлвуд-Ричардсон коэффициенттері, индукцияланған модульдер, плэтизм, топтардың өрмек көбейтіндісі (wreath product), топтардың тік көбейтіндісі, графтардың автоморфизмдер тобы.

1. Кіріспе

Алгебраның көпбейнелерін зерттеу заманауи алгебраның маңызды есептерінің бірі. Берілген алгебраның көпбейнелер есебі алгебраның еркін әрі мульти-сызықты бөлігін S_n-модульге жіктеу есебіне алып келеді.

Ғылымда кейбір еркін алгебралардың S_n-модульге жіктелуі белгілі. Мысалға, айталық F_n^{assoc}, F_n^{leib}, F_n^{zinb} сәйкесінше еркін ассоциативті, еркін Лейбниц және еркін Цинбиел алгебраларының мульти-сызықты бөліктері болсын. Онда бұл еркін алгебралардың мульти-сызықты бөліктері S_n-модуль ретінде $\mathbb{C}S_n$-ге изоморфты.

Айталық F_n^{lie} еркін Ли алгебрасының мульти-сызықты бөлігі болсын. Онда бұл еркін алгебраның келтірілмейтін (жіктелмейтін) S_n-модульдерге жіктелуі [1] жұмысында толығымен сипатталған.

Ключевые слова: *n-арная алгебра, n-арное дерево, свободная алгебра, S_n-модуль, числа Костки, коэффициенты Литтлвуда-Ричардсона, индуцированные модули, плэтизм, сплетение групп, прямое произведение групп, группа автоморфизмов графа.*

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Айталық F_n^{bicom}, F_n^{nov} еркін бикоммутативті және еркін Новиков алгебраларының мульти-сызықты бөліктері болсын. Онда бұл еркін алгебраларының S_n-модульдік құрылымы толығымен [2], [3] жұмысында сипатталған.

Еркін анти-коммутативті алгебраның F_n^{acom} мульти-сызықты бөлігінің S_n-модульдік құрылымдары туындатушылыр саны $1 \leq n \leq 7$ болғанда [1] жұмысында зерттелген.

Ал M. Bremner-дің [5] мақаласында еркін анти-коммутативті n-арлы алгебраның мулти-сызықты бөлігі S_n-модуль ретінде қарастырылган. Мұнда бірдей типті жақшалар ішкі векторлық кеңістік құрайды және S_n-модуль болады. Кейбір типті жақшалар үшін S_n-модульдік құрылымы толығымен сипатталған.

Бұл жұмыста еркін коммутативті n-арлы алгебраны S_n-модуль ретінде қарастырамыз. Мұнда да бірдей типті жақшалар ішкі векторлық кеңістік құрайды және S_n-модуль. Кейбір бірдей типті жақшалардан туындатылған векторлық кеңістіктерді S_n-модуль ретінде қарастырамыз және жіктелмейтін модульдерінің еселіктерін толығымен сипаттаймыз.

2. Базис ережесі

Анықтама: *Айталық A векторлық кеңістік болсын және A-да $\omega : A^n \to A$ операциясы анықталсын. Онда (A, ω) жұбын n-арлы алгебра немесе n-алгебра деп атаймыз.*

Егер $n = 2$ болса және төмендегі тепе-теңдік орындалса

$$\omega(a, b) = \omega(b, a),$$

онда бұл алгебраны 2-арлы немесе бинарлы коммутативті алгебра деп атаймыз.

Егер $n = 3$ болса және төмендегі тепе-теңдік орындалса

$$\omega(a, b, c) = \omega(a, c, b) = \omega(b, a, c) = \omega(b, c, a) = \omega(c, a, b) = \omega(c, b, a),$$

онда бұл алгебраны 3-арлы немесе тернарлы коммутативті алгебра деп атаймыз.

Егер $n = k$ болса және кез келген $\sigma \in S_k$, үшін төмендегі тепе-теңдік орындалса

$$\omega(a_1, a_2, \ldots, a_k) = \omega(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}),$$

онда бұл алгебраны k-арлы коммутативті алгебра деп атаймыз.

$F_n^k = F_n^k(X)$ арқылы еркін n-арлы коммутативті алгебраның мультисызықты векторлық бөлігін белгілейміз, мұндағы k мономдағы көбейтулер саны немесе n-арлы коммутативті жақшалар саны, $X = \{a_1, a_2, \dots, a_{kn-(k-1)}\}$ әріптер немесе туындатушылар жиыны. Бұл жұмыста $k = 1, 2, 3, 4$ тең n-арлы коммутативті алгебралар қарастырылады.

Айталық $X = \{a_1, a_2, \dots, a_n\}$ болсын. Онда F_n^1 векторлық кеңістігінің базистік элементі төмендегідей мономмен беріледі

$$(a_1 a_2 \dots a_n).$$

Айталық $X = \{a_1, a_2, \dots, a_{2n-1}\}$ болсын. Онда F_n^2 векторлық кеңістігінің базистік элементтері төмендегідей мономдармен беріледі

$$(a_{i_1} a_{i_2} \dots a_{i_{n-1}} (a_{j_1} a_{j_2} \dots a_{j_n}))$$

мұнда $i_1 < i_2 \dots < i_{n-1}$ және $j_1 < j_2 \dots < j_n$, $i_m, j_m \in \{1, 2, 3, \dots, 2n-1\}$.

Айталық $X = \{a_1, a_2, \dots, a_{3n-2}\}$ болсын. Онда F_n^3 векторлық кеңістігінің базистік элементтері төмендегідей мономдармен беріледі

Бірінші тип:

$$(a_{i_1} a_{i_2} \dots a_{i_{n-1}} (a_{j_1} a_{i_2} \dots a_{j_{n-1}} (a_{k_1} a_{k_2} \cdots a_{k_n})))$$

мұнда $i_1 < i_2 \dots < i_{n-1}$, $j_1 < j_2 \dots < j_{n-1}$ және $k_1 < k_2 \dots < k_n$, $i_m, j_m, k_m \in \{1, 2, 3, \dots, 3n-2\}$.

Екінші тип:

$$(a_{i_1} a_{i_2} \dots a_{i_{n-2}} ((a_{j_1} a_{i_2} \dots a_{j_n})(a_{k_1} a_{k_2} \cdots a_{k_n}))$$

мұнда $i_1 < i_2 \dots < i_{n-2}$, $j_1 < j_2 \dots < j_n$, $k_1 < k_2 \dots < k_n$ және $j_1 < k_1$, $i_m, j_m, k_m \in \{1, 2, 3, \dots, 3n-2\}$.

Айталық $X = \{a_1, a_2, \dots, a_{4n-3}\}$ болсын. Онда F_n^4 векторлық кеңістігінің базистік элементтері төмендегідей мономдармен беріледі

Бірінші тип:

$$(a_{i_1} a_{i_2} \dots a_{i_{n-1}} (a_{j_1} a_{j_2} \dots a_{j_{n-1}} (a_{k_1} a_{k_2} \cdots a_{k_{n-1}} (a_{l_1} a_{l_2} \dots a_{l_n}))))$$

мұнда $i_1 < i_2 \cdots < i_{n-1}$, $j_1 < j_2 \cdots < j_{n-1}$, $k_1 < k_2 \cdots < k_{n-1}$ және $l_1 < l_2 \cdots < l_n$, $i_m, j_m, k_m, l_m \in \{1, 2, 3, \ldots, 4n-3\}$.

Екінші тип:

$$(a_{i_1}a_{i_2}\ldots a_{i_{n-2}}(a_{j_1}a_{j_2}\ldots a_{j_n})(a_{k_1}a_{k_2}\ldots a_{k_{n-1}}(a_{l_1}a_{l_2}\cdots a_{l_n})))$$

мұнда $i_1 < i_2 \cdots < i_{n-2}$, $j_1 < j_2 \cdots < j_n$, $k_1 < k_2 \cdots < k_{n-1}$ және $l_1 < l_2 \cdots < l_n$, $i_m, j_m, k_m, l_m \in \{1, 2, 3, \ldots, 4n-3\}$.

Үшінші тип:

$$(a_{i_1}a_{i_2}\ldots a_{i_{n-1}}(a_{j_1}a_{j_2}\ldots a_{j_{n-2}}(a_{k_1}a_{k_2}\ldots a_{k_n})(a_{l_1}a_{l_2}\cdots a_{l_n})))$$

мұнда $i_1 < i_2 \cdots < i_{n-1}$, $j_1 < j_2 \cdots < j_{n-2}$, $k_1 < k_2 \cdots < k_n$, $l_1 < l_2 \cdots < l_n$ және $k_1 < l_1$, $i_m, j_m, k_m, l_m \in \{1, 2, 3, \ldots, 4n-3\}$.

Мысалы:

Айталық $X = \{a, b\}$ болсын. Онда F_2^1 кеңістігінің базистік элементтері

$$(ab)$$

болады.

Айталық $X = \{a, b, c\}$ болсын. Онда F_2^2 кеңістігінің базистік элементтері

$$(a(bc)), (b(ac)), (c(ab))$$

болады.

Айталық $X = \{a, b, c, d\}$ болсын. Онда F_2^3 кеңістігінің базистік элементтері

$$(a(b(cd))), (a(c(bd))), (a(d(bc))), (b(a(cd))), (b(c(ad))), (b(d(ac))),$$

$$(c(a(bd))), (c(b(ad))), (c(d(ab))), (d(a(bc))), (d(b(ac))), (d(c(ab))),$$

$$((ab)(cd)), ((ac)(bd)), ((ad)(bc))$$

болады.

Айталық $X = \{a, b, c\}$ болсын. Онда F_3^1 кеңістігінің базистік элементтері

$$(abc)$$

болады.

Айталық $X = \{a, b, c, d, e\}$ болсын. Онда F_3^2 кеңістігінің базистік элементтері

$$(ab(cde)), (ac(bde)), (ad(bce)), (ae(bcd)), (bc(ade)),$$

$$(bd(ace)), (be(acd)), (cd(abe)), (ce(abd)), (de(abc))$$

болады.

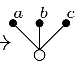
Кез келген $n \geq 1$ және $k \geq 1$ үшін F_n^k векторлық кеңістігінің өлшемі төмендегі формуламен есептелінеді

$$\frac{(kn)!}{k!(n!)^k}.$$

Бұл формуланың дәлелдеуі [5] жұмыста келтірілген.

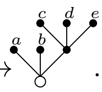
F_n^k кеңістігінің әр элементіне k ішкі төбеден тұратын n-арлы нонпланар боялған түбірлі ағаштарды сәйкес қоюға болады.

Мысалы: Егер $(abc) \in F_3^1$ болса, онда

$(abc) \leftrightarrow$ 

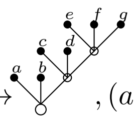
.

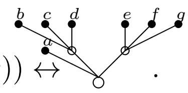
Егер $(ab(cde)) \in F_3^2$ болса, онда

$(ab(cde)) \leftrightarrow$ 

.

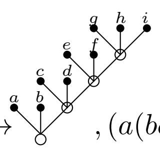
Егер $(ab(cd(efg))), (a(bcd)(efg)) \in F_3^3$ болса, онда

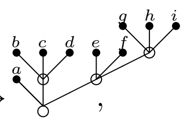
$(ab(cd(efg))) \leftrightarrow$ 

, $(a(bcd)(efg)) \leftrightarrow$ 

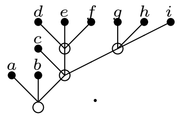
.

Егер $(ab(cd(ef(ghi)))), (a(bcd)(ef(ghi))), (ab(c(def)(ghi))) \in F_3^4$ болса, онда

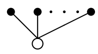
$(ab(cd(ef(ghi)))) \leftrightarrow$ 

, $(a(bcd)(ef(ghi))) \leftrightarrow$ 

,

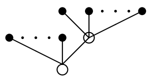
$(ab(c(def)(ghi))) \leftrightarrow$  .

Сонымен F_n^1 векторлық кеңістігіміздің элементтері



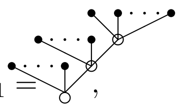
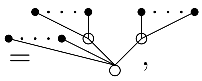
типті ағашпен туындатылған. Мұнда түптен шыққан жапырақтар саны n-ге тең.

Ал F_n^2 векторлық кеңістігіміздің элементтері



типті ағашпен туындатылған. Мұнда бірінші ішкі төбеден (түп) шыққан жапырақтар саны $n-1$, ал екінші ішкі төбеден шыққан жапырақтар саны n-ге тең.

Ал F_n^3 векторлық кеңістігіміздің элементтері

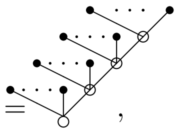
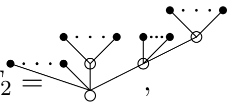
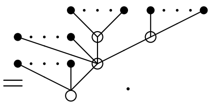
$T_1 =$  , $\qquad T_2 =$  ,

типті ағаштармен туындатылған. T_1 және T_2 ағаштарымен туындатылған векторлық кеңістіктерімізді сәйкесінше M^{T_1}, M^{T_2} арқылы белгілейік. Мұнда M^{T_1} мен M^{T_2} қиылысуы бос жиын және инвариантты ішкі кеңістіктер. Сондықтан F_n^3 кеңістігін

$$F_n^3 = M^{T_1} \oplus M^{T_2}$$

арқылы жаза аламыз. Мұнда T_1 бірінші типті ағашының бірінші ішкі төбесінен (түптен) шыққан жапырақтар саны $n-1$, ал екінші ішкі төбесінен шыққан жапырақтар саны $n-1$, ал үшінші ішкі төбесінен шыққан жапырақтар саны n-ге тең. T_2 екінші типті ағашының бірінші ішкі төбесінен (түптен) шыққан жапырақтар саны $n-2$, ал екінші ішкі төбесінен шыққан жапырақтар саны n, ал үшінші ішкі төбесінен шыққан жапырақтар саны n-ге тең.

Ал F_n^4 векторлық кеңістігіміздің элементтері

$T_1 =$  , $T_2 =$  , $T_3 =$  .

типті ағаштармен туындатылған. T_1, T_2 және T_3 ағаштарымен туындатылған векторлық кеңістіктерімізді сәйкесінше M^{T_1}, M^{T_2} және M^{T_3} арқылы белгілейік. Мұнда - M^{T_1}, M^{T_2} және M^{T_3} векторлық кеңістіктерінің өзара қиылысуы бос жиын және инвариантты ішкі кеңістіктер. Сондықтан F_n^4 кеңістігін

$$F_n^4 = M^{T_1} \oplus M^{T_2} \oplus M^{T_3}$$

арқылы жаза аламыз.

3. Негізгі нәтижелер

Негізгі теореманы келтірмес бұрын, симметриялық топтың векторлық кеңістікке әсерін анықтап алайық. Бұл жұмыста топтың әсері табиғи түрде анықталған. Мысалы, айталық $m = (a_1a_2(a_3a_4a_5)) \in F_3^2$, $\sigma = (123)(45) \in S_5$ болсын, онда әсер төмендегідей анықталады

$$\sigma(m) = (a_2a_3(a_1a_5a_4)).$$

Енді негізгі нәтижелерді келтірейік.

Теорема: *Айталық $n = 1, 2, 3, ...$ болсын. Онда*

a) $F_n^1 \cong S^{(n)}$,

b) $F_n^2 \cong \bigoplus_{\lambda \vdash 2n-1} K_{\lambda(n,n-1)} S^\lambda$,

c) $F_n^3 \cong \bigoplus_{\lambda \vdash 3n-2} (K_{\lambda(n,n-1,n-1)} + \sum_{i=0} c^\lambda_{(n-2)(2n-2i,2i)}) S^\lambda$,

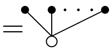
d) $F_n^4 \cong (K_{\lambda(n,n-1,n-1,n-1)} + K_{\lambda(n,n,n-1,n-2)} + \sum_{i=0}^{n-2} \sum_{j=0} c^\lambda_{(2n-3-i,i)(2n-2j,2j)}) S^\lambda$ *болады,*

мұндағы $K_{\lambda\mu}$ – Костка саны, $c^\nu_{\lambda\mu}$ – Литтлвуд-Ричардсон коэффициенті.

Костка сандары және Литтлвуд-Ричардсон коэффициенттері жайлы мәліметті толығырақ [6] жұмысынан көруге болады.

Дәлелдеуі:

a) Бір төбеден шыққан жапырақтардың орын ауыстыруы және бір төбеден шыққан тең ішкі ағаштардың орын ауыстыруы мономның толтырылуын (базис ережесі) өзгертпейді. Демек, төмендегі n-жапырақтан тұратын түбірлі ағаштың

$T =$ 

симметриялар тобы немесе T-ның автоморфизмдер тобы

$$\Gamma(T) = S_n$$

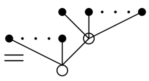
түрінде анықталады. Мұнда жапырақтар саны n-ге тең. Онда S_n-модульге жіктелуі төмендегі формуламен есептелінеді

$$Ind_{S_n}^{S_n}(S^{(n)}),$$

мұндағы $S^{(n)}$ $(n) \vdash n$ бөліктеуіне сәйкес келетін Шпехт модулі [6]. Демек

$$F_n^1 \cong S^{(n)}.$$

b) F_n^2 векторлық кеңістігі

$T =$ 

түбірлі ағашымен туындатылған. Мұнда жапырақтар саны $2n-1$-ге тең. Бұл T түбірлі ағашының автоморфизмдер тобы

$$\Gamma(T) = S_n \times S_{n-1}$$

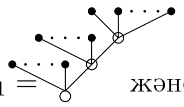
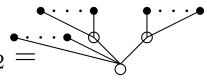
түрінде анықталады. Мұндағы $G \times H$ дегеніміз G мен H топтарының тік көбейтіндісі (direct product). Онда F_n^2 кеңістігінің S_{2n-1}-модульге жіктелуі төмендегі формуламен есептелінеді

$$Ind_{S_n \times S_{n-1}}^{S_{2n-1}}(S^{(n)} \otimes S^{(n-1)}).$$

Яғни

$$F_n^2 \cong \bigoplus_{\lambda \vdash 2n-1} K_{\lambda(n,n-1)} S^{\lambda}.$$

c) F_n^3 векторлық кеңістігі

$T_1 =$  және $T_2 =$ 

түбірлі ағаштарымен туындатылған. Мұнда T_1 мен T_2 ағаштарының жапырақтар саны $3n-2$-ге тең. T_1 мен T_2 түбірлі ағаштарының автоморфизмдер тобы сәйкесінше

$$\Gamma(T_1) = S_n \times S_{n-1} \times S_{n-1}, \quad \Gamma(T_2) = S_2[S_n] \times S_{n-2}$$

түрінде анықталады . Мұндағы $G[H]$ дегеніміз G мен H топтарының өрмек көбейтіндісі (wreath product). Онда M^{T_1} кеңістігінің S_{3n-2} – модульге жіктелуі төмендегідей есептелінеді

$$Ind_{S_n \times S_{n-1} \times S_{n-1}}^{S_{3n-2}}(S^{(n)} \otimes S^{(n-1)} \otimes S^{(n-1)}).$$

Яғни

$$M^{T_1} \cong \bigoplus_{\lambda \vdash 3n-2} K_{\lambda(n,n-1,n-1)} S^{\lambda}.$$

Ал M^{T_2} кеңістігінің S_{3n-2}-модульге жіктелуі төмендегідей есептелінеді

$$Ind_{S_2[S_n] \times S_{n-2}}^{S_{3n-2}}(S^{(n)} \otimes S^{(n)} \otimes S^{(n-2)}).$$

[7] жұмыста

$$Ind_{S_2[S_n]}^{S_{2n}}(S^{(n)} \otimes S^{(n)}) = S^{(2n)} \oplus S^{(2n-2,2)} \oplus S^{(2n-4,4)} \oplus \dots$$

плэтизм формуласы дәлелденген. Сондықтан

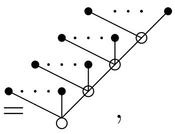
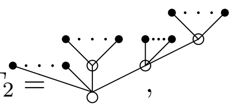
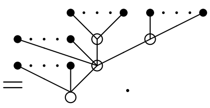
$$M^{T_2} = \bigoplus_{\lambda \vdash 3n-2} (c^{\lambda}_{(n-2)(2n)} + c^{\lambda}_{(n-2)(2n-2,2)} + c^{\lambda}_{(n-2)(2n-4,4)} + \dots) S^{\lambda}.$$

Демек,

$$F_n^3 = M^{T_1} \oplus M^{T_2} \cong$$

$$\cong \bigoplus_{\lambda \vdash 3n-2} (K_{\lambda(n,n-1,n-1)} + c^{\lambda}_{(n-2)(2n)} + c^{\lambda}_{(n-2)(2n-2,2)} + c^{\lambda}_{(n-2)(2n-4,4)} + \dots) S^{\lambda}.$$

d) F_n^4 векторлық кеңістігі

$T_1 =$  , $T_2 =$  , $T_3 =$  .

түбірлі ағаштарымен туындатылған. Мұнда T_1, T_2 және T_3 ағаштарының жапырақтар саны $4n-3$ болады. T_1, T_2 және T_3 түбірлі ағаштарының автоморфизмдер тобы сәйкесінше

$$\Gamma(T_1) = S_n \times S_{n-1} \times S_{n-1} \times S_{n-1}, \quad \Gamma(T_2) = S_n \times S_n \times S_{n-1} \times S_{n-2},$$

$$\Gamma(T_2) = S_2[S_n] \times S_{n-1} \times S_{n-2}$$

түрінде анықталады. Онда M^{T_1} кеңістігінің S_{4n-3}-модульге жіктелуі төмендегідей есептелінеді

$$Ind_{S_n \times S_{n-1} \times S_{n-1} \times S_{n-1}}^{S_{4n-3}} (S^{(n)} \otimes S^{(n-1)} \otimes S^{(n-1)} \otimes S^{(n-1)}).$$

Яғни

$$M^{T_1} \cong \bigoplus_{\lambda \vdash 4n-3} K_{\lambda(n,n-1,n-1,n-1)} S^{\lambda}$$

болады.

M^{T_2} кеңістігінің S_{4n-3}-модульге жіктелуі төмендегідей есептелінеді

$$Ind_{S_n \times S_n \times S_{n-1} \times S_{n-2}}^{S_{4n-3}} (S^{(n)} \otimes S^{(n)} \otimes S^{(n-1)} \otimes S^{(n-2)}).$$

Яғни

$$M^{T_2} \cong \bigoplus_{\lambda \vdash 4n-3} K_{\lambda(n,n,n-1,n-2)} S^{\lambda}$$

болады.

Ал M^{T_3} кеңістігінің S_{4n-3}-модульге жіктелуі төмендегідей есептелінеді

$$Ind_{S_2[S_n] \times S_{n-1} \times S_{n-2}}^{S_{4n-3}} (S^{(n)} \otimes S^{(n)} \otimes S^{(n-1)} \otimes S^{(n-2)}).$$

Бізге

$$Ind_{S_2[S_n]}^{S_{2n}} (S^{(n)} \otimes S^{(n)}) = S^{(2n)} \oplus S^{(2n-2,2)} \oplus S^{(2n-4,4)} \oplus \ldots$$

плэтизм формуласы және

$$Ind^{2n-3}_{S_{n-1}\times S_{n-2}}(S^{(n-1)}\otimes S^{(n-2)}) = S^{(2n-3)}\oplus S^{(2n-4,1)}\oplus S^{(2n-5,2)}\oplus\cdots\oplus S^{(n-1,n-2)}$$

формуласы белгілі.

Сондықтан

$$M^{T_3} = \bigoplus_{\lambda\vdash 4n-3}(c^{\lambda}_{(2n-3)(2n)} + c^{\lambda}_{(2n-3)(2n-2,2)} + c^{\lambda}_{(2n-3)(2n-4,4)} + \dots$$

$$+c^{\lambda}_{(2n-4,1)(2n)} + c^{\lambda}_{(2n-4,1)(2n-2,2)} + c^{\lambda}_{(2n-4,1)(2n-4,4)} + \dots$$

$$+c^{\lambda}_{(2n-5,2)(2n)} + c^{\lambda}_{(2n-5,2)(2n-2,2)} + c^{\lambda}_{(2n-5,2)(2n-4,4)} + \dots$$

$$\dots + c^{\lambda}_{(n-1,n-2)(2n)} + c^{\lambda}_{(n-1,n-2)(2n-2,2)} + c^{\lambda}_{(n-1,n-2)(2n-4,4)} + \dots)S^{\lambda} =$$

$$= (\sum_{i=0}^{n-2}\sum_{j=0} c^{\lambda}_{(2n-3-i,i)(2n-2j,2j)})S^{\lambda}.$$

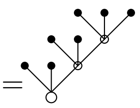
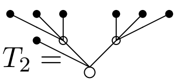
Демек,

$$F^4_n \cong M^{T_1} \oplus M^{T_2} \oplus M^{T_3} \cong$$

$$\cong (K_{\lambda(n,n-1,n-1,n-1)} + K_{\lambda(n,n,n-1,n-2)} + \sum_{i=0}^{n-2}\sum_{j=0} c^{\lambda}_{(2n-3-i,i)(2n-2j,2j)})S^{\lambda}. \quad \square$$

Графтардың автоморфизмдер тобы мен топтардың өрмек көбейтіндісі (wreath product) туралы мәліметтерді [8], [9] жұмыстарынан толығырақ көруге болады.

Мысалы: Айталық $n = 3$, $k = 2$ болсын. Онда F^2_3 кеңістігі

$T_1 =$  жане $T_2 =$ 

түбірлі ағаштарымен туындатылады. Демек,

$$F^2_3 \cong M^{T_1} \oplus M^{T_2},$$

мұндағы

$$M^{T_1} \cong \bigoplus_{\lambda\vdash 7} K_{\lambda(3,2,2)}S^{\lambda} =$$

$$S^{(7)} \oplus 2S^{(6,1)} \oplus 3S^{(5,2)} \oplus S^{(5,1,1)} \oplus 2S^{(4,3)} \oplus 2S^{(4,2,1)} \oplus S^{(3,3,1)} \oplus S^{(3,2,2)},$$

$$M^{T_2} \cong \bigoplus_{\lambda \vdash 7} (c^{\lambda}_{(1)(6)} + c^{\lambda}_{(1)(4,2)}) S^{\lambda} =$$

$$= S^{(7)} \oplus S^{(6,1)} \oplus S^{(5,2)} \oplus S^{(4,3)} \oplus S^{(4,2,1)}.$$

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Жахаев Б.К. ПРЕДСТАВЛЕНИЯ ГРУППЫ S_n НА НЕКОТОРЫХ ДЕРЕВЬЯХ В СВОБОДНОЙ КОММУТАТИВНОЙ n-АЛГЕБРЕ

В работе свободная коммутативная n-арная алгебра рассматривается как S_n-модуль. Базисные элементы свободной коммутативной n-арной алгебры порождаются нон-планарными n-арными деревьями. Полностью описываются S_n-модульные разложения некоторых n-арных деревьев в свободной коммутативной n-алгебре.

Zhakhayev B.K. REPRESENTATIONS OF S_n ON SOME TREES IN FREE COMMUTATIVE n-ALGEBRA

In this work free commutative n-ary algebra is considered as S_n-module. Basic elements of free commutative n-ary algebra are generated by non-planar n-ary trees. S_n-module decompositions of some n-ary trees in free commutative n-algebra are fully described.

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О КОГОМОЛОГИИ АЛГЕБРАИЧЕСКИХ ГРУПП И ИХ АЛГЕБР ЛИ В ПОЛОЖИТЕЛЬНОЙ ХАРАКТЕРИСТИКЕ

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Посвящается 60-летнему юбилею академика А. С. Джумадильдаева

Аннотация: Когомологическая теория алгебраических групп и их алгебр Ли в положительной характеристике является одним из интенсивно исследуемых направлений современной алгебры. Получены много значимых и интересных результатов. В статье дается обзор основных результатов развития когомологий простых модулей простых односвязных алгебраических групп и их алгебр Ли над алгебраически замкнутым полем положительной характеристики.

Ключевые слова: Алгебраическая группа, алгебра Ли, когомология, простой модуль.

1. Введение

1.1. Пусть G – простая односвязная алгебраическая группа над алгебраически замкнутым полем k характеристики $p > 0$ и $\mathfrak{g}$ – алгебра Ли группы G, T – максимальный тор в G, $B \supset T$ – подгруппа Бореля группы G, соответствующая отрицательным корням, U – унипотентный радикал B. Алгебры Ли B и U соответственно обозначим через $\mathfrak{u}$ и $\mathfrak{b}$. Ядро морфизма Фробениуса

$$F : G \to G,$$

Keywords: *Algebraic group, Lie algebra, cohomology, simple module.*
2010 Mathematics Subject Classification: 20G05, 20G10, 20G40, 17B45, 17B56.
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рассматриваемое как ядро морфизма групповых схем, обозначается через G_1. Ядра морфизма Фробениуса для B, U и T соответственно обозначаются через B_1, U_1 и T_1. Известно, что теория представления G_1 и теория представления алгебры Ли $\mathfrak{g}$ группы G эквивалентны.

Обозначим через Φ систему корней группы G относительно (G, T). Множество положительных и отрицательных корней соответственно обозначим через Φ^+ и Φ^- и пусть Δ – множество простых корней. Для системы корней ранга l пусть $\alpha_1, \cdots, \alpha_l$ – простые корни и $\lambda_1, \cdots, \lambda_l$ – фундаментальные веса. Обозначим целочисленную рещетку весов, порожденную фундаментальными весами, через $X(T)$ (аддитивная группа характеров максимального тора T) и пусть $X_+(T)$ – множество доминантных весов, $X_1(T)$ – множество ограниченных весов.

Скалярное произведение на евклидовом пространстве $\mathbb{E}$, порожденное системой корней Φ, обозначается через $\langle , \rangle$. Двойственным к $\alpha \in \Phi$ корнем является $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

Пусть α_0 – максимальный корень и $\widetilde{\alpha}_0$ – максимальный короткий корень. Действие группы Вейля W системы Φ на группу характеров $X(T)$ определяется по правилу $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, где $s_\alpha \in W$, $\alpha \in \Phi$. Если обозначить полусумму положительных корней через ρ, то другое действие группы Вейля, часто используемое в теории представлений, задается по формуле $w \cdot \lambda = w(\lambda + \rho) - \rho$, где $w \in W$, $\lambda \in X(T)$.

Аффинная группа Вейля W_p порождается всеми аффинными отражениями $s_{\alpha,np}$, где $\alpha \in \Phi^+$, $n \in \mathbb{Z}$. В дальнейшем мы будем пользоваться действием W_p на $X(T)$, определяемым формулой

$$s_{\alpha,np} \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha + np\alpha.$$

Для любого $\lambda \in X(T)$ существует одномерный B-модуль k_λ и индуцированный G-модуль $H^0(\lambda) = Ind_B^G(k_\lambda)$; $H^0(\lambda) \neq 0$ тогда и только тогда, когда $\lambda \in X_+(T)$. Пусть $V(\lambda)$ – модуль Вейля над G со старшим весом λ, тогда $H^0(\lambda) \cong V(-w_0\lambda)^*$, т.е. индуцированный модуль $H^0(\lambda)$ является G-модулем, двойственным модулю Вейля со старшим весом $-w_0(\lambda)$. При этом простой G-модуль $L(\lambda)$ будет простым цоколем $H^0(\lambda)$ или простым фактор-модулем $V(\lambda)$ по максимальному подмодулю [1, п. 5.7].

Пусть V – рациональный G-модуль. Обозначим через $V^{(r)}$ кручение Фробениуса V степени r. Более того, существует единственный $r \geq 1$ такой, что $V^{(-r)}$ есть G-модуль, на котором G_1 действует нетривиально.

Если вес ν лежит в замыкании нижнего фундаментального алькова аффинной группы Вейля, то для него будем использовать запись через фундаментальные веса: $\nu = m_1\lambda_1 + m_2\lambda_2$, где λ_1, λ_2 – фундаментальные веса и $m_1, m_2 \in \ ZZ$.

В основном, исследования когомологий простых модулей простых односвязных алгебраических групп и их алгебр Ли в положительной характеристике проводились в следующей последовательности:

1) вычисление $H^n(\mathfrak{u}, k)$;
2) вычисление $H^n(B_1, k_\lambda)$;
3) вычисление $H^n(G_1, H^0(\lambda))$;
4) вычисление $H^n(\mathfrak{g}, H^0(\lambda))$;
5) описание структуры $H^0(\lambda)$;
6) вычисление $H^n(G, L(\lambda))$;
7) вычисление $H^n(\mathfrak{g}, L(\lambda))$.

1.2. Над полем характеристики нуль $H^n(\mathfrak{u}, k)$ был вычислен Б. Костантом в [2]. В положительной характеристике, если $p \geq h$, где $h = \langle \rho, \widetilde{\alpha}_0^\vee \rangle + 1$ – число Кокстера, из результатов работ Э. М. Фридландера и Б. Дж. Паршаля [3], П. Поло и Дж. Тилоуина [4] следует, что формальные характеры этих когомологических групп совпадают с формальными характерами соответствующих групп характеристики нуль. В малых характеристиках известны только структуры малых когомологических групп $H^1(\mathfrak{u}, k)$ и $H^2(\mathfrak{u}, k)$. $H^1(\mathfrak{u}, k)$ полностью вычислены Е. К. Янценом в [5]. Для неспециальных характеристик поля $H^2(\mathfrak{u}, k)$ был описан в диссертации автора [6] и дополнены в работах К. П. Бенделя, Д. К. Накано и К. Пиллена [7] ($p > 2$), К. Б. Райта [8] ($p = 2$).

Для больших характеристик поля, когда $p \geq h$, начальные три этапа реализованы полностью в известных работах Х. Х. Андерсена и Е. К. Янцена [9], Э. Фридландера и Б. Паршалля [10], С. Кумара, Н. Лаурицена и Дж. Томсена [11]. В малых характеристиках поля вычисления проводились для $n = 1, 2$. Когомологические группы

$$H^1(\mathfrak{u}, k),\ H^1(B_1, k_\lambda),\ H^1(G_1, H^0(\lambda)) \cong H^1(\mathfrak{g}, H^0(\lambda))$$

вычислены Е. К. Янценом в [5] для всех положительных характеристик.

Вторые группы когомологии

$$H^2(\mathfrak{u}, k),\ H^2(B_1, k_\lambda),\ H^2(G_1, H^0(\lambda))$$

вычислены в диссертации Ш. Ш. Ибраева [6] (неспециальные положительные характеристики) и в работах К. П. Бенделя, Д. К. Накано и К. Пиллена [7] ($p > 2$), К. Б. Райта [8] ($p = 2$).

В общем случае, структура когомологии $H^n(\mathfrak{g}, H^0(\lambda))$ мало изучены. Как уже отмечалось выше, когомологий $H^1(\mathfrak{g}, H^0(\lambda))$ вычислены полностью Е. К. Янценом в [5]. Другие результаты связаны с конкретными модулями.

Наиболее простым является случай, когда $H^0(\lambda)$ является неприводимым G-модулем. Хорошо известно, что ограниченный G-модуль $H^0(\lambda)$ неприводим, если λ лежит внутри нижнего фундаментального алькова. Тогда когомология $H^n(\mathfrak{g}, H^0(\lambda))$ нетривиальна только в следующих случаях:

- $\lambda = (p-2)\lambda_1$ и $\Phi = A_1$, $p > 2$;
- $\lambda = 0$.

Первый случай соответствует классической простой трехмерной алгебре Ли $\mathfrak{sl}_2(k)$ и ее когомологии с коэффициентами в $H^0((p-2)\lambda_1)$ и в $H^0(0)$ и полностью описан А. С. Джумадильдаевым при $p > 3$ [12].

Пусть $\lambda = 0$. Первая группа когомологий тривиального одномерного модуля подробно рассмотрена в [5]. Когомология $H^2(\mathfrak{g}, k)$ эквивалентна пространству классов эквивалентных центральных расширений алгебры Ли $\mathfrak{g}$. Эти когомологии вычислены Ван дер Калленом [13] и Ш. Ш. Ибраевым [14].

Случай $\lambda = \alpha_0$ также представляет особый интерес в связи с структурными вопросами и вопросами классификации конечномерных модулярных алгебр Ли. Действительно, нетрудно заметить, что $H^0(\lambda)$ изоморфен коприсоединенному модулю $\mathfrak{g}^*$. А также известно, что $\mathfrak{g}^* \cong \mathfrak{g}$, если $\mathfrak{g}$ – простая алгебра Ли. Тогда когомологии $H^1(\mathfrak{g}, H^0(\alpha_0))$ и $H^2(\mathfrak{g}, H^0(\alpha_0))$ соответственно могут быть интерпретированы как пространства внешних дифференцирований и локальных деформаций алгебры Ли $\mathfrak{g}$. Они изучались в работах Р. Блока [15], Б. Ю. Вейсфеллера и В. Г. Каца [16], А. Н. Рудакова [17], А. С. Джумадильдаева [18], С. А. Кириллова, М. И. Кузнецова и Н. Г. Чебочко [19, 20, 21] и Ш. Ш. Ибраева [22]. Согласно результатам этих работ для простой классической алгебры Ли $\mathfrak{g}$ (факторалгебры

Ли исключаются) когомологии $H^1(\mathfrak{g}, H^0(\alpha_0))$ и $H^2(\mathfrak{g}, H^0(\alpha_0))$ тривиальны кроме случая, когда $H^2(\mathfrak{g}, H^0(\alpha_0))^{(-1)} \cong H^0(\lambda_1)$ для классической алгебры Ли типа B_2 над алгебраически замкнутым полем характеристики 3. Заметим, что простая фактоалгебра Ли классической модулярной алгебры Ли по центру соответствует простому модулю $L(\alpha_0)$ над G и будет рассмотрена ниже в связи с когомологиями простых модулей.

О структуре модуля $H^0(\lambda)$. Пусть $V(\lambda)$ – модуль Вейля над G со старшим весом λ, тогда $H^0(\lambda) \cong V(-w_0(\lambda))^*$, т.е. индуцированный модуль $H^0(\lambda)$ является G-модулем, двойственным модулю Вейля со старшим весом $-w_0(\lambda)$. Согласно [23, часть II, п. 4.14] когомологии односвязных простых алгебраических групп и их алгебр Ли над алгебраически замкнутым полем положительной характеристики с коэффициентами в простых конечномерных модулях зависят от структур модуля Вейля. Классификация конечномерных простых модулей была получена К. Шевалле [24] (см. также [23, часть II, п. 2.7]). Каждый простой модуль однозначно (с точностью до изоморфизма) определяется своим старшим весом. Его можно описать как фактор-модуль модуля Вейля соответствующего старшего веса по максимальному подмодулю. В общем случае характеры простых модулей известны для достаточно больших характеристик поля. Для подробного ознакомления можно изучить работы Дж. Люстига [25, 26], Х. Андерсена, Е. К. Янцена и В. Зергеля [27], М. Кашивара и Т. Танисаки [28, 29], М. Каждана и Дж. Люстига [30, 31], Р. Безрукавникова, И. Мирковича и Д. Румынина [32], П. Фибига [33, 34] и недавние контрпримеры Г. Вильямсона к гипотезе Люстига в больших характеристиках. Явное описание структуры модулей Вейля получено либо для малых алгебраических групп, либо для некоторых классов модулей Вейля. Простые модули над SL_2 были описаны А. Н. Рудаковым и И. Р. Шафаревичем [35], Р. Картером и Э. Клайном [36]. Композиционные факторы модуля Вейля для SL_3 и Sp_4 частично вычислены Б. Браденом в [37]. Для алгебраических групп Sp_4 $(p > 0)$, G_2 $(p > 5)$ и SL_4 $(p > 3)$ характеры и структура максимальных подмодулей модулей Вейля, старшие веса которых сильно связаны с доминантными весами внутри нижнего фундаментального алькова аффинной группы Вейля, вычислены Е. К. Янценом соответственно в работах [38, стр.139–141], [39, стр.294–298] (заметим, что простые модули ограниченными старшими весами не исчерпываются этими модулями). Случай

$p = 3$ для группы G_2 описан Т. Спрингером в работе [40]. Модули Вейля с простыми максимальными подмодулями получены в работах Е. К. Янцена [41] и Дж. О'Хэллоран [42, теорема 1.3]. Композиционные факторы модулей Вейля над Sp_n со старшими фундаментальными весами вычислены в работах А. Премета и И. Супруненко [43], А. М. Адамовича [44].

Когомологии $H^n(G, L(\lambda))$. Полное описание когомологий простых модулей получены либо для малых групп, либо для некоторых малых модулей. Например, вычислены когомологии первой степени SL_2 (Э. Клайн [45]), SL_3 (С. Йехия [46]), Sp_4 (Дж. Йе [47]), G_2, $p \geq 13$ (Дж. Лиу, Дж. Йе [48]), SO_7 (Ш. Ш. Ибраев [49]); когомологии второй степени SL_2 и SL_3 (Д. Стюарт [50], [51]), Sp_4, $p > 7$, G_2, $p \geq 7$ и SO_7 (Ш. Ш. Ибраев [52], [53], [54]). Когомологии простых модулей третьей степени описаны для SL_2, $p > 3$ (Ш. Ш. Ибраев [55]), SL_3, $p > 3$, Sp_4, $p > 5$ и G_2, $p > 11$ (А. С. Джумадильдаев, Ш. Ш. Ибраев [66]). Примеры нетривиальных третьих когомологий простых модулей содержатся в работе Дж. О'Хэллоран [42, предл. 2.1], в которой описаны когомологии простых модулей, связанные с модулями Вейля с простыми максимальными подмодулями в области ограниченных весов.

В общем случае младшие когомологии простых модулей известны либо для достаточно больших характеристик поля, либо для некоторых классов простых модулей малых размерностей. Формулы вычисления расширения простых модулей, включая тривиальный одномерный модуль, получены Х. Андерсеном [56], К. Бенделем, Д. Накано и К. Пилленом в [57, Теорема 2.5] для больших характеристик $p \geq 3h - 3$, где h – число Кокстера. Когомологии первой степени простых модулей над Sp_{2n} с фундаментальными старшими весами вычислены в работах А. С. Клещева и Дж. Шета [58, 59]. Первые когомологии простых модулей с минимальными доминантными старшими весами были вычислены Э. Клайном, Б. Паршаллом и Л. Скоттом [60, 61]. В [62] коллективом авторов последний результат был расширен для всех доминантных старших весов, меньших или равных фиксированному фундаментальному весу, за исключением некоторых малых характеристик поля, зависящих от типов систем корней. Развивая методику, примененную в предыдущей работе, теми же авторами были получены аналогичные результаты для вторых групп когомологий [63]. В работе Дж. МакНинча вычислены вторые когомологии простых модулей, размерности которых не превышают характеристику поля [64].

Когомологии $H^n(\mathfrak{g}, L(\lambda))$. Эти когомологии известны только для алгебр Ли ранга 1 и 2, а также для некоторых простых модулей. Когомологии $\mathfrak{g} = \mathfrak{sl}_2(k)$ с коэффициентами в простых модулях при $p > 3$ вычислены А.С. Джумадильдаевым [12]. В характеристике $p = 3$ когомологии нетривиальны также для простых модулей $L(0) = H^0(0) \cong k$ и $L(\lambda_1) = H^0(\lambda_1)$. Аналогичные вычисления когомологии $\mathfrak{g} = \mathfrak{sl}_3(k)$ с коэффициентами в простых модулях при $p > 3$ были проделаны Ш. Ш. Ибраевым (не опубликовано). Вторые когомологии простых модулей для классических алгебр Ли ранга 2 были вычислены А. С. Джумадильдаевым и Ш. Ш. Ибраевым [65] (с исправлениями в [66, 1.2]).

Когомологии $H^1(\mathfrak{g}, L(\lambda))$ известны в следующих случаях (Е. К. Янцен и Ш. Ш. Ибраев):

- $\Phi = A_2, A_3$, $p \geq 3$ и $n = 1$ [5, 6.10)];
- характеристка поля не специальна и не делит индекса связанности, λ – крохотный вес [5, предл.4.9(b)],;
- $\Phi = A_l$ и $\lambda \in \{p\lambda_1 - \alpha_1, p\lambda_l - \alpha_l\}$ [5, предл.4.10(b)];
- $\Phi = G_2$ и $p = 2$ [5, предл.5.3];
- $\Phi = G_2$, $p = 3$ и $\lambda = \lambda_2$ [5, предл.5.10];
- $\Phi = C_l$, $l \geq 1$, $p = 2$ и $\lambda = 0$ [5, предл.6.2];
- $\Phi = A_1$, $p = 2$ и $\lambda = \lambda_1$ [5, предл.6.4(a)];
- $\Phi = A_l$, $l \geq 2$, $p = 2$ и $\lambda \in \{\lambda_2, \lambda_{l-2}\}$ [5, предл.6.4(b)];
- $\Phi = A_3$,, $p = 2$ и $\lambda = \lambda_2$ [5, предл.6.4(c)];
- $\Phi = A_l$ и $\lambda = r\lambda_1$, $0 \leq r < p$ [5, предл.6.5];
- $n = 1$ и $\lambda = \widetilde{\alpha}_0$ [5, предл.6.6];
- $\lambda = \alpha_0$ [5, предл.6.7 – 6.9];
- $\lambda = s_{\widetilde{\alpha}_0,p} \cdot 0$ и $p \geq h$ [14, предл.3.1].

Пусть $\mathfrak{g}$ – классическая алгебра Ли над алгебраически замкнутым полем k характеристики $p > 0$ и $C_{\mathfrak{g}}$ – ее центр.

Лемма 1. *Пусть $\overline{\mathfrak{g}}$ – простая факторалгебра Ли алгебры Ли $\mathfrak{g}$ по центру $C_{\mathfrak{g}}$. Тогда $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$.*

Доказательство. В пространстве $\overline{\mathfrak{g}}$ можно ввести структуру модуля над алгебрами Ли $C_{\mathfrak{g}}$, $\mathfrak{g}$ и $\overline{\mathfrak{g}}$:

$C_{\mathfrak{g}} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}}$, $(c, \overline{a}) \mapsto \mu(c)\overline{a}$, где μ – ненулевая линейная форма на $C_{\mathfrak{g}}$;

$\mathfrak{g} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},\ (a_1, \overline{a}_2) \mapsto \overline{[a_1, a_2]},\ a_1 \in \mathfrak{g},\ \overline{a}_2 \in \overline{\mathfrak{g}};$
$\overline{\mathfrak{g}} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},\ (\overline{a}_1, \overline{a}_2) \mapsto \overline{[a_1, a_2]},\ \overline{a}_1,\ \overline{a}_2 \in \overline{\mathfrak{g}}.$

Короткая точная последовательность коцепных комплексов

$$0 \to (C^*(C_{\mathfrak{g}}, \overline{\mathfrak{g}}), d) \to (C^*(\mathfrak{g}, \overline{\mathfrak{g}}), d) \to (C^*(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}), d) \to 0$$

дает длинную точную когомологическую последовательность

$$\cdots \to H^0(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) \to H^1(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \to H^1(\mathfrak{g}, \overline{\mathfrak{g}}) \to H^1(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) \to \cdots .$$

Так как $H^j(C_{\mathfrak{g}}, \overline{\mathfrak{g}}) = 0$ для всех $j \geq 0$ [67, лемма 4.2], то из точности последней длинной когомологической последовательности следует, что $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}}) \cong H^n(\mathfrak{g}, \overline{\mathfrak{g}})$. Лемма 1 доказана.

Согласно Лемме 1 вычисление когомологии $H^n(\overline{\mathfrak{g}}, \overline{\mathfrak{g}})$ приводится к вычислению $H^n(\mathfrak{g}, \overline{\mathfrak{g}})$. Когомологии $H^1(\overline{\mathfrak{g}}, \overline{\mathfrak{g}})$ вычислены в работах Д. С. Пермякова [68] и в [14, предл.4.2]. Когомологии $H^2(\overline{\mathfrak{g}}, \overline{\mathfrak{g}})$ вычислены в работе Н. Г. Чебочко [21].

2. Когомологии $H^n(\mathfrak{u}, k)$

Основой методики вычисления когомологии $H^n(G, L(\lambda))$ и $H^n(\mathfrak{g}, L(\lambda))$ является структура когомологии $H^n(\mathfrak{u}, k)$.

Теорема 1 [3, 4]. *Если $p \geq h-1$, то имеют место следующие изоморфизм T-модулей:*

$$H^n(\mathfrak{u}, k) \cong \bigoplus_{w \in W,\, l(w)=n} -w \cdot 0.$$

В малых характеристиках результаты получены для $n = 1, 2$.

Предложение 1 [5, предл. 2.1]. *Имеют место следующие изоморфизм B-модулей:*

$$H^1(U_1, k) \cong H^1(\mathfrak{u}, k) \cong (\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])^*.$$

Предложение 1 доказано для всех характеристик поля. Если $p > 3$, то второй изоморфизм хорошо известный факт. Первый изоморфизм и утверждения следующего предложения можно легко получить, используя первый квадрант спектральной последовательности Фридландера–Паршаля

$$E_2^{i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes H^j(\mathfrak{u}, k) \Longrightarrow H^{2i+j}(U_1, k).$$

Предложение 2 [6, 3, 4]. *Если $p > 3$, то имеют место следующие изоморфизм T-модулей:*

$$H^2(\mathfrak{u}, k) \cong \bigoplus_{w \in W,\, l(w)=2} -w \cdot 0.$$

Таким образом, если $p > 3$, то согласно Предложениям 1 и 2 Теорема 1 верна и при $n = 1, 2$. В характеристиках $p = 2$ и $p = 3$ появляются дополнительные 2-коциклы. Они получены в работах [6, 7, 8]. Если $p = 3$, то дополнительные классы нетривиальных коциклов имеются в следующих случаях:

$\Phi = B_l, l \geq 3 :\ \alpha_{n-2} + 2\alpha_{n-1} + 3\alpha_n;$

$\Phi = C_l, l \geq 3 :\ \alpha_{n-2} + 3\alpha_{n-1} + \alpha_n;$

$\Phi = F_4 :\ \alpha_1 + 2\alpha_2 + 3\alpha_3,\ \alpha_2 + 3\alpha_3 + \alpha_4;$

$\Phi = G_2 :\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 3\alpha_2,\ 6\alpha_1 + 3\alpha_2,\ 4\alpha_1 + 3\alpha_2.$

В случае характеристики $p = 2$ дополнительные классы коциклов можно найти в работах [6, теорема 2.1] (неспециальные характеристики) [8, приложение A].

3. Когомологии $H^n(G, L(\lambda))$

Для вычисления когомологий $L(\lambda)$ используется спектральная последовательность Линдона-Хохшильда-Серра относительно расширения групп $1 \to G^1 \to G \to G/G^1 \to 1$:

$$E_2^{ij} = H^i(G/G^1, H^j(G^1, L(\lambda))) \Rightarrow H^{i+j}(G, L(\lambda)). \tag{1}$$

Согласно Салливану [69, п.1, с. 768]

$$H^i(G/G^1, H^j(G^1, L(\lambda))) \cong H^i(G, H^j(G^1, L(\lambda))^{(-1)}).$$

Следовательно,

$$E_2^{ij} \cong H^i(G, H^j(G^1, L(\lambda))^{(-1)}). \tag{2}$$

Пусть $X_1(T)$ – множество ограниченных доминантных весов. Если $\lambda \in X_+(T)$, то его можно представить в виде $\lambda = \lambda^0 + p\mu$, где $\lambda^0 \in X_1(T)$, $\mu \in X_+(T)$ и по теореме Стейнберга о тензорном произведении $L(\lambda) = L(\lambda^0) \otimes L(\mu)^{(1)}$. Тогда $H^j(G^1, L(\lambda))^{(-1)} = H^j(G^1, L(\lambda^0) \otimes L(\mu)^{(1)})^{(-1)}$. Так как для G-модулей V и W имеет место изоморфизм G-модулей [69, п.1, с. 767]

$$H^j(G^1, V \otimes W^{(1)}) \cong H^j(G^1, V) \otimes W^{(1)},$$

то

$$H^j(G^1, L(\lambda))^{(-1)} \cong H^j(G^1, L(\lambda^0))^{(-1)} \otimes L(\mu).$$

Используя полученный изоморфизм для $H^j(G^1, L(\lambda))^{(-1)}$ в формуле (2), получим

$$E_2^{ij} \cong H^i(G, H^j(G^1, L(\lambda^0))^{(-1)} \otimes L(\mu)). \tag{3}$$

Если E_∞^{nm} – стабильное значение точки (n, m) спектральной последовательности (1), то

$$H^n(G, L(\lambda)) = \bigoplus_{i+j=n} E_\infty^{ij}. \tag{4}$$

Методика вычисления когомологии $H^n(G, L(\lambda))$, основаннная на формулах (3) и (4), впервые была применена Э. Фридландером и Б. Паршаллом для вычисления когомологий $H^n(G, V^{(1)})$, где G – односвязная простая алгебраическая группа над полем положительной характеристики, и V – конечномерный рациональный G-модуль, допускающий хорошую фильтрацию [10].

Определим следующие множества простых конечномерных G-модулей:

$$M(\lambda^0) = \{L(\lambda) \,|\, Ext^1_G(L(\lambda^0), L(\lambda)) \neq 0,\ \lambda \in X_+(T)\},\ \lambda^0 \in X_1(T);$$

$$M_i = \{L(\lambda^0 + p\mu) \,|\, E_2^{2-i,i} = H^{2-i}(G, H^i(G^1, L(\lambda^0))^{(-1)} \otimes L(\mu)) \neq 0,$$
$$\lambda^0 \in X_1(T),\ \mu \in X_+(T)\},\ i = 0, 1, 2;$$

$$N_i = \{L(\lambda^0 + p\mu) \,|\, E_2^{3-i,i} = H^{3-i}(G, H^i(G^1, L(\lambda^0))^{(-1)} \otimes L(\mu)) \neq 0,$$
$$\lambda^0 \in X_1(T),\ \mu \in X_+(T)\},\ i = 0, 1, 2, 3.$$

В случае односвязных простых алгебраических групп ранга 2 явные описания множеств $M(\mu^0)$, $\mu^0 \in X_1(T)$, M_i, $i = 0, 1, 2$; N_i, $i = 0, 1, 2, 3$, известны.

Пусть $G = SL_3$ и $p > 3$. Порядок фундаментальной группы системы R равен $|\pi| = 3$. Поэтому в области ограниченных весов существуют $|W|/|\pi| = 2$ алькова аффинной группы Вейля W_p. Обозначим их через C_1 и C_2. Тогда

$C_1 = \{\nu \in X(T) \,|\, 0 < \langle \nu + \rho, \alpha^\vee \rangle < p$ для всех $\alpha \in R^+\}$,

$C_2 = \{\omega_0(\nu) = s_{\widetilde{\alpha}_0, p} \cdot \nu \,|\, \nu \in C_1\}$.

Пусть $G = Sp_4$ и $p > 5$. В области ограниченных весов существуют 4 алькова аффинной группы Вейля W_p. Обозначим их через C_1, C_2, C_3 и C_4 [38, с. 139]. Тогда

$C_1 = \{\nu \in X(T) \,|\, 0 < \langle \nu + \rho, \alpha^\vee \rangle < p \text{ для всех } \alpha \in R^+\}$,

$C_2 = \{\omega_0(\nu) = s_{\widetilde{\alpha}_0,p} \cdot \nu \,|\, \nu \in C_1\}$,

$C_3 = \{\omega_1(\nu) = s_{\widetilde{\alpha}_0,p} s_{\alpha_2,0} \cdot \nu \,|\, \nu \in C_1\}$,

$C_4 = \{\omega_2(\nu) = s_{\widetilde{\alpha}_0,p} s_{\alpha_2,0} s_{\alpha_1,0} \cdot \nu \,|\, \nu \in C_1\}$.

Будем рассматривать элементы еще двух альковов, не входящих в область ограниченных весов:

$C_5 = \{\beta_0(\nu) = s_{\widetilde{\alpha}_0,p} s_{\alpha_2,0} s_{\widetilde{\alpha}_0,p} \cdot \nu \,|\, \nu \in C_1\}$,

$C_6 = \{\delta_0(\nu) = s_{\widetilde{\alpha}_0,p} s_{\alpha_2,0} s_{\alpha_1,0} s_{\widetilde{\alpha}_0,p} \cdot \nu \,|\, \nu \in C_1\}$.

Пусть $G = G_2$ и $p > 7$. В области ограниченных весов существуют 12 алькова аффинной группы Вейля W_p. Следуя Янцену [38, с. 139-140], обозначим их через $C_1 – C_8, C_{11}, C_{13}, C_{15}, C_{16}$. Альковы $C_9, C_{10}, C_{12}, C_{14}$ не входят в область ограниченных весов. Рассмотрим следующие элементы W_p :

$w^0 = s_{\widetilde{\alpha}_0,p}$, $w^1 = s_{\widetilde{\alpha}_0,p} s_{\alpha_1,0}$, $w^2 = s_{\widetilde{\alpha}_0,p} s_{\alpha_1,0} s_{\alpha_2,0}$, $w^3 = s_{\widetilde{\alpha}_0,p} s_{\alpha_1,0} s_{\alpha_2,0} s_{\alpha_1,0}$,

$w^4 = s_{\widetilde{\alpha}_0,p} s_{\alpha_1,0} s_{\alpha_2,0} s_{\alpha_1,0} s_{\alpha_2,0}$, $w^5 = s_{\widetilde{\alpha}_0,p} s_{\alpha_1,0} s_{\alpha_2,0} s_{\alpha_1,0} s_{\alpha_2,0} s_{\alpha_1,0}$.

Тогда по определению

$C_1 = \{\nu \in X(T) \,|\, 0 < \langle \nu + \rho, \alpha^\vee \rangle < p \text{ для всех } \alpha \in R^+\}$,

$C_{i+2} = \{\omega_i(\nu) = w^i \cdot \nu \,|\, \nu \in C_1\}$, $i = 0, 1, 2, 3$;

$C_6 = \{\beta_0(\nu) = w^3 w^0 \cdot \nu \,|\, \nu \in C_1\}$;

$C_7 = \{\omega_4(\nu) = w^4 \cdot \nu \,|\, \nu \in C_1\}$;

$C_8 = \{\delta_0(\nu) = w^4 w^0 \cdot \nu \,|\, \nu \in C_1\}$;

$C_9 = \{\omega_5(\nu) = w^5 \cdot \nu \,|\, \nu \in C_1\}$;

$C_{2i+9} = \{\delta_i(\nu) = w^4 w^i \cdot \nu \,|\, \nu \in C_1\}$, $i = 1, 2, 3$;

$C_{2i+10} = \{\gamma_i(\nu) = w^5 w^i \cdot \nu \,|\, \nu \in C_1\}$, $i = 0, 1, 2$;

$C_{16} = \{\beta_1(\nu) = w^4 w^3 w^0 \cdot \nu \,|\, \nu \in C_1\}$.

Мы также будем рассматривать элементы следующих двух альковов:

$C_{17} = \{\gamma_3(\nu) = w^5 w^3 \cdot \nu \,|\, \nu \in C_1\}$, $C_{18} = \{\beta_2(\nu) = w^5 w^3 w^0 \cdot \nu \,|\, \nu \in C_1\}$.

3.1. Расширения простых модулей для G. Для алгебраических групп ранга 2 известны следующие результаты.

Предложение 3 [51]. *Пусть G – односвязная простая алгебраическая группа SL_3 над алгебраически замкнутым полем k характеристики $p > 3$,*

$\lambda^0 \in \{0,\, \lambda_1,\, \lambda_2,\, \alpha_0\}$, $\mu \in X_+(T)$, λ_1, λ_2 – *фундаментальные веса. Тогда*

$$Ext^1_G(L(\lambda^0), L(\mu)) \cong \begin{cases} k, & \text{если } \mu \in M(\lambda^0); \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M(0) = \{L(\omega_0(0))^{(r)}, L(\omega_0((p{-}3)\lambda_2))^{(r)} \otimes L(\lambda_1)^{(r+1)}, L(\omega_0((p{-}3)\lambda_1))^{(r)} \otimes L(\lambda_2)^{(r+1)}, r \geq 0\},$$

$$M(\lambda_1) = \{L(\omega_0(\lambda_1)), L(\omega_0((p{-}4)\lambda_2)) \otimes L(\lambda_1)^{(1)},\; L(\omega_0((p{-}4)\lambda_1{+}\lambda_2)) \otimes L(\lambda_2)^{(1)}\} \cup$$
$$\cup \{L(\lambda_1) \otimes L(\gamma)^{(1)} \,|\, L(\gamma) \in M(0)\},$$

$$M(\lambda_2) = \{L(\gamma)^* \,|\, L(\gamma) \in M(\lambda_1)\},$$

$$M(\alpha_0) = \{L(\omega_0(\lambda_1 + \lambda_2)),\; L(\omega_0(\lambda_1 + (p-5)\lambda_2)) \otimes L(\lambda_1)^{(1)},$$
$$L(\omega_0((p-5)\lambda_1 + \lambda_2)) \otimes L(\lambda_2)^{(1)}\} \cup \{L(\alpha_0) \otimes L(\gamma)^{(1)} \,|\, L(\gamma) \in M(0)\}.$$

Предложение 4 [47]. *Пусть G – односвязная простая алгебраическая группа Sp_4 над алгебраически замкнутым полем k характеристики $p > 5$, $\lambda^0 \in \{0, \lambda_1, \lambda_2, \alpha_0\}$ и $\mu \in X_+(T)$, λ_1, λ_2 – фундаментальные веса. Тогда*

$$Ext^1_G(L(\lambda^0), L(\mu)) \cong \begin{cases} k, & \text{если } \mu \in M(\lambda^0); \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M(0) = \{L(\omega_0(0))^{(r)}, L(\omega_2(0))^{(r)} \otimes L(\lambda_2)^{(r+1)}, L(\omega_1((p{-}4)\lambda_1))^{(r)} \otimes L(\lambda_1)^{(r+1)}, r \geq 0\},$$

$$M(\lambda_1) = \{L(\omega_0(\lambda_1)), L(\omega_2(\lambda_1)) \otimes L(\lambda_2)^{(1)},\; L(\omega_1((p-5)\lambda_1)) \otimes L(\lambda_1)^{(1)}\} \cup$$
$$\cup \{L(\lambda_1) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\},$$

$$M(\lambda_2) = \{L(\omega_0(\lambda_2)), L(\omega_2(\lambda_2)) \otimes L(\lambda_2)^{(1)},\; L(\omega_1((p-6)\lambda_1 + \lambda_2)) \otimes L(\lambda_1)^{(1)}\} \cup$$
$$\cup \{L(\lambda_2) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\},$$

$$M(\alpha_0) = \{L(\omega_0(\alpha_0)), L(\omega_2(\alpha_0)) \otimes L(\lambda_2)^{(1)},\; L(\omega_1((p-6)\lambda_1)) \otimes L(\lambda_1)^{(1)}\} \cup$$
$$\cup \{L(\alpha_0) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\}.$$

Предложение 5 [48, 53]. *Пусть G – односвязная простая алгебраическая группа G_2 над алгебраически замкнутым полем k характеристики $p > 7$, $\lambda^0 \in \{0, \lambda_1, \lambda_2\}$ и $\mu \in X_+(T)$, λ_1, λ_2 – фундаментальные веса. Тогда*

$$Ext^1_G(L(\lambda^0), L(\mu)) \cong \begin{cases} k, & \text{если } \mu \in M(\lambda^0); \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M(0) = \{L(\omega_0(0))^{(r)},\ L(\delta_3(0))^{(r)},\ L(\delta_3(0))^{(r)} \otimes L(\lambda_2)^{(r+1)},$$
$$L(\omega_4(0))^{(r)} \otimes L(\lambda_1)^{(r+1)},\ L(\delta_1(0))^{(r)} \otimes L(\lambda_1)^{(r+1)},\ r \geq 0\},$$
$$M(\lambda_1) = \{L(\omega_0(\lambda_1)),\ L(\delta_3(\lambda_1)),\ L(\delta_3(\lambda_1)) \otimes L(\lambda_2)^{(1)},\ L(\omega_4(\lambda_1)) \otimes L(\lambda_1)^{(1)},$$
$$L(\delta_1(\lambda_1)) \otimes L(\lambda_1)^{(1)}\} \cup \{L(\lambda_1) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\},$$
$$M(\lambda_2) = \{L(\omega_0(\lambda_2)),\ L(\delta_3(\lambda_2)),\ L(\delta_3(\lambda_2)) \otimes L(\lambda_2)^{(1)},\ L(\omega_4(\lambda_2)) \otimes L(\lambda_1)^{(1)},$$
$$L(\delta_1(\lambda_2)) \otimes L(\lambda_1)^{(1)}\} \cup \{L(\lambda_2) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\}.$$

3.2. Вторые когомологии простых модулей для G. Эти когомологии полностью описаны для групп ранга 1 и 2.

Теорема 2 [51]. *Пусть G – односвязная простая алгебраическая группа SL_3 над алгебраически замкнутым полем k характеристики $p > 3$ и $L(\lambda)$ – простой G-модуль со старшим весом $\lambda \in X_+(T)$, λ_1, λ_2 – фундаментальные веса. Тогда*

$$H^2(G, L(\lambda)) \cong \begin{cases} k, & \text{если } \lambda \in \bigcup_{i=0}^{2} M_i, \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M_2 = \{L(\alpha_0)^{(1)},\ L((p-3)\lambda_1) \otimes L(\lambda_2)^{(1)},\ L((p-3)\lambda_1) \otimes L(\lambda_1)^{(1)}\},$$
$$M_1 = \{L(\omega_0(0)) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(0)\} \cup$$
$$\cup \{L(\omega_0((p-3)\lambda_1)) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(\lambda_2)\} \cup$$
$$\cup \{L(\omega_0((p-3)\lambda_2)) \otimes L(\mu)^{(1)} \,|\, L(\mu) \in M(\lambda_1)\},$$
$$M_0 = \{L(\mu)^{(d)} \,|\, L(\mu) \in M_1 \cup M_2,\ d \geq 1\}.$$

Теорема 3 [52]. *Пусть G – односвязная простая алгебраическая группа Sp_4 над алгебраически замкнутым полем k характеристики $p > 5$ и $L(\lambda)$ – простой G-модуль со старшим весом $\lambda \in X_+(T)$, λ_1, λ_2 – фундаментальные веса. Тогда*

$$H^2(G, L(\lambda)) \cong \begin{cases} k, & \text{если } \lambda \in \bigcup_{i=0}^{2} M_i, \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M_2 = \{L(\alpha_0)^{(1)}, L(\omega_0((p-4)\lambda_1)) \otimes L(\lambda_1)^{(1)}, L(\omega_1(0)),$$

$$L(\omega_1(0)) \otimes L(\lambda_2)^{(1)}, L(\omega_2((p-4)\lambda_1)) \otimes L(\lambda_1)^{(1)}\},$$

$$M_1 = \{L(\omega_0(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(0)\} \cup \{L(\omega_2(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(\lambda_2)\} \cup$$

$$\cup \{L(\omega_1((p-4)\lambda_1)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(\lambda_1)\},$$

$$M_0 = \{L(\mu)^{(d)} \mid L(\mu) \in M_1 \cup M_2, \, d \geq 1\}.$$

Теорема 4 [53]. *Пусть G – односвязная простая алгебраическая группа G_2 над алгебраически замкнутым полем k характеристики $p > 7$ и $L(\lambda)$ – простой G-модуль со старшим весом $\lambda \in X_+(T)$, λ_1, λ_2 – фундаментальные веса. Тогда*

$$H^2(G, L(\lambda)) \cong \begin{cases} k, & \text{если } \lambda \in \bigcup_{i=0}^{2} M_i \setminus \{L(\beta_1(0))^{(d)}, \, d \geq 0\}, \\ k \oplus k, & \text{если } \lambda \in \{L(\beta_1(0))^{(d)}, \, d \geq 0\}, \\ 0 & \text{в остальных случаях,} \end{cases}$$

где

$$M_2 = \{L(\lambda_2)^{(1)}, L(\omega_1(0)), L(\omega_3(0)) \otimes L(\lambda_1)^{(1)}, L(\delta_0(0)) \otimes L(\lambda_1)^{(1)}, L(\delta_2(0)),$$

$$L(\delta_2(0)) \otimes L(\lambda_1)^{(1)}, L(\delta_2(0)) \otimes L(\lambda_2)^{(1)}, L(\beta_1(0)), L(\beta_1(0)) \otimes L(\lambda_2)^{(1)}\},$$

$$M_1 = \{L(\omega_0(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(0)\} \cup \{L(\omega_4(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(\lambda_1)\} \cup$$

$$\cup \{L(\delta_1(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(\lambda_1)\} \cup \{L(\delta_3(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(0) \cup M(\lambda_2)\},$$

$$M_0 = \{L(\mu)^{(d)} \mid L(\mu) \in M_1 \cup M_2, \, d \geq 1\}.$$

3.3. Когомологии $H^3(G, L(\lambda))$. Эти когомологии полностью описаны для простых алгебраических групп ранга 2.

Теорема 5 [66]. *Пусть G – односвязная простая алгебраическая группа SL_3 над алгебраически замкнутым полем k характеристики $p > 3$ и $L(\lambda)$ – простой G-модуль. Тогда*

$$H^3(G, L(\lambda)) \cong \begin{cases} k, & \text{если } L(\lambda) \in \bigcup_{i=0}^{3} N_i \setminus \{L(\omega_0(0))^{(s)} \otimes L(\alpha_0)^{(s+1)},\ s \geq 0\}; \\ k \oplus k, & \text{если } L(\lambda) \in \{L(\omega_0(0))^{(s)} \otimes L(\alpha_0)^{(s+1)},\ s \geq 0\}; \\ 0 & \text{в остальных случаях.} \end{cases}$$

Теорема 6 [66]. *Пусть G – односвязная простая алгебраическая группа Sp_4 над алгебраически замкнутым полем k характеристики $p > 5$ и $L(\lambda)$ – простой G-модуль. Тогда*

$$H^3(G, L(\lambda)) \cong \begin{cases} k, & \text{если } L(\lambda) \in \bigcup_{i=0}^{3} N_i; \\ 0 & \text{в остальных случаях.} \end{cases}$$

Определим следующие множества простых G-модулей:

$$L_3 = \{L(\delta_1(0)) \otimes L(\lambda_2)^{(1)}, L(\delta_1(0)) \otimes L(\lambda_1)^{(1)}, L(\delta_3(0))\},$$

$$L_2 = \{L(\beta_1(0)) \otimes L(\mu)^{(1)} \mid L(\mu) \in M(0)\},$$

$$L_1 = \{L(\omega_0(0)) \otimes L(\beta_1(0))^{(d+2)},\ d \geq 0\} \cup$$

$$\cup \{L(\mu) \otimes L(\lambda_1)^{(1)} \otimes L(\beta_1(0))^{(d+2)} \mid \mu = \omega_4(0), \delta_1(0),\ d \geq 0\} \cup$$

$$\cup \{L(\delta_3(0)) \otimes L(\lambda_2)^{(1)} \otimes L(\beta_1(0))^{(d+2)},\ d \geq 0\} \cup \{L(\delta_3(0)) \otimes L(\beta_1(0))^{(d+2)},\ d \geq 0\},$$

$$L_0 = \{L(\mu)^{(s)} \mid L(\mu) \in \cup_{i=1}^{3} L_i,\ s \geq 1\}.$$

Используя эти обозначения, мы можем теперь сформулировать основной результат настоящей работы для группы типа G_2.

Теорема 7 [66]. *Пусть G – односвязная простая алгебраическая группа G_2 над алгебраически замкнутым полем k характеристики $p > 11$ и $L(\lambda)$ – простой G-модуль. Тогда*

$$H^3(G, L(\lambda)) \cong \begin{cases} k, & \text{если } L(\lambda) \in \bigcup_{i=0}^{3} N_i \setminus \bigcup_{i=0}^{3} L_i; \\ k \oplus k, & \text{если } L(\lambda) \in \bigcup_{i=0}^{3} L_i; \\ 0 & \text{в остальных случаях.} \end{cases}$$

4. Когомологии $H^n(\mathfrak{g}, L(\lambda))$

В категории ограниченных модулей теория представлений G^1 и теория представлений алгебры Ли $\mathfrak{g}$ эквивалентны [23, часть I, п. 9.6]. Следовательно, когомологии ограниченного модуля для G^1 и соответствующие ограниченные когомологии алгебры Ли $\mathfrak{g}$ также эквивалентны. Ограниченные когомологии алгебр Ли для ограниченных модулей были введены Хохшильдом в [70]. В этой же работе была построена точная последовательность, устанавливающая связь между ограниченными и обычными когомологиями алгебр Ли. Известно, что особые модули (модули с нетривиальными когомологиями) ограниченных алгебр Ли ограничены [12]. Поэтому последовательность Хохшильда также устанавливает связь между когомологиями G^1 с коэффициентами в ограниченных модулях и соответствующими когомологиями алгебры Ли $\mathfrak{g}$ группы G. Для любой ограниченной алгебры Ли существует по крайней мере один модуль с нетривиальной обычной второй когомологией [71].

Теорема 8 [12]. *Пусть $\mathfrak{g} = A_1$ $(p > 2)$ и V – неприводимый $\mathfrak{g}$-модуль. Тогда $H^2(\mathfrak{g}, V) = 0$, кроме следующих случаев:*

$$H^1(\mathfrak{g}, L((p-2)\lambda_1)) \cong H^2(\mathfrak{g}, L((p-2)\lambda_1)) \cong L(\lambda_1)^{(1)},\ H^0(\mathfrak{g}, k) \cong H^3(\mathfrak{g}, k) \cong k.$$

Теорема 9 [65, 66]. *Пусть $\mathfrak{g} = A_2$, B_2, G_2 $(p > h)$ и V – неприводимый $\mathfrak{g}$-модуль. Тогда $H^2(\mathfrak{g}, V) = 0$, кроме следующих случаев:*

(a) $\mathfrak{g} = A_2$, $H^2(\mathfrak{g}, L((p-3)\lambda_1)) \cong L(\lambda_1)^{(1)}$, $H^2(\mathfrak{g}, L((p-3)\lambda_2)) \cong L(\lambda_2)^{(1)}$;

(b) $\mathfrak{g} = B_2$, $H^2(\mathfrak{g}, L((p-3)\lambda_1 + 2\lambda_2)) \cong L(\lambda_1)^{(1)} \oplus L(0)^{(1)}$,

$$H^2(\mathfrak{g}, L(\lambda_1 + (p-4)\lambda_2)) \cong L(\lambda_2)^{(1)}, H^2(\mathfrak{g}, L((p-2)(\lambda_1 + \lambda_2))) \cong L(\lambda_2)^{(1)};$$

(c) $\mathfrak{g} = G_2$, $H^2(\mathfrak{g}, L((p-6)\lambda_1 + \lambda_2)) \cong L(0)^{(1)}$,

$$H^2(\mathfrak{g}, L((p-5)\lambda_1 + 2\lambda_2)) \cong L(\lambda_1)^{(1)},$$

$$H^2(\mathfrak{g}, L(4\lambda_1 + (p-3)\lambda_2)) \cong L(\lambda_1)^{(1)} \oplus L(\lambda_2)^{(1)} \oplus L(0)^{(1)},$$

$$H^2(\mathfrak{g}, L(3\lambda_1 + (p-4)\lambda_2)) \cong L(\lambda_1)^{(1)},$$

$$H^2(\mathfrak{g}, L((p-2)(\lambda_1 + \lambda_2))) \cong L(\lambda_2)^{(1)} \oplus L(0)^{(1)} \oplus L(0)^{(1)}.$$

Согласно этой теореме в случае нетривиального простого ограниченного модуля соответствующие вторые когомологии G^1 и $\mathfrak{g}$ совпадают.

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Ыбыраев Ш.Ш. ОҢ СИПАТТАМАДАҒЫ АЛГЕБРАЛЫҚ ТОПТАР МЕН ОЛАРДЫҢ ЛИ АЛГЕБРАЛАРЫНЫҢ КОГОМОЛОГИЯЛАРЫ ТУРАЛЫ

Оң сипаттамадағы алгебралық топтар мен олардың Ли алгебраларының когомологиялық теориясы қазіргі алгебрадағы қарқынды зерттеліп жатқан бағыттырдың бірі болып саналады. Көптеген маңызды және қызықты нәтижелер алынған. Мақалада оң сипаттамалы алгебралық түйықталған өрістегі жәй бірбайланысты алгебралық топтар мен олардың Ли алгебраларының қарапайым модульдерінің когомологияларының дамуының негізгі нәтижелеріне шолу жасалған.

Ibraev Sh.Sh. ON THE COHOMOLOGY OF ALGEBRAIC GROUPS AND THEIR LIE ALGEBRAS IN POSITIVE CHARACTERISTIC

The cohomology theory of algebraic groups and their Lie algebras in positive characteristic is one of the intensively studied areas of the modern algebra. There are a lot of important and interesting results. In this paper we give a short overview of the main results of the cohomology theory of simple modules for simple and simply connected algebraic groups and their Lie algebras over an algebraically closed field of positive characteristic.

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О ГАМИЛЬТОНОВЫХ АЛГЕБРАХ ЛИ ХАРАКТЕРИСТИКИ 2*

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Аннотация: Приводится инвариантное построение неальтернирующих гамильтоновых алгебр Ли четной характеристики, соответствующих общим гамильтоновым формам с полиномиальными коэффициентами. Вычисляются когомологии комплекса разделенных степеней симметрических дифференциальных форм. Рассматриваются примеры фильтрованных неальтернирующих гамильтоновых алгебр Ли, в том числе алгебр, допускающих невырожденные дифференцирования.

Ключевые слова: Алгебры Ли характеристики два, гамильтоновы алгебры Ли, невырожденные дифференцирования.

1. Введение

А.С. Джумадильдаев является ведущим специалистом в области теории деформаций модулярных алгебр Ли, оказавшим большое влияние на развитие исследований в этом направлении. Отметим в качестве примера, что уже одна из первых его работ [1], посвященная деформациям классических алгебр Ли серий A_n, B_n, C_n, D_n характеристики 3, получила дальнейшее продолжение в исследованиях нижегородских математиков, описавших локальные деформации всех классических алгебр Ли характеристики $p \geq 3$ и алгебр Ли с однородной системой корней характеристики 2 [2]-[4].

Keywords: *Lie algebra of characteristic two, Hamiltonian Lie algebra, non-singular derivations.*

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В настоящих заметках мы рассмотрим ряд вопросов о гамильтоновых алгебрах Ли над полем характеристики 2, имеющих, в частности, отношение к фильтрованным деформациям градуированных алгебр Ли. Интерес к таким алгебрам обусловлен рядом причин. Во-первых, над полем четной характеристики существуют необычные гамильтоновы алгебры – неальтернирующие, соответствующие симметрическим дифференциальным формам. Серия градуированных алгебр Ли такого типа была построена в работе [5]. Во-вторых, построение фильтрованных деформаций градуированных алгебр Ли и их реализаций является составной частью проблемы классификации простых модулярных алгебр Ли. Кроме того, гамильтоновы алгебры Ли являются одной из универсальных математических структур, имеющей важное значение в математике. Удивительным образом они находят применение даже в далеких, на первый взгляд, областях. Так, после работ А. Шалева и Е.И. Зельманова [6], [7], посвященных гипотезам о коклассах, гамильтоновы алгебры Ли стали играть важную роль в теории конечных p-групп и про-p-групп. Развитие техники, связанной с алгебрами Ли, привело к возникновению нового направления в теории градуированных алгебр Ли над полями характеристики p ([8]-[9]). Особый интерес в этой области представляют простые модулярные алгебры Ли, допускающие невырожденное дифференцирование. С учетом классификации простых модулярных алгебр Ли характеристики $p > 3$ (Р. Блок, Р. Вилсон, Г. Штраде, А. Премет и др.) все простые алгебры Ли характеристики $p > 3$, имеющие невырожденные дифференцирования, к настоящему время известны (см. [10]-[12]), также как и среди алгебр тех же типов в случае характеристик 2 и 3.

Неальтернирующие гамильтоновы алгебры Ли интенсивно исследовались школой Д. Лейтеса (S. Bouarrouj, U. Yier, M. Messaoudene, П. Грозман, А. Лебедев, И. Щепочкина) в направлении распространения идей и методов теории суперагебр Ли на случай алгебр Ли четной характеристики. Был построен комплекс симметрических дифференциальных форм в разделенных степенях, что привело к более естественному определению градуированных неальтернирующих гамильтоновых алгебр Ли, проведен анализ неальтернирующих алгебр с точки зрения продолжений Картана, рассмотрены некоторые алгебры Воличенко (см. [13]-[15]).

В настоящих заметках дается общее геометрическое, т.е. независящее от координат или, иначе, от конкретных образующих алгебры разделенных степеней, определение комплекса разделенных степеней симметрических дифференциальных форм. Естественность разделенных, а не обычных степеней для дифференциальных форм объясняется наличием структуры алгебры Хопфа на симметрической алгебре свободного A-модуля W специальных дифференцирований исходной алгебры разделенных степеней $A = A(n : m)$. Мы вычисляем когомологии комплекса симметрических дифференциальных форм. Далее дается определение общих неальтернирующих гамильтоновых форм и соответствующих алгер Ли. Идея геометризации неклассических простых модулярных алгебр Ли разрабатывалась первым автором с конца 80-х годов, начиная с рассмотрения алгебры разделенных степеней как естественной структуры, связанной с распределениями над алгеброй срезанных многочленов [16]. Большое влияние здесь оказала работа В.Г. Каца о фильтрованных алгебрах Ли картановского типа [17], в которой было показано, что все фильтрованные деформации алгебр Ли картановского типа, соответствующих стандартным дифференциальным формам, являются алгебрами, соответствующими более общим формам того же вида. Для гамильтоновых алгебр Ли геометрический подход позволил выделить инвариантные характеристики гамильтоновой формы с полиномиальными коэффициентами, определяющие класс изоморфизма общих гамильтоновых алгебр Ли, – значение формы в нуле и ее когомологический класс [18]. Полностью задачу классификации гамильтоновых форм решил С.М. Скрябин (см. [19] для случая полиномиальных форм). Здесь мы устанавливаем простоту общих неальтернирующих алгебр Ли $P(n : m, \omega)$ в стабильном случае $m \neq 1$.

Последняя часть заметок посвящена построению невырожденных дифференцирований неальтернирующих гамильтоновых алгебр Ли. Здесь приводится общая конструкция, основанная на описании максимальных торов алгебры W [20]. Разложение неальтернирующей алгебры Ли на корневые пространства относительно невырожденного дифференцирования указывает на связь неальтернирующих алгебр с алгебрами Блока [21]. Подробное исследование общих неальтернирующих алгебр Ли будет изложено в последующих работах.

Всюду в дальнейшем основное поле F предполагается алгебраически замкнутым характеристики $p = 2$.

2. Симметрические дифференциальные формы

Над полем характеристики 2 комплекс симметрических дифференциальных форм построен в [13], [15]. Здесь мы используем элементарную технику алгебр Хопфа для обоснования структуры алгебры разделенных степеней. Пусть $W = W(n : m)$ – алгебра Ли специальных дифференцирований алгебры разделенных степеней $A = A(n : m)$. Обозначим через $S(W) = S^0 + S^1 + S^2 + \dots$ симметрическую алгебру свободного A-модуля W со стандартной градуировкой. На $S(W)$ имеется структура алгебры Хопфа над A с коумножением $\delta(D) = D \otimes 1 + 1 \otimes D$, антиподальным отображением $S(D) = -D$ и коединицей $\varepsilon(D) = 0$, $D \in W$. Пространство Ω_s, двойственное к S(W) и состоящее из линейных функционалов, непрерывных относительно $\mathfrak{m}$-адической топологии, где $\mathfrak{m}$ – ядро ε, является коммутативной алгеброй разделенных степеней. Например, для базиса A-модуля W, состоящего из частных производных $\partial_1, \dots, \partial_n$, получаем базис $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ симметрической алгебры $S(W)$ и двойственный базис $dx^{(\beta)} = dx_1^{(\beta_1)} \cdots dx_n^{(\beta_n)}$ алгебры Ω_s,

$$dx^{(\alpha)} dx^{(\beta)} = \binom{\alpha+\beta}{\alpha} dx^{(\alpha+\beta)}.$$

Для $f \in A$ полагаем $(f dx_i)^{(k)} = f^k dx_i^{(k)}$ и распространяем операции возведения в разделенные степени на произвольные 1-формы так, чтобы выполнялось соотношение $(\omega_1 + \omega_2)^{(k)} = \sum_{s=0}^{k} \omega_1^{(s)} \omega_2^{(k-s)}$. Очевидно, $\Omega_s = S^{0*} + S^{1*} + S^{2*} + \dots$ – градуированная алгебра, $S^{0*} = A\varepsilon$, ε – единица в Ω_s и в дальнейшем обозначается через 1. По определению симметрическому k-линейному над A отображению ψ из W в A соответствует единственное A-линейное отображение $\phi \in S^{k*}$. Таким образом, S^{k*} можно рассматривать как A-модуль симметрических k-полилинейных над A отображений из W в A. Напомним, что умножение в Ω_s сопряжено коумножению в $S(W)$, в частности, для 1-форм $\omega_1,\ \omega_2 \in S^{1*}$

$$\omega_1\omega_2(D_1,\ D_2) = \omega_1 \otimes \omega_2(\delta(D_1 D_2)) = \omega_1 \otimes \omega_2((D_1 \otimes 1 + 1 \otimes D_1)(D_2 \otimes 1 + 1 \otimes D_2)) =$$

$$= \omega_1 \otimes \omega_2(D_1 D_2 \otimes 1 + D_1 \otimes D_2 + D_2 \otimes D_1 + 1 \otimes D_1 D_2) =$$

$$= \omega_1(D_1)\omega_2(D_2) + \omega_1(D_2)\omega_2(D_1).$$

При $p = 2$ получаем $\omega^2 = 0$. Для 1-формы ω, $D_1, \ldots, D_k \in W$, можно показать, что $\omega^{(k)}(D_1, \ldots, D_k) = \omega(D_1) \ldots \omega(D_k)$.

Алгебра Ли W действует стандартным образом на $S(W)$ и на двойственном модуле Ω_s. При $p = 2$ определен внешний дифференциал d, который является дифференцированием алгебры Ω_s:

$$df(D) = D(f),\ D \in W,\ d(\omega^{(k)}) = \omega^{(k-1)} d\omega,\ d^2 = 0.$$

Над полем характеристики 2 мы рассматриваем кососимметрические формы как симметрические формы такие, что $\omega(D_1, \ldots, D_k) = 0$, если $D_i = D_j$ для некоторых $i \neq j$. Поэтому комплекс $(\Omega_s,\ d)$ содержит в качестве подкомплекса стандартный комплекс де Рама (Ω, d) над алгеброй разделенных степеней A. Для симметрических дифференциальных форм также имеет место формула гомотопии Картана:

$$D\omega = (d\omega) \lrcorner D + d(\omega \lrcorner D).$$

Теорема 1. *Пусть x_i, $i = 1, \ldots, n$, – стандартные переменные алгебры разделенных степеней $A = A(n : m)$ над полем F характеристики 2, $\overline{x_i} = x_i^{(2^{m_i}-1)}$, $(\Omega_s,\ d)$ – комплекс симметрических дифференциальных форм над A, $B(n)$ – градуированная алгебра разделенных степеней над полем F от переменных $(dx_i)^{(2)}$, $i = 1, \ldots, n$, степени 2, $(\Omega,\ d)$ – комплекс де Рама над A.*
i) Кольцо когомологий $H^*(\Omega_s)$ является тензорным произведением градуированных алгебр

$$H^*(\Omega_s) = B(n) \otimes_F H^*(\Omega),$$

ii) $\dim H^i(\Omega_s) = \binom{n+i-1}{i}$,
iii) $H^1(\Omega_s) =< [\overline{x_i} dx_i],\ i = 1, \ldots, n >$,
iv)$H^2(\Omega_s) =< [(dx_i)^{(2)}],\ i = 1, \ldots, n,\ [\overline{x_i x_j} dx_i dx_j],\ 1 \leq i < j \leq n >$.

Из строения комплекса $(\Omega_s,\ d)$ получаем i). Теперь легко найти ряд Пуанкаре комплекса Ω_s:

$$P(t) = \sum_{i \geq 0} \dim H^i(\Omega_s) t^i = (1-t)^{-n},$$

откуда следует ii). iii)-iv) следуют из i) и хорошо известного описания когомологий де Рама над алгеброй разделенных степеней:

$H^k(\Omega) =< [\overline{X_I} dX_I],\ I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\},\ i_1 < \ldots < i_k >$. Здесь $\overline{X_I} = \overline{x_{i_1}} \ldots \overline{x_{i_k}},\ dX_I = dx_{i_1} \ldots dx_{i_k}$.

Замечание. Описание когомологий де Рама над алгеброй разделенных степеней может быть получено из леммы Пуанкаре и известно, по крайней мере, с 70-х годов XX века. Например, в кандидатской диссертации Я.С. Крылюка (1978) вычислены когомологии подкрученного (в частности, стандартного) комплекса де Рама [22, Лемма 1.1]. В качестве ссылки можно указать монографию Х. Штраде [23, Proposition 6.4.4], где рассматривается случай четного числа переменных, что несущественно.

3. Гамильтоновы алгебры Ли

Гамильтоновой формой над полем характеристики 2 будем называть симметрическую замкнутую 2-форму ω над алгеброй разделенных степеней $A = A(n : m)$, $\omega = \sum_{i \leq j} \omega_{ij} dx^{(\varepsilon_i + \varepsilon_j)}$, $\omega_{ij} \in A$. Отметим, что мы рассматриваем кососимметрические формы как частный случай симметрических форм. Симметрическая, но не кососимметрическая форма будет называться неальтернирующей. Пусть $a_{ij} = \omega_{ij}(0)$ – свободный член многочлена ω_{ij}, $\omega(0)$ – симметрическая дифференциальная форма с постоянными коэффициентами a_{ij}. Согласно теореме 1 iii) 2-форма ω замкнута тогда и только тогда, когда $\omega_{ii} = a_{ii}$ и кососимметрическая форма $\sum_{i<j} \omega_{ij} dx^{(\varepsilon_i + \varepsilon_j)}$ замкнута. Матрица формы ω в базисе $\{\partial_i\}$ свободного A-модуля $W = W(n : m)$ имеет вид $(\omega) = (\omega(\partial_i, \partial_j)) = (\omega_{ij})$, $\omega_{ij} = \omega_{ji}$ при $i > j$. Форма невырождена, если ее матрица обратима. Невырожденность ω равносильна невырожденности $\omega(0)$. Как известно, невырожденная симметрическая форма на векторном пространстве над совершенным полем характеристики 2 в кососимметрическом случае допускает симплектический базис и пространство четномерно, а в неальтернирующем случае – ортонормированный базис [20]. Поэтому будем считать, что линейная часть $\omega(0)$ симметрической дифференциальной формы ω имеет канонический вид.

Гамильтонова алгебра Ли $\tilde{P}(n : m, \omega)$ состоит из векторных полей, сохраняющих неальтернирующую гамильтонову форму ω, $\tilde{P}(n : m, \omega) = \{D \in W(n : m) |\ D\omega = 0\}$. Так же, как в классическом случае, гамильтоново векторное поле $D \in \tilde{P}(n : m, \omega)$ определяется гамильтонианом

$F \in \tilde{A} = A + < x_i^{(2^{m_i})}, \ i = 1, \ldots, n >$, который определен с точностью до константы,

$$D = D_F = \sum_{i,j=1}^{n} \tilde{\omega}_{ij} \partial_i F \partial_j.$$

$$[D_F, D_G] = D_{\{F,G\}}, \ \{F, G\} = D_F(G) = \sum_{i,j=1}^{n} \tilde{\omega}_{ij} \partial_i F \partial_j G.$$

Здесь $(\tilde{\omega}_{ij}) = (\omega ij)^{-1}$. Алгебра Ли $\tilde{P}(n : m, \omega))$ отождествляется с пространством $\tilde{A}/F$ со скобкой Пуассона $\{F, G\}$. Отметим, что для неальтернирующей гамильтоновой формы алгебра Пуанкаре $(\tilde{A}, \ \{\ ,\ \})$, вообще говоря, является алгеброй Лейбница, но не алгеброй Ли. Алгебра Ли $\tilde{P}(n : m, \omega)$ имеет фильтрацию, индуцированную естественной фильтрацией алгебры W. Соответствующая ассоциированная градуированная алгебра Ли изоморфна градуированной алгебре Ли $P''(n : m)$ работы [5]. Алгебра Ли $P(n : m, \omega) = A/F$ является идеалом коразмерности n в алгебре $\tilde{P}(n : m, \omega)$, для которого ассоциированная градуированная алгебра Ли изоморфна алгебре Ли $P(n : m)$. Следующая теорема является непосредственным следствием теоремы 2.2 работы [5].

Теорема 2. *Пусть ω – замкнутая невырожденная неальтернирующая гамильтонова форма. Если $m \neq 1$, то $P(n : m, \omega)$ – простая алгебра Ли размерности $2^{|m|} - 1$.*

Рассмотрим примеры неальтернирующих гамильтоновых алгебр Ли для малых n и $m = 1$. Легко проверить непосредственно, что $P(2 : 1, \omega)$, $\omega = (dx_1)^2 + (dx_2)^2 + x_1 x_2 dx_1 dx_2$ – простая трехмерная алгебра Ли, изоморфная альтернирующей гамильтоновой алгебре $H(2 : 1, \overline{\omega})$, соответствующей форме $\overline{\omega} = (1 + x_1 x_2) dx_1 dx_2$ а также алгебре Ли $W(1 : 2)'$. Пусть $n = 3$, $m = (1, 1, 1)$,

$$\omega = dx_1 dx_2 + dx_3^{(2)} + a x_1 x_2 dx_1 dx_2 + b x_1 x_3 dx_1 dx_3 + c x_2 x_3 dx_2 dx_3,$$

$$(\omega) = \begin{pmatrix} 0 & 1 + a x_1 x_2 & b x_1 x_3 \\ 1 + a x_1 x_2 & 0 & c x_2 x_3 \\ b x_1 x_3 & c x_2 x_3 & 1 \end{pmatrix},$$

$$\omega^{-1} = \begin{pmatrix} 0 & 1+ax_1x_2 & cx_2x_3 \\ 1+ax_1x_2 & 0 & bx_1x_3 \\ cx_2x_3 & bx_1x_3 & 1 \end{pmatrix}.$$

Получаем алгебру $P(3,1,\omega)$ размерности 7 с умножением

$$\{f,g\} = (1+ax_1x_2)(\partial_1 f\partial_2 g + \partial_2 f\partial_1 g) + cx_2x_3(\partial_1 f\partial_3 g + \partial_3 f\partial_1 g) + \\ + bx_1x_3(\partial_2 f\partial_3 g + \partial_3 f\partial_2 g) + \partial_3 f\partial_3 g.$$

Вычисления, выполненные вторым автором, показывают, что если $b \neq 0$ или $c \neq 0$, то $P(3,1,\omega)$ – простая алгебра Ли. Согласно компьютерным вычислениям, проведенным Б. Эйк [25], здесь, как и в случае $n = 2$, новых простых алгебр Ли нет.

4. Невырожденные дифференцирования неальтернирующих гамильтоновых алгебр Ли

Вопрос о существовании невырожденных дифференцирований алгебр Ли картановского типа (и алгебр Меликяна), в частности, гамильтоновых алгебр Ли, соответствующих кососимметрическим гамильтоновым формам, рассматривался в работах [10]-[12]. Было доказано, что гамильтонова алгебра допускает невырожденное дифференцирование тогда и только тогда, когда когомологический класс соответствующей гамильтоновой формы $\sum a_{ij}\overline{x_i x_j}$ невырожден, т.е. $\det(a_{ij}) \neq 0$.

В качестве примера, приведем построение неальтернирующих гамильтоновых алгебр Ли характеристики 2, допускающих невырожденное дифференцирование. Согласно [12] алгебра Ли L допускает невырожденное дифференцирование тогда и только тогда, когда p-алгебра Ли $DerL$ содержит тор T максимальной размерности, все веса которого в алгебре L отличны от нуля. По построению неальтернирующие гамильтоновы алгебры Ли являются транзитивными подалгебрами алгебры Ли $W = W(n:m)$. Поэтому естественно начать построение с максимального тора алгебры W. Полное описание максимальных торов в p-замыкании алгебры Ли $W = W(n:m)$ получено в [20]. Следуя теореме 4 [20], выберем замкнутые 1-формы $\omega_1, \ldots, \omega_n \in \Omega^1(A)$ над $A = A(n:m)$ такие, что $\{\omega_1(0), \ldots, \omega_n(0)\}$ образуют базис пространства $\Omega^1(0)$, а их классы когомологий $\{[\omega_1], \ldots, [\omega_n]\}$ – базис первой группы когомологий де Рама

$H^1(\Omega)$. Очевидно, $\{\omega_1, \ldots, \omega_n\}$ – базис свободного A-модуля Ω^1. Пусть $\{D_1, \ldots, D_n\}$ – двойственный базис A-модуля W, V – линейная оболочка этого базиса над полем F. Согласно теореме 4 [20] V – абелева транзитивная подалгебра в W и ее p-замыкыние $\overline{V}$ в $DerW$ является тором максимальной размерности $|m| = m_1 + \cdots + m_n$. Кроме того, алгебра инвариантов тора $\overline{V}$ состоит из констант, $A_0 = F$. В частности, все дифференцирования D_i полупросты и алгебра A является прямой суммой одномерных весовых подпространств относительно V, все веса образуют элементарную абелеву группу Γ порядка $q = 2^{|\mathbf{m}|}$, $A = \oplus_{\beta \in \Gamma} A_\beta$ (подробности см в [20]). Очевидно, тор $\overline{V}$ содержит элемент D такой, что $D^q = D$, $A_\beta = A_b, D|_{A_b} = b \cdot id,\ b \in \mathbb{F}_q$.

Перейдем к построению неальтернирующей гамильтоновой алгебры Ли. Положим $\omega = \omega_1^{(2)} + \ldots + \omega_n^{(2)}$, тогда ω – неальтернирующая гамильтонова форма и $\tilde{P}(n : m, \omega)) = V \oplus P(n : m, \omega)$. Умножение в $P(n : m, \omega)$ можно записать так:

$$\{F, G\} = \sum_{i=1}^{n} D_i(F) D_i(G).$$

Алгебра Ли $P^{(1)}(n : m, \omega)$ – простая алгебра Ли с невырожденным дифференцированием D. Действительно, выберем базис $\{u_b\}$ в A/F,

$$A/F = \oplus_{b \in \mathbb{F}^*} A_b,\ A_b = < u_b > .$$

Так как D содержится в p-оболочке абелевой подалгебры $V \subset \tilde{P}(n : m, \omega)$, а при отождествлении $\tilde{P}(n : m, \omega)$ с A/F присоединенное действие элементов из V совпадает с действием соответствующих дифференцирований из $V \subset W$ на A/F, то и элементы из p-оболочки $\overline{V}$ действуют на $P(n : m, \omega)$, как на соответствующие гамильтонианы: $[D, D_G] = D_{D(G)}$. Другими словами, для $u_b \in P(n : m, \omega) D(u_b) = b u_b$, следовательно, D – невырожденное дифференцирование. В эту схему включаются также градуированные алгебры Ли $P(n : m)$. Действительно, стандартную форму $\omega = (dx_1)^{(2)} + \ldots + (dx_n)^{(2)}$ можно записать в виде $\omega = (\omega_1)^{(2)} + \ldots + (\omega_n)^{(2)}$, $\omega_i = (1 + x_i^{(2^{m_i}-1)}) dx_i$. Отметим, что для случая $m = 1$ получаем структуру алгебры Капланского [26] на $P^{(1)}(n : 1)$, как доказано в [5].

В базисе $\{u_b\}$ умножение в $P(n : m, \omega)$ задается правилом $\{u_a, u_b\} = f(a, b) u_{a+b}$, где $f(a, b)$ – симметрическая функция на $\mathbb{F}_q^*$ со значениями в F.

Может оказаться, что функция $f(a,b)$ биаддитивна и тогда $P^{(1)}(n:m,\omega)$ – алгебра Блока [21]. Связь между неальтернирующей гамильтоновой формой ω и соответствующими ей функциями $f(a,b)$ является более тонким вопросом.

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Кузнецов М.И., Кондратьева А.В., Чебочко Н.Г. 2* СИПАТТАМАСЫНЫҢ ГАМИЛЬТОНДЫ ЛИ АЛГЕБРАЛАРЫ ТУРАЛЫ

Полиномиалды коэффициенттері бар жалпы гамильтондық формаларға сәйкес келетін жұп сипаттамалы алмаспайтын гамильтонды Ли алгебраларының инвариантты тұрғызылуы келтіріледі. Симметриялы дифференциалдық формалардың айырылған дәрежелерінің кешенінің когомологиялары есептелінеді. Сүзгіленген алмаспайтын гамильтонды Ли алгебраларының мысалдары, атап айтқанда, азғынбаған дифференциалдауды рұқсат ететін алгебралар қарастырылады.

Kuznetsov M.I., Kondratyeva A.V., Chebochko N.G. ON HAMILTONIAN LIE ALGEBRAS OF CHARACTERISTIC 2*

The invariant construction of nonalternating Hamiltonian Lie algebras of even characteristic is given. Cohomology of complex of divided powers of symmetric differential forms is calculated. The examples of simple filtered nonalternating Hamiltonian Lie algebras including those admitting nonsingular derivations are considered.

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SEMIFIELD SPREADS AND BENT FUNCTIONS

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Annotation: In this paper we study bent functions which are linear on elements of spreads in affine planes. In particular, we study bent functions of spreads related to symplectic presemifields. We also study such functions from the viewpoint of pseudo-planar functions.

Keywords: Finite semifields, spreads, pseudo-planar functions, bent functions.

1. Introduction

Bent functions were introduced by Rothaus [1] but were already studied by Dillon [2] as difference sets. A bent function is a Boolean function with an even number of variables which achieves the maximum possible distance from affine functions [3]. Bent functions have relations to coding theory, cryptography, sequences, combinatorics and designs theory [3, 4, 5].

Dillon [2] introduced bent functions related to partial spreads of $\mathbb{F}_{2m} \times \mathbb{F}_{2m}$. He constructed bent functions that are constant on elements of a spread. This approach was further studied in [6, 7]. Dillon also introduced a class H of bent functions that are linear on elements of a Desarguesian spread. In [8] it is shown that there is one-to-one correspondence between these bent functions and oval polynomials (o-polynomials) from finite geometry. In [9, 10] this approach was extended to other types of spreads, and bent functions which are affine on the elements of spreads, were studied. In this paper we study bent functions which are linear on elements of spreads and calculate their duals.

Ключевые слова: *Конечные полуполя, расслоения, псевдо-планарные функции, бент-функции.*

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The paper is organised as follows. We firstly recall in Section 2 definitions and notations concerning semifields, spreads and bent functions. Next, in Section 3 we investigate bent functions which are linear on the elements of spreads. In Section 4 we consider bent functions related to pseudo-planar functions. Finally, in Section 5 we study equivalence of bent functions obtained from isotopic semifields. In particular, we show that semifields isotopic to finite fields produce EA-equivalent bent functions.

2. Preliminary considerations and notations

Let $\mathbb{F}_{p^m}$ and $\mathbb{F}_p$ be finite fields of orders p^m and p respectively. Consider $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ as a $2m$-dimensional vector space over $\mathbb{F}_p$. A spread of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ is a family of $p^m + 1$ subspaces of dimension m such that every nonzero point of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ lies in a unique subspace.

Spreads can be constructed using (pre)-semifields [11]. A presemifield $(S, +, *)$ is a vector space under operation $+$, with additional operation $*$, satisfying the following axioms:

(S1) $x * (y + z) = x * y + x * z$ and $(x + y) * z = x * z + y * z$, for all x, y, $z \in S$.

(S2) $x * y = 0$ implies $x = 0$ or $y = 0$.

Presemifield is a semifield if it has a multiplicative identity. One can define a presemifield S by taking elements of a finite field $\mathbb{F}_p$ and introducing new multiplication operation $*$:

$$x * y = xy + \sum_{i<j} a_{ij}(x^{2^i} y^{2^j} + x^{2^j} y^{2^i}).$$

Two presemifields $(S, +, *)$ and $(S', +, \star)$ are called isotopic if there exist three bijective linear mappings L, M, $N : S \to S'$ such that

$$L(x * y) = M(x) \star N(y)$$

for any $x, y \in S$. If $M = N$ then presemifields are called strongly isotopic. Every presemifield is isotopic to a semifield.

Let $F = \mathbb{F}_p$, a finite field of p^m elements. We define a $\mathbb{F}_p$-bilinear form $B : F \times F \to \mathbb{F}_p$ by $B(x, y) = Tr(xy)$, and an alternating form on $(F \times F) \times (F \times F)$ by

$$\langle (x, y), (x', y') \rangle = B(x, y') - B(y, x').$$

Let $S = (F, +, *)$ be a presemifield with respect to operation $*$. Dual presemifield $S^d = (F, +, \star)$ is defined by operation

$$x \star y = y * x.$$

With presemifield $S = (F, +, *)$ one can associate a spread, a collection of subspaces $\{(0, y) \mid y \in F\}$ and $\{(x, x * z) \mid m \in F\}$, $z \in F$. Transpose presemifield $S^t = (F, +, \circ)$ of the presemifield S is defined as a presemifield whose associated spread is orthogonal (dual) to the spread of S with respect to the alternating form $\langle \cdot, \cdot \rangle$, that is,

$$\langle (x, x * z), (y, y \circ z) \rangle = 0$$

for any $x, y, z \in F$. It is equivalent to

$$B(x, y \circ z) = B(x * z, y).$$

A presemifield is called symplectic, if its associated spread is symplectic (that is, every subspace from spread is isotropic with respect to the alternating form $\langle \cdot, \cdot \rangle$). This means

$$0 = \langle (x, x \circ z), (y, y \circ z) \rangle = B(x, y \circ z) - B(x \circ z, y)$$

for any $x, y, z \in F$. Equivalently,

$$B(x, y \circ z) = B(x \circ z, y) \tag{1}$$

for any $x, y, z \in F$.

Using operations S^d and S^t one can get at most 6 isotopy classes of presemifields, which is called the Knuth [12, 13] orbit $\mathcal{K}(S)$ of the presemifield S:

$$\mathcal{K}(S) = \{[S], [S^d], [S^t], [S^{dt}], [S^{td}], [S^{dtd}] = [S^{tdt}]\}.$$

A presemifield $S = (F, +, *)$ is called commutative, if the operation $*$ of multiplication is commutative. A presemifield S is commutative if and only if $S = S^d$, and a presemifield S is symplectic if and only if $S = S^t$. Therefore, Knuth orbit of a commutative (symplectic) presemifield contains at most three classes. If presemifield S is commutative then S^{td} is symplectic, and if S is symplectic then S^{dt} is commutative. If presemifield S is commutative then its transpose S^t is dual to symplectic presemifield S^{td}.

If $L : F \to F$ is a $\mathbb{F}_p$-linear map, its adjoint operator L^* with respect to the form B is defined as a unique linear operator satisfying the following condition:

$$B(L(x), y) = B(x, L^*(y)), \quad \text{for all } x, y \in F.$$

Equality (1) means that all right multiplication mappings $R_z(x) = x \circ z$ of a symplectic presemifield are self-adjoint with respect to B.

Starting from a symplectic presemifield $(F, +, \circ)$, one can construct a commutative presemifield in the following way [14, 15]. Consider the linear map $L_z : F \to F$, $L_z(x) = z \circ x$. Let L_z^* be the adjoint operator of L_z with respect to the form B:

$$B(z \circ x, y) = B(L_z(x), y) = B(x, L_z^*(y)).$$

We introduce new operation $*$ by

$$z * y = L_z^*(y),$$

so

$$B(z \circ x, y) = B(x, z * y).$$

Then $(F, +, *)$ is a commutative presemifield. Similarly, starting from commutative presemifield $(F, +, *)$ and putting $L_z(x) = z * x$, one can get a symplectic presemifield $(F, +, \circ)$:

$$B(z * x, y) = B(L_z(x), y) = B(x, L_z^*(y)) = B(x, z \circ y).$$

From now on we consider only even characteristic case and put $F = \mathbb{F}_p$, the Galois field with 2^n elements. Let $\mathbb{F}_2^n$ be the $\mathbb{F}_2$-vector space of dimension n. We shall endow $\mathbb{F}_2^n$ with the structure of field $\mathbb{F}_{2^n}$. A Boolean function on $\mathbb{F}_{2^n}$ is a mapping from $\mathbb{F}_{2^n}$ to the prime field $\mathbb{F}_2$.

If f is a Boolean function defined on $\mathbb{F}_{2^n}$, then the Walsh transform of f is defined as follows:

$$W_f(u) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+Tr(ux)}.$$

Bent functions can be defined in terms of the Walsh transform as follows. Let n be an even integer. A Boolean function f on $\mathbb{F}_{2^n}$ is said to be bent if its Walsh transform satisfies $W_f(u) = \pm 2^{n/2}$ for all $u \in \mathbb{F}_{2^n}$.

Given a bent function f over $\mathbb{F}_{2^n}$, we can always define its dual function, denoted by $\tilde{f}$, when considering the signs of the values of the Walsh transform $W_f(u)$ of f. More precisely, $\tilde{f}$ is defined by the equation:

$$(-1)^{\tilde{f}(x)} 2^{n/2} = W_f(x).$$

The dual of a bent function is bent again, and $\tilde{\tilde{f}} = f$.

Boolean functions $f, g : \mathbb{F}_{2^n} \to \mathbb{F}_2$ are extended-affine equivalent (in brief, EA-equivalent) if there exist an affine permutation L of $\mathbb{F}_{2^n}$ and an affine function $\ell : \mathbb{F}_{2^n} \to \mathbb{F}_2$ such that $g(x) = (f \circ L)(x) + \ell(x)$.

If Boolean functions f and g are EA-equivalent and f is bent then g is bent too.

The bivariate representation of Boolean functions makes sense only when n is an even integer, which is the case for bent functions. For $n = 2m$ we identify $\mathbb{F}_{2^n}$ with $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ and consider the input to f as an ordered pair (x, y) of elements of $\mathbb{F}_{2^m}$. The function f being Boolean, its bivariate representation can be written in the (non unique) form $f(x, y) = Tr(P(x, y))$, where $P(x, y)$ is a polynomial in two variables over $\mathbb{F}_{2^m}$. In this paper we shall only consider functions in their bivariate representation.

3. Spreads and bent functions

We recall the construction of bent functions from [16]. Let $L_z : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ be a linear function for any $z \in \mathbb{F}_{2^m}$. Consider a spread whose elements are the subspace $\{(0, y) \mid y \in \mathbb{F}_{2^m}\}$ and 2^m subspaces $\{(x, L_z(x)) \mid x \in \mathbb{F}_{2^m}\}$. These subspaces form a spread if and only if the mapping $z \mapsto L_z(x) = y$ is a permutation of $\mathbb{F}_{2^m}$. Denote by Γ_x the inverse of this bijection, that is, $\Gamma_x(y) = z$. A Boolean function on $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ is linear on the elements of the spread if and only if there exists a function $G : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ and an element $\mu \in \mathbb{F}_{2^m}$ such that, for every $y \in \mathbb{F}_{2^m}$,

$$f(0, y) = Tr(\mu y), \tag{2}$$

and for every $x, z \in \mathbb{F}_{2^m}$,

$$f(x, L_z(x)) = Tr(G(z)x). \tag{3}$$

Up to EA-equivalence, one can assume that $\mu = 0$. Indeed, one can add the linear function $g(x, y) = Tr(\mu y)$ to f; this changes μ into 0 and $G(z)$ into $G(z) + L_z^*(\mu)$, where L_z^* is the adjoint operator of L_z, since for $y = L_z(x)$ one has $Tr(\mu y) = B(\mu, y) = B(\mu, L_z(x)) = B(L_z^*(\mu), x) = Tr(L_z^*(\mu)x)$.

We take $\mu = 0$ in expression (2), and relation (3) becomes

$$f(x, y) = Tr(G(z)x) = Tr(G(\Gamma_x(y))x). \tag{4}$$

We recall that a function $H : F \to F$ is called 2-to-1, if preimage $H^{-1}(y)$ of any element $y \in F$ consists of 0 or 2 points.

Theorem 3.1 ([16], Theorem 2). *Consider a spread of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ whose elements are 2^m subspaces of the form $\{(x, L_z(x)) \mid x \in \mathbb{F}_{2^m}\}$, where, for every $z \in \mathbb{F}_{2^m}$, function L_z is linear, and the subspace $\{(0, y) \mid y \in \mathbb{F}_{2^m}\}$. For every $x \in \mathbb{F}_{2^m}^*$, let us denote by Γ_x the inverse of the permutation $z \mapsto L_z(x) = y$. A boolean function defined by equation (4) is bent if and only if G is a permutation and, for every $b \neq 0$ the function $G(z) + L_z^*(b)$ is 2-to-1, where L_z^* is the adjoint operator of L_z.*

An example of such function $G(x)$ was introduced in [9] in a particular case of spreads related to symplectic semifields. We calculate the dual of the corresponding bent function.

Theorem 3.2 [24]. *Let $(F, +, \circ)$ be a symplectic presemifield, and $(F, +, *)$ be its corresponding commutative presemifield. Consider a spread of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ whose elements are subspaces $\{(0, y) \mid y \in \mathbb{F}_{2^m}\}$ and $\{(x, z \circ x) \mid x \in \mathbb{F}_{2^m}\}$, $z \in \mathbb{F}_{2^m}$. For every $x \in \mathbb{F}_{2^m}^*$, denote by Γ_x the inverse of the permutation $z \mapsto z \circ x = y$, and put $G(z) = z * z$. For every $c \in \mathbb{F}_{2^m}$, let Z_c be the image of the map $z \mapsto z * (z + c)$, and χ_c be the characteristic function of the set $Z_c \times \{c\}$. Then a boolean function defined by equation (4) is bent, and its dual function is*

$$\tilde{f} = 1 + \sum_{c \in \mathbb{F}_{2^m}} \chi_c.$$

Denote $R_x(z) = z \circ x = y$. Let $\Gamma_x = R_x^{-1}$ be the inverse function, so $z = R_x^{-1}(y)$. Let $G(z) = z * z$. Then function from (4) can be rewritten as

$$f(x, y) = Tr(G(z)x) = B(z * z, x) = B(z, z \circ x) =$$

$$= B(z,y) = B(\Gamma_x(y),y) = Tr(R_x^{-1}(y)y). \tag{5}$$

If the multiplication * in a commutative presemifield $(F,+,*)$ is given by

$$x * y = xy + \sum_{i<j} a_{ij}(x^{2^i}y^{2^j} + x^{2^j}y^{2^i})$$

then $G(z) = z * z = z^2$.

Now we consider spreads of symplectic presemifields and present a result similar to Theorem 3.

Theorem 3.3 [24]. *Let $(F,+,\circ)$ be a symplectic presemifield. Consider a spread of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ whose elements are subspaces $\{(0,y) \mid y \in \mathbb{F}_{2^m}\}$ and $\{(x, x\circ z) \mid x \in \mathbb{F}_{2^m}\}$, $z \in \mathbb{F}_{2^m}$. For every $x \in \mathbb{F}_{2^m}^*$, denote by Γ_x the inverse of the permutation $z \mapsto x \circ z = y$, and put $G(z) = \sqrt{z}$. For every $c \in \mathbb{F}_{2^m}$, let Z_c be the image of the map $z \mapsto \sqrt{z} + c \circ z$, and χ_c be the characteristic function of the set $Z_c \times \{c\}$. Then a boolean function defined by equation (4) is bent, and its dual function is*

$$\tilde{f} = 1 + \sum_{c\in\mathbb{F}_{2^m}} \chi_c.$$

We recall that, in case of finite fields F, a function $G(x) : F \to F$ is an o-polynomial if G is a permutation and $G(x) + xb$ is a 2-to-1 function for any nonzero b. We call a function $G : F \to F$ an o-polynomial for the presemifield $S = (F,+,*)$ if G is a permutation and $G(x) + x * b$ is a 2-to-1 function for any nonzero b. Consider affine semifield plane $\{(x,y) \mid x,y \in F\}$. If $G(x)$ is o-polynomial then "curve"$y = G(x)$ intersects with "line"$y = x * b + a$ in one point if $b = 0$, and 0 or 2 points if $b \neq 0$.

Theorem 3.4 [24]. *Let $S = (\mathbb{F}_{2^m},+,*)$ be a presemifield, and $S^t = (\mathbb{F}_{2^m},+,\circ)$ be its corresponding transpose presemifield. Let $G : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$ be a linear o-polynomial for the presemifield S. Then the adjoint map G^* is an o-polynomial for the presemifield S^t.*

Example 3.1. Recall the construction of Kantor-Williams presemifields [14, 17]. Let $F = \mathbb{F}_{2^m}$ and let $m > 1$ be odd. Let $F = F_0 \supset F_1 \supset \cdots \supset F_n$ be chain of subfields, T_i be trace map from F to F_i, and $\zeta_i \in F^*$. The commutative

Kantor presemifield is given by operation

$$x * y = xy + \left(x \sum_{i=1}^{n} T_i(\zeta_i y) + y \sum_{i=1}^{n} T_i(\zeta_i x) \right)^2,$$

and the corresponding Kantor-Williams symplectic presemifield is given by operation

$$x \circ y = xy + y^{2^{m-1}} \sum_{i=1}^{n} T_i(\zeta_i x) + \sum_{i=1}^{n} \zeta_i T_i(xy^{2^{m-1}}).$$

If we take $G(z) = z * z = z^2$, then function $f(x, y) = Tr(\Gamma_x(y)y)$ is bent. This expression is in implicit form, to make it explicit we have to find the explicit expression for $\Gamma_x(y)$. Let us calculate Γ_x for the simplest case $n = 1$. Denote $T = T_1$, $F_1 = \mathbb{F}_{2^k}$, $\zeta = \zeta_1$. Then

$$z \circ x = zx + \sqrt{x}T(\zeta z) + \zeta T(z\sqrt{x}) = y,$$

$$z = \frac{y}{x} + \frac{\sqrt{x}}{x}T(\zeta z) + \frac{\zeta}{x}T(z\sqrt{x}),$$

$$T(\zeta z) = T(\frac{\zeta y}{x}) + T(\frac{\zeta\sqrt{x}}{x})T(\zeta z) + T(\frac{\zeta^2}{x})T(z\sqrt{x}),$$

$$T(z\sqrt{x}) = T(\frac{y\sqrt{x}}{x}) + T(\zeta z) + T(\frac{\zeta\sqrt{x}}{x})T(z\sqrt{x}).$$

Therefore,

$$\begin{cases} [T(\frac{\zeta\sqrt{x}}{x}) + 1]T(\zeta z) & + & T(\frac{\zeta^2}{x})T(z\sqrt{x}) & = & T(\frac{\zeta y}{x}), \\ 1 \cdot T(\zeta z) & + & [T(\frac{\zeta\sqrt{x}}{x}) + 1]T(z\sqrt{x}) & = & T(\frac{y\sqrt{x}}{x}). \end{cases}$$

Determinant of this linear system is equal to

$$[T(\frac{\zeta\sqrt{x}}{x}) + 1]^2 + T(\frac{\zeta^2}{x}) = 1,$$

so

$$T(\zeta z) = T(\frac{\zeta y}{x})[T(\frac{\zeta\sqrt{x}}{x}) + 1] + T(\frac{y\sqrt{x}}{x})T(\frac{\zeta^2}{x}),$$

$$T(z\sqrt{x}) = [T(\frac{\zeta\sqrt{x}}{x}) + 1]T(\frac{y\sqrt{x}}{x}) + 1 \cdot T(\frac{\zeta y}{x}).$$

Therefore,

$$\Gamma_x(y) = z \quad = \quad \frac{y}{x} + \frac{\sqrt{x}}{x}\left(T(\frac{\zeta y}{x})T(\frac{\zeta\sqrt{x}}{x}) + T(\frac{\zeta y}{x}) + T(\frac{y\sqrt{x}}{x})T(\frac{\zeta^2}{x})\right)$$
$$+\frac{\zeta}{x}\left(T(\frac{\zeta\sqrt{x}}{x})T(\frac{y\sqrt{x}}{x}) + T(\frac{y\sqrt{x}}{x}) + T(\frac{\zeta y}{x})\right).$$

For the general case we could use Theorem 4 below.

Remark 3.1. For finite fields an o-polynomial corresponds to a hyperoval. It is generally known that the function $G(z) = z * z$ from Theorem 3 determines a hyperoval for commutative semifield planes (see, for example, [18]). The function $G(z) = \sqrt{z}$ from Theorem 3 corresponds to hyperovals for semifields which are transpose to commutative semifields (and equivalently they are dual to symplectic semifields [14, 15]).

Remark 3.2. Computations show that for the case of Knuth [12] commutative presemifield $(\mathbb{F}_{2^5}, +, *)$, where the product is given by

$$x * y = xy + x^2Tr(y) + y^2Tr(x),$$

the functions $G_1(z) = z^8$, $G_2(x) = z + z^2 + z^4$ and $G_3(x) = z^2 + z^4 + z^8$ give other examples of o-polynomials. The corresponding Kantor-Williams symplectic presemifield $(\mathbb{F}_{2^5}, +, \circ)$ is given by operation

$$x \circ y = xy + y^{2^{m-1}}Tr(x) + Tr(xy^{2^{m-1}}).$$

Then by Theorem 3 the adjacent maps $G_1^*(z) = z^4$, $G_2^*(z) = z + z^8 + z^{16}$ and $G_3^*(z) = z^4 + z^8 + z^{16}$ give examples of o-polynomials for the transpose of Knuth presemifield (i. e. dual presemifield of the mentioned Kantor-Williams symplectic presemifield).

Remark 3.3. In the finite field case, if $G(z)$ is an o-polynomial, then the function $G^{-1}(z)$ is an o-polynomial as well [8]. Using examples from Remark 2 we see that, in general, in case of proper semifield, i. e. a finite semifield which is not a field, the polynomial $G^{-1}(z)$ might not to be o-polynomial neither for the semifield or its transpose.

4. Pseudo-planar functions

Commutative semifields of odd characteristics are in one-to-one correspondence with planar functions [11]. There are no planar functions over fields of even characteristic, however one can define modified planar functions which carry similar properties in some sense [19, 20]. Let F be a finite field of characteristic two. We call a function $f : F \to F$ pseudo-planar if the map

$$x \mapsto f(x+a) + f(x) + ax$$

is a permutation of F for each $a \in F^*$.

Theorem 4.1 ([4], Theorem 9). *Let F be a finite field of characteristic two.*

1. If $(F, +, *)$ is a commutative presemifield with multiplication given by

$$x * y = xy + \sum_{i<j} a_{ij}(x^{2^i} y^{2^j} + x^{2^j} y^{2^i})$$

then $f(x) = \sum_{i<j} a_{ij} x^{2^i+2^j}$ is a pseudo-planar function and $x * y = xy + f(x+y) + f(x) + f(y)$.

2. If $(F, +, *)$ is a commutative presemifield then there exist strongly isotopic commutative presemifield $(F, +, \star)$ and pseudo-planar function f such that $x \star y = xy + f(x+y) + f(x) + f(y)$. Therefore, up to isotopism, any commutative semifield can be described by pseudo-planar functions.

3. Let f be a pseudo-planar function. Then $(F, +, *)$ with multiplication $x * y = xy + f(x+y) + f(x) + f(y)$ is a presemifield if and only if f is a quadratic function.

Since every commutative presemifield can be obtained from a pseudo-planar function, we study now spreads and related bent functions from the viewpoint of pseudo-planar functions.

Theorem 4.2. *Let $f(x) = \sum a_{ij} x^{2^i+2^j}$ be a pseudo-planar function, $a_{ii} = 0$ for all i. Then the operation*

$$x * y = xy + \sum a_{ij}(x^{2^i} y^{2^j} + x^{2^j} y^{2^i})$$

defines the corresponding commutative presemifield $(S, +, *)$, and the operation

$$x \circ y = xy + \sum a_{ij}^{2^{-j}} x^{2^{i-j}} y^{2^{-j}} + \sum a_{ij}^{2^{-i}} x^{2^{j-i}} y^{2^{-i}}$$

defines the corresponding symplectic presemifield $(S, +, \circ)$.

Proof. Expression for $x * y$ follows from Theorem 4. Furthermore,

$$\begin{aligned} B(x * z, y) &= B(xz + \sum a_{ij}(x^{2^i} z^{2^j} + x^{2^j} z^{2^i}), y) = \\ &= Tr(xzy + \sum a_{ij}(x^{2^i} z^{2^j} y + x^{2^j} z^{2^i})y) = \\ &= Tr(zxy + z\sum a_{ij}^{2^{-j}} x^{2^{i-j}} y^{2^{-j}} + z\sum a_{ij}^{2^{-i}} x^{2^{j-i}} y^{2^{-i}}) = \\ &= B(z, xy + \sum a_{ij}^{2^{-j}} x^{2^{i-j}} y^{2^{-j}} + \sum a_{ij}^{2^{-i}} x^{2^{j-i}} y^{2^{-i}}). \end{aligned}$$

Therefore,

$$x \circ y = xy + \sum a_{ij}^{2^{-j}} x^{2^{i-j}} y^{2^{-j}} + \sum a_{ij}^{2^{-i}} x^{2^{j-i}} y^{2^{-i}}. \quad \Box$$

In the following we study appropriate bent functions for all remaining known pseudo-planar functions [21].

Example 4.1. Let us consider pseudo-planar function $f(x) = ax^{2^k+1}$, $m = 2k$, $a \in \mathbb{F}_{2^k}^*$, $Tr_k(a) = 0$, where Tr_k is the trace function from $\mathbb{F}_{2^k}$ to $\mathbb{F}_2$ (see [19]). By Theorem 4 corresponding commutative presemifield gives $G(z) = z * z = z^2$ and we also get symplectic presemifield with operation

$$x \circ y = xy + ax^{2^k} y^{2^k} + ax^{2^k} y.$$

If $z \circ x = zx + az^{2^k} x^{2^k} + az^{2^k} x = y$, then

$$\begin{cases} xz + (ax^{2^k} + ax)z^{2^k} = y, \\ (ax^{2^k} + ax)z + x^{2^k} z^{2^k} = y^{2^k}. \end{cases}$$

Therefore,

$$\Gamma_x(y) = z = \frac{\begin{vmatrix} y & ax^{2^k} + ax \\ y^{2^k} & x^{2^k} \end{vmatrix}}{\begin{vmatrix} x & ax^{2^k} + ax \\ ax^{2^k} + ax & x^{2^k} \end{vmatrix}} = \frac{x^{2^k} y + (ax^{2^k} + ax) y^{2^k}}{x^{2^k+1} + (ax^{2^k} + ax)^2},$$

$$f(x,y) = Tr(\Gamma_x(y)y) = Tr\left(\left[\frac{x^{2^k}y + (ax^{2^k} + ax)y^{2^k}}{x^{2^k+1} + (ax^{2^k} + ax)^2}\right] y\right).$$

Symplectic presemifield $S = (F, +, \circ)$ from this example is isotopic to a finite field. We show now that corresponding bent function is EA-equivalent to a bent function obtained from a finite field.

We recall that quadratic equation $at^2 + bt + c = 0$ has two roots in $\mathbb{F}_{2^k}$ if and only if $b \neq 0$ and $Tr_k(ac/b^2) = 0$. Therefore, quadratic equation $at^2 + t + a = 0$ has two roots and we denote one root by e, so $ae^2 + e + a = 0$. Denote $\rho = 2^k$ and define

$$L(x) = ex + x^\rho,$$
$$M(x) = ax + \frac{a}{e}x^\rho,$$
$$N(x) = e^2x + x^\rho.$$

Then

$$\begin{aligned} L(x \circ y) &= L(xy + ax^\rho y^\rho + ax^\rho y) = \\ &= e(xy + ax^\rho y^\rho + ax^\rho y) + (x^\rho y^\rho + axy + axy^\rho) = \\ &= (e + a)xy + (ea + 1)x^\rho y^\rho + eax^\rho y + axy^\rho, \end{aligned}$$

$$M(x) \cdot N(y) = (ax + \frac{a}{e}x^\rho)(e^2y + y^\rho) =$$
$$= ae^2xy + \frac{a}{e}x^\rho y^\rho + eax^\rho y + axy^\rho.$$

Therefore, M, N, L is an isotopism from $S = (F, +, \circ)$ into the finite field F:

$$L(x \circ y) = M(x) \cdot N(x).$$

We see that $M^* = M$ since $B(M^*(x), y) = B(x, M(y)) = B(x, ay + \frac{a}{e}y^\rho) = Tr(x(ay + \frac{a}{e}y^\rho)) = Tr((ax + \frac{a}{e}x^\rho)y) = B(ax + \frac{a}{e}x^\rho, y)$. Then $(M^*)^{-1} = L$ since

$$LM^*(x) = L(ax + \frac{a}{e}x^\rho) = e(ax + \frac{a}{e}x^\rho) + (ax + \frac{a}{e}x^\rho) = x.$$

Consider a bent function

$$f(x,y) = B(z,y) = B(\frac{y}{x}, y).$$

Since
$$z\circ x = L^{-1}(M(z)\cdot N(x)) = y, \quad z = M^{-1}\frac{1}{N(x)}L(y),$$
we have
$$\hat{f}(x,y) = B(z,y) = B(M^{-1}\frac{1}{N(x)}L(y),y) =$$
$$= B(\frac{1}{N(x)}L(y),(M^*)^{-1}(y)) = B(\frac{1}{N(x)}L(y),L(y)).$$
So the maps $x\mapsto N(x)$, $y\mapsto L(y)$ determine EA-equivalence of functions f and $\hat{f}$.

Example 4.2. Consider pseudo-planar function $f(x) = ax^{2^{2k}+2^{4k}}$, $m = 6k$, $a\in\mathbb{F}_{2^m}^*$, a is a $(4k-1)$-th power but not a $3(4^k-1)$-th power [22]. Denote $\rho = 2^{2k}$. By Theorem 4 corresponding commutative presemifield gives $G(z) = z*z = z^2$ and we also get symplectic presemifield with operation
$$x\circ y = xy + a^{\rho^2}x^\rho y^{\rho^2} + a^\rho x^{\rho^2}y^\rho.$$
If $z\circ x = zx + a^{\rho^2}z^\rho x^{\rho^2} + a^\rho z^{\rho^2}x^\rho = y$, then
$$\begin{cases} xz + a^{\rho^2}x^{\rho^2}z^\rho + a^\rho x^\rho z^{\rho^2} = y,\\ a^{\rho^2}x^{\rho^2}z + x^\rho z^\rho + axz^{\rho^2} = y^\rho,\\ a^\rho x^\rho z + axz^\rho + x^{\rho^2}z^{\rho^2} = y^{\rho^2}, \end{cases}$$
Therefore,
$$\Gamma_x(y) = z = \frac{\begin{vmatrix} y & a^{\rho^2}x^{\rho^2} & a^\rho x^\rho\\ y^\rho & x^\rho & ax\\ y^{\rho^2} & ax & x^{\rho^2}\end{vmatrix}}{\begin{vmatrix} x & a^{\rho^2}x^{\rho^2} & a^\rho x^\rho\\ a^{\rho^2}x^{\rho^2} & x^\rho & ax\\ a^\rho x^\rho & ax & x^{\rho^2}\end{vmatrix}} =$$
$$= \frac{(a^2x^2 + x^{\rho+\rho^2})y + (a^{1+\rho}x^{1+\rho} + a^{\rho^2}x^{2\rho^2})y^\rho + (a^\rho x^{2\rho} + a^{1+\rho^2}x^{1+\rho^2})y^{\rho^2}}{a^2x^3 + a^{2\rho}x^{3\rho} + a^{2\rho^2}x^{3\rho^2} + x^{1+\rho+\rho^2}},$$
$$f(x,y) = Tr(\Gamma_x(y)y) =$$

$$=Tr\left(\left[\frac{(a^2x^2+x^{\rho+\rho^2})y+(a^{1+\rho}x^{1+\rho}+a^{\rho^2}x^{2\rho^2})y^{\rho}+(a^{\rho}x^{2\rho}+a^{1+\rho^2}x^{1+\rho^2})y^{\rho^2}}{a^2x^3+a^{2\rho}x^{3\rho}+a^{2\rho^2}x^{3\rho^2}+x^{1+\rho+\rho^2}}\right]y\right).$$

Example 4.3. We consider pseudo-planar function $f(x) = x^{2^k+1} + x^{2^{2k}+2^k}$, $m = 3k$, $k \not\equiv 2 \pmod 3$ (see [23]). Denote $\rho = 2^k$. By Theorem 4 the corresponding commutative presemifield gives $G(z) = z * z = z^2$ and the symplectic presemifield is given by operation

$$x \circ y = xy + x^{\rho}(y + y^{\rho^2}) + x^{\rho^2}(y^{\rho} + y^{\rho^2}).$$

If $z \circ x = zx + z^{\rho}(x + x^{\rho^2}) + z^{\rho^2}(x^{\rho} + x^{\rho^2}) = y$, then

$$\begin{cases} xz & + & (x+x^{\rho^2})z^{\rho} & + & (x^{\rho}+x^{\rho^2})z^{\rho^2} & = & y, \\ (x+x^{\rho^2})z & + & x^{\rho}z^{\rho} & + & (x+x^{\rho})z^{\rho^2} & = & y^{\rho}, \\ (x^{\rho}+x^{\rho^2})z & + & (x+x^{\rho})z^{\rho} & + & x^{\rho^2}z^{\rho^2} & = & y^{\rho^2}, \end{cases}$$

Therefore,

$$\Gamma_x(y) = z = \frac{\begin{vmatrix} y & x+x^{\rho^2} & x^{\rho}+x^{\rho^2} \\ y^{\rho} & x^{\rho} & x+x^{\rho} \\ y^{\rho^2} & x+x^{\rho} & x^{\rho^2} \end{vmatrix}}{\begin{vmatrix} x & x+x^{\rho^2} & x^{\rho}+x^{\rho^2} \\ x+x^{\rho^2} & x^{\rho} & x+x^{\rho} \\ x^{\rho}+x^{\rho^2} & x+x^{\rho} & x^{\rho^2} \end{vmatrix}} =$$

$$=\frac{(x^2+x^{2\rho}+x^{\rho+\rho^2})y+(x^{1+\rho}+x^{2\rho}+x^{2\rho^2}+x^{\rho+\rho^2})y^{\rho}+(x^2+x^{1+\rho}+x^{1+\rho^2}+x^{2\rho})y^{\rho^2}}{x^3+x^{3\rho}+x^{3\rho^2}+x^{1+2\rho}+x^{2+\rho^2}+x^{\rho+2\rho^2}+x^{1+\rho+\rho^2}},$$

$$f(x,y) \;=\; Tr(\Gamma_x(y)y).$$

Example 4.4. Let us consider pseudo-planar function $f(x) = x^{2^{2k}+1} + x^{2^{2k}+2^k}$, $m = 3k$, $k \not\equiv 1 \pmod 3$ (see [23]). Denote $\rho = 2^k$. By Theorem 4 corresponding commutative presemifield gives $G(z) = z * z = z^2$ and the symplectic presemifield is given by operation

$$x \circ y = xy + x^{\rho}(y^{\rho} + y^{\rho^2}) + x^{\rho^2}(y + y^{\rho}).$$

If $z \circ x = zx + z^\rho(x^\rho + x^{\rho^2}) + z^{\rho^2}(x + x^\rho) = y$, then

$$\begin{cases} xz + (x^\rho + x^{\rho^2})z^\rho + (x + x^\rho)z^{\rho^2} = y, \\ (x^\rho + x^{\rho^2})z + x^\rho z^\rho + (x + x^{\rho^2})z^{\rho^2} = y^\rho, \\ (x + x^\rho)z + (x + x^{\rho^2})z^\rho + x^{\rho^2} z^{\rho^2} = y^{\rho^2}, \end{cases}$$

Therefore,

$$\Gamma_x(y) = z = \frac{\begin{vmatrix} y & x^\rho + x^{\rho^2} & x + x^\rho \\ y^\rho & x^\rho & x + x^{\rho^2} \\ y^{\rho^2} & x + x^{\rho^2} & x^{\rho^2} \end{vmatrix}}{\begin{vmatrix} x & x^\rho + x^{\rho^2} & x + x^\rho \\ x^\rho + x^{\rho^2} & x^\rho & x + x^{\rho^2} \\ x + x^\rho & x + x^{\rho^2} & x^{\rho^2} \end{vmatrix}} =$$

$$= \frac{(x^2 + x^{2\rho^2} + x^{\rho+\rho^2})y + (x^2 + x^{1+\rho} + x^{2\rho^2} + x^{1+\rho^2})y^\rho + (x^{2\rho} + x^{1+\rho^2} + x^{\rho+\rho^2} + x^{2\rho^2})y^{\rho^2}}{x^3 + x^{3\rho} + x^{3\rho^2} + x^{2+\rho} + x^{1+2\rho^2} + x^{2\rho+\rho^2} + x^{1+\rho+\rho^2}},$$

$$f(x,y) = Tr(\Gamma_x(y)y).$$

Example 4.5. Consider a function $f(x) = a^{2^{2k}+1}x^{2^{2k}+1} + a^{-(2^k+1)}x^{2^k+1}$, $m = 3k$. It is pseudo-planar [23] if and only if

$$Tr_k^{3k}((a^{2^{2k}+2^k} + a^{-2^{2k}-2^k-2})(a^{2^k+1} + \epsilon^{2^k-1})\epsilon^{2^k+2} + a^{2^k-2^{2k}}\epsilon^3 + \epsilon) \neq 0$$

for all $\epsilon \in \mathbb{F}^*_{2^{3k}}$, where Tr_k^{3k} is the trace function from $\mathbb{F}_{2^{3k}}$ to $\mathbb{F}_{2^k}$.

Denote $\rho = 2^k$. By Theorem 4 corresponding commutative presemifield gives $G(x) = x * x = x^2$ and we get symplectic presemifield with operation

$$\begin{aligned} x \circ y &= xy + a^{1+\rho}x^\rho y^\rho + a^{1+\rho^2}x^{\rho^2}y + a^{-(1+\rho^2)}x^{\rho^2}y^{\rho^2} + a^{-(1+\rho)}x^{\rho^2}y = \\ &= xy + (a^{1+\rho}y^\rho + a^{-(1+\rho)}y)x^\rho + (a^{1+\rho^2}y + a^{-(1+\rho^2)}y^{\rho^2})x^{\rho^2}. \end{aligned}$$

Therefore,

$$z \circ x = xz + (a^{1+\rho}x^\rho + a^{-(1+\rho)}x)z^\rho + (a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2})z^{\rho^2} = y,$$

and then

$$\begin{cases} xz + (a^{1+\rho}x^{\rho} + a^{-(1+\rho)}x)z^{\rho} + (a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2})z^{\rho^2} = y, \\ (a^{1+\rho}x^{\rho} + a^{-(1+\rho)}x)z + x^{\rho}z^{\rho} + (a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho})z^{\rho^2} = y^{\rho}, \\ (a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2})z + (a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho})z^{\rho} + x^{\rho^2}z^{\rho^2} = y^{\rho^2}, \end{cases}$$

Denoting

$$\Delta = \begin{pmatrix} x & a^{1+\rho}x^{\rho} + a^{-(1+\rho)}x & a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2} \\ a^{1+\rho}x^{\rho} + a^{-(1+\rho)}x & x^{\rho} & a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho} \\ a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2} & a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho} & x^{\rho^2} \end{pmatrix},$$

$$\Delta_1 = \begin{pmatrix} y & a^{1+\rho}x^{\rho} + a^{-(1+\rho)}x & a^{1+\rho^2}x + a^{-(1+\rho^2)}x^{\rho^2} \\ y^{\rho} & x^{\rho} & a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho} \\ y^{\rho^2} & a^{\rho+\rho^2}x^{\rho^2} + a^{-(\rho+\rho^2)}x^{\rho} & x^{\rho^2} \end{pmatrix},$$

we have

$$\Gamma_x(y) = \frac{\det(\Delta_1)}{\det(\Delta)},$$

$$f(x,y) = Tr(\Gamma_x(y)y).$$

Now we consider the general case. Let $f(x) = \sum_{i<j} a_{ij}x^{2^i+2^j}$ be a pseudo-planar function and let $(F, +, \circ)$ be the corresponding symplectic presemifield. Let $L_z(x) = z \circ x$. Let us to find the compositional inverse Γ_x of the map

$L_z : z \mapsto L_z(x) = y$. One has

$$z \circ x = zx + \sum_{i<j} a_{ij}^{2^{m-j}} z^{2^{m+i-j}} x^{2^{m-j}} + \sum_{i<j} a_{ij}^{2^{m-i}} z^{2^{j-i}} x^{2^{m-i}} =$$

$$= zx + \sum_{\substack{1 \le j \le m-1 \\ 1 \le s \le m-1 \\ s+j \ge m}} a_{s-m+j,j}^{2^{m-j}} z^{2^s} x^{2^{m-j}} + \sum_{\substack{0 \le i \le m-2 \\ 1 \le s \le m-1 \\ s+i \le m-1}} a_{i,i+s}^{2^{m-i}} z^{2^s} x^{2^{m-i}} =$$

$$= zx + \sum_{\substack{1 \le s \le m-1 \\ 1 \le t \le m-1 \\ t \le s}} a_{s-t,m-t}^{2^t} z^{2^s} x^{2^t} + \sum_{\substack{1 \le s \le m-1 \\ 2 \le t \le m \\ t \ge s+1}} a_{m-t,s+m-t}^{2^t} z^{2^s} x^{2^t} =$$

$$= xz + \sum_{s=1}^{m-1} \left(\sum_{1 \le t \le s} a_{s-t,m-t}^{2^t} x^{2^t} + \sum_{s+1 \le t \le m} a_{m-t,s+m-t}^{2^t} x^{2^t} \right) z^{2^s}.$$

We have to get z from the equation $z \circ x = y$. Denote

$$b_0 = x,$$

$$b_s = \sum_{1 \le t \le s} a_{s-t,m-t}^{2^t} x^{2^t} + \sum_{s+1 \le t \le m} a_{m-t,s+m-t}^{2^t} x^{2^t}, \quad 1 \le s \le m-1.$$

Then

$$b_0 z + b_1 z^{2^1} + b_2 z^{2^2} + \cdots + b_{m-1} z^{2^{m-1}} = y.$$

Therefore,

$$\begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{m-1} \\ b_{m-1}^2 & b_0^2 & b_1^2 & \dots & b_{m-2}^2 \\ b_{m-2}^{2^2} & b_{m-1}^{2^2} & b_0^{2^2} & \dots & b_{m-3}^{2^2} \\ \dots & \dots & \dots & \dots & \dots \\ b_1^{2^{m-1}} & b_2^{2^{m-1}} & b_3^{2^{m-1}} & \dots & b_0^{2^{m-1}} \end{pmatrix} \begin{pmatrix} z \\ z^{2^1} \\ z^{2^2} \\ \dots \\ z^{2^{m-1}} \end{pmatrix} = \begin{pmatrix} y \\ y^{2^1} \\ y^{2^2} \\ \dots \\ y^{2^{m-1}} \end{pmatrix}.$$

Denote

$$\Delta = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{m-1} \\ b_{m-1}^2 & b_0^2 & b_1^2 & \dots & b_{m-2}^2 \\ b_{m-2}^{2^2} & b_{m-1}^{2^2} & b_0^{2^2} & \dots & b_{m-3}^{2^2} \\ \dots & \dots & \dots & \dots & \dots \\ b_1^{2^{m-1}} & b_2^{2^{m-1}} & b_3^{2^{m-1}} & \dots & b_0^{2^{m-1}} \end{pmatrix},$$

$$\Delta_1 = \begin{pmatrix} y & b_1 & b_2 & \dots & b_{m-1} \\ y^2 & b_0^2 & b_1^2 & \dots & b_{m-2}^2 \\ y^{2^2} & b_{m-1}^{2^2} & b_0^{2^2} & \dots & b_{m-3}^{2^2} \\ \dots & \dots & \dots & \dots & \dots \\ y^{2^{m-1}} & b_2^{2^{m-1}} & b_3^{2^{m-1}} & \dots & b_0^{2^{m-1}} \end{pmatrix}.$$

Then

$$\Gamma_x(y) = \frac{\det(\Delta_1)}{\det(\Delta)}. \tag{6}$$

The above discussion leads to the following result.

THEOREM 4.3. *Let $f(x) = \sum_{i<j} a_{ij} x^{2^i+2^j}$ be a pseudo-planar function. Then the operation*

$$x \circ y = xy + \sum_{i<j} a_{ij}^{2^{m-j}} x^{2^{m+i-j}} y^{2^{m-j}} + \sum_{i<j} a_{ij}^{2^{m-i}} x^{2^{j-i}} y^{2^{m-i}}$$

defines the corresponding symplectic presemifield $(F, +, \circ)$. Consider a spread of $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$ whose elements are the subspace $\{(0, y) \mid y \in \mathbb{F}_{2^m}\}$ and subspaces $\{(x, z \circ x) \mid x \in \mathbb{F}_{2^m}\}$, $z \in \mathbb{F}_{2^m}$. Then the Boolean function defined by (3) and (6) is bent.

5. Equivalence of bent functions

In this section we consider equivalence questions. We study weather we get EA-equivalent bent functions if we change symplectic semifield to an isotopic symplectic semifield. We recall some facts from [24]. Let $\hat{S} = (F, +, \hat{\circ})$ and $S = (F, +, \circ)$ be isotopic symplectic presemifields:

$$L(x \hat{\circ} y) = M(x) \circ N(y),$$

$$B(x \circ y, z) = B(x, z \circ y),$$

$$B(x\hat{\circ}y, z) = B(x, z\hat{\circ}y).$$

For the sake of readability, let us denote by $\bar{L}$ the adjoint operator of L. Then we have

$$\begin{aligned} B(L^{-1}(M(x) \circ N(y)), z) &= B(x\hat{\circ}y, z) = \\ &= B(x, z\hat{\circ}y) = \\ &= B(x, L^{-1}(M(z) \circ N(y)) = \\ &= B(\bar{L}^{-1}(x), M(z) \circ N(y)) = \\ &= B(\bar{L}^{-1}(x) \circ N(y), M(z)) = \\ &= B(\bar{M}(\bar{L}^{-1}(x) \circ N(y)), z). \end{aligned}$$

Therefore,

$$L^{-1}(M(x) \circ N(y)) = \bar{M}(\bar{L}^{-1}(x) \circ N(y)),$$

$$M(x) \circ N(y) = L\bar{M}(\bar{L}^{-1}(x) \circ N(y)).$$

Denoting $\varphi = M\bar{L}$, $a = \bar{L}^{-1}(x)$, $b = N(y)$, we have

$$\varphi(a) \circ b = \bar{\varphi}(a \circ b).$$

Define

$$K^+(S) = \{\varphi \in \mathrm{End}_{\mathbb{F}_p}(S) \mid \varphi(a) \circ b = \bar{\varphi}(a \circ b) \text{ for all } a, b \in S\}.$$

So we proved the following

Lemma 5.1. *Let S be a symplectic presemifield with respect to a form $\langle \cdot, \cdot \rangle$. Let $\hat{S}$ and S be isotopic presemifields with isotopism (M, N, L). Then $\hat{S}$ is symplectic with respect to the form $\langle \cdot, \cdot \rangle$ if and only if $M\bar{L} \in K^+(S)$.*

We recall that the left nucles of a semifield $(S, +, \circ)$ is defined as

$$N_l(S) = \{c \in S \mid c \circ (x \circ y) = (c \circ x) \circ y \text{ for all } x, y \in S\}.$$

Lemma 5.2. *Let S be a symplectic semifield, $c \in N_l(S)$, $L_c(y) = c \circ y$. Then the map $\varphi = L_c$ is self-adjoint with respect to the form B, that is, $\bar{\varphi} = \varphi$.*

Proof. We have

$$B(x, \bar{\varphi}(y)) = B(\varphi(x), y) = B(c \circ x, y) = B(c, y \circ x) = B(c \circ (y \circ x), 1) =$$

$$= B((c \circ y) \circ x, 1) = B(c \circ y, x) = B(x, c \circ y).$$

Hence $\bar{\varphi}(y) = c \circ y = \varphi(y)$, which completes the proof. □

An immediate consequence is the following.

Corollary 5.1. *Let S be a symplectic semifield, $c \in N_l(S)$, $L_c(y) = c \circ y$. Then*

$$K^+(S) \supseteq \{L_c \mid c \in N_l(S)\}.$$

Proof. Let $c \in N_l(S)$. We have $L_c(a) \circ b = (c \circ a) \circ b = c \circ (a \circ b) = L_c(a \circ b) = \bar{L}_c(a \circ b)$, according to Lemma 5. □

The previous lemma shows that the set $K^+(S)$ is a nonzero subspace.

Consider bent function

$$f(x, y) = B(z, y) = B(R_x^{-1}(y), y).$$

Since

$$z \hat{\circ} x = L^{-1}(M(z) \circ N(x)) = y, \quad z = M^{-1} R^{-1}_{N(x)} L(y),$$

we have

$$\hat{f}(x, y) = B(z, y) = B(M^{-1} R^{-1}_{N(x)} L(y), y) = B(R^{-1}_{N(x)} L(y), \bar{M}^{-1}(y)).$$

It seems that in general functions f and $\hat{f}$ should not be EA-equivalent, but it does hold under some suitable assumption:

Corollary 5.2. *If $L\bar{M} = id$ then f and $\hat{f}$ are EA-equivalent bent functions.*

Proof. We have

$$\hat{f}(x, y) = B(R^{-1}_{N(x)} L(y), \bar{M}^{-1}(y)) =$$

$$= B(R^{-1}_{N(x)} L(y), (L\bar{M})^{-1} L(y)) = B(R^{-1}_{N(x)} L(y), L(y)),$$

so the maps $x \mapsto N(x)$, $y \mapsto L(y)$ determine EA-equivalence of functions f and $\hat{f}$. □

Now we consider the case of presemifields isotopic to finite fields.

Theorem 5.1. *Let $\hat{S} = (F, +, \hat{\circ})$ be a symplectic presemifield isotopic to a finite field $S = (F, +, \circ)$. If $L\bar{M} = \lambda \cdot id$ for some $\lambda \in F$ then f and $\hat{f}$ are EA-equivalent bent functions.*

Proof. Let $S = (F, +, \circ)$ is a finite field F. Let's calculate the subspace $K^+(F)$. Let linear function φ be defined by linearized polynomial

$$\varphi(a) = \sum_{i=0}^{m-1} \beta_i a^{p^i},$$

where $\beta_i \in F$. We show that

$$\bar{\varphi}(a) = \sum_{i=0}^{m-1} \beta_i^{p^{m-i}} a^{p^{m-i}}.$$

Indeed,

$$B(\bar{\varphi}(a), b) = B(a, \varphi(b)) = Tr\left(a \sum_{i=0}^{m-1} \beta_i b^{p^i}\right) =$$

$$= Tr\left(\sum_{i=0}^{m-1} b\beta_i^{p^{m-i}} a^{p^{m-i}}\right) =$$

$$= B\left(\sum_{i=0}^{m-1} \beta_i^{p^{m-i}} a^{p^{m-i}}, b\right).$$

For $\varphi \in K^+(S) = K^+(F)$ we have

$$\varphi(a) \cdot b = \bar{\varphi}(ab),$$

which means

$$\sum_{i=0}^{m-1} \beta_i a^{p^i} \cdot b = \sum_{i=0}^{m-1} \beta_i^{p^{m-i}} (ab)^{p^{m-i}}.$$

Then we have $\beta_i = 0$ for all $i \neq 0$, hence $\varphi(a) = \bar{\varphi}(a) = \beta_0 a$. Therefore,

$$K^+(F) = \{\lambda \cdot id \mid \lambda \in F\}.$$

Now we consider the function $\hat{f}(x, y)$. Since $L\bar{M} \in K^+(F)$ we have $L\bar{M} =$

$M\bar{L} = \lambda \cdot id$ for some $\lambda \in F$. Let $\lambda^{-1} = \alpha^2$. Then we have

$$\begin{aligned}\hat{f}(x,y) = B(R_{N(x)}^{-1}L(y), \bar{M}^{-1}(y)) = \\ = B(R_{N(x)}^{-1}L(y), (L\bar{M})^{-1}L(y)) = \\ = B(R_{N(x)}^{-1}L(y), \alpha^2 L(y)) = \\ = B(\frac{1}{N(x)} \cdot \alpha L(y), \alpha L(y)),\end{aligned}$$

so, the maps $x \mapsto N(x)$, $y \mapsto \alpha L(y)$ determine EA-equivalence of functions f and $\hat{f}$. □

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Әбдіхалықов Қанат Серікұлы ЖАРТЫЛАЙ ӨРІСТЕРДЕН ТУЫНДАҒАН ҚАБАТТАСУЛАР ЖӘНЕ БЕНТ-ФУНКЦИЯЛАР

Бұл мақалада аффиндік жазықтықтарының қабаттасуларының элементтерінде сызықты болатын бент-функцияларды зерттейміз. Соның ішінде, біз симплекстік жартылай өрістерден туындаған қабаттасулармен байланысқан бент-функцияларды қарастырамыз. Сонымен қатар, біз мұндай функцияларды псевдо-планарлы функциялар тұрғысынан зерттейміз.

Абдухаликов Канат Серикович РАССЛОЕНИЯ, ПОРОЖДЕННЫЕ ПОЛУПОЛЯМИ И БЕНТ-ФУНКЦИИ

В данной работе мы исследуем бент-функции, линейные на элементах расслоений в аффинных плоскостях. В частности, мы изучаем бент-функции, связанные с расслоениями, порожденными симплектическими полуполями. Мы также изучаем такие функции с точки зрения псевдо-планарных функций.

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MAXWELL EQUATIONS,THEIR HAMILTONIAN AND BIQUATERNIONIC FORMS AND PROPERTIES OF THEIR SOLUTIONS

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Annotation: The properties of system of Maxwell equations (MEqs.) of classic electrodynamics are considered and some imperfection of them are discussed. Their modification on the basis of a hamiltonian and biquaternionic form has been offered, which gives the connected hyperbolic system. This system liquidates these shortcomings. Solutions of modified MEqs are constructed, which possess some new properties. In particular, the electromagnetic waves are considered, which describe longitudinal electromagnetic waves of different polarization. Such waves are observed in nature but don't be described in classic electrodynamics. By use distribution theory the shock EM-waves with jump of tension on wave fronts are considered and conditions on fronts of shock waves are presented.

Keywords: Maxwell equations, biquaternions, biwave equation, electromagnetic wave, shock EM-wave.

Processes of electromagnetic waves diffraction are described by the Maxwell equations (MEqs), which make theoretical base of modern electrodynamics and quantum mechanics. There is an extensive bibliography connected with development of different methods for solving these equations and boundary value problems for them, applications of their solutions to a wide class of problems of electrodynamics and electrical equipment.

Fine property of these equations is an opportunity to write down them in a complex form which reduces the number of the equations twice. This hamiltonian form of Maxwell equations (one vectorial equation for currents and one scalar equation for charges) is used not so widely. But reduction of the number of the equations simplifies procedure of creation of their solutions

Ключевые слова: *Уравнения Максвелла, бикватернион, биволновое уравнение, электромагнитная волна, ударная ЭМ-волна.*

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by use the hamiltonian form. From this form it is easy to make a transition to a biquaternionic form MEqs in which this system is equivalent to one biquaternionic wave equation. This feature is noticed long ago and many works are devoted to representation of MEqs in algebras of hypercomplex numbers [1-6]. However, all these equations possess a number of mathematical features which do this system poorly connected and of a mixed hyperbolic-elliptic type, that contradict the wave nature of distribution of electromagnetic waves in environments. Besides they don't describe longitudinal electromagnetic waves, which are observed in practice. Therefore various attempts of improvement of this system for the description of the observed electromagnetic phenomena are made [7-8].

In this article some imperfection of the system of Maxwell equations (MEqs.) are discussed and their modification on the basis of a biquaternionic form of these equations has been offered, which liquidates these shortcomings. Introduction of scalar α-fields in biquaternion of intensity of electromagnetic field (EM-field) allows to unite the vector and scalar MEqs in one connected system 8 equations. Modified Maxwell equations are the new strong hyperbolic system and we construct their solutions by use biquaternionic potentials and generalized function theory [9].

Solutions of modified MEqs possess the different new properties which are considered here. In particular, they describe longitudinal electromagnetic waves. Also longitudinal EM-waves of different polarization for the modified Maxwell equations are constructed and some of their properties are described. Moreover, the shock EM-waves with jump of tension on wave fronts are considered and conditions on fronts of shock waves are presented.

1. Maxwell equations

The classic system of Maxwell equations has the form:

$$\begin{gathered} rot\,E = -\frac{\partial B}{\partial t}, \\ rot\,H = \frac{\partial D}{\partial t} + j^E(x,t), \end{gathered} \tag{1}$$

$$\begin{gathered} div\,D = \rho^E(x,t), \quad D = \varepsilon E, \\ div\,H = 0, \quad B = \mu H, \end{gathered} \tag{2}$$

$x \in R^3,\ t \in R^1$. Here electric conductivity ε and magnetic permeability μ are constants in isotropic EM-medium. Vectors E, H are the tentions of electric and magnetic fields, B is magnetic induction, D is electric displacement, $j^E(x,t)$ is the density of electric currents, ρ^E is the density of electric charges.

In the theory of boundary value problems of electrodynamics the symmetric form of MEqs is commonly used which is invariant to exchange

$$E \leftrightarrow H,\ \ -\varepsilon \leftrightarrow \mu$$

symmetric form of Maxwell equations:

$$-\varepsilon\,\partial_t E + rotH = j^E(x,t), \qquad \mu\ \partial_t H + rotE = j^H(x,t), \tag{3}$$

$$\varepsilon divE = \rho^E(x,t), \quad -\mu\ divH = \rho^H(x,t). \tag{4}$$

It's equivalent to MEqs when magnetic charges and currents are equal to zero:

$$\rho^H = 0, \quad j^H = 0. \tag{5}$$

The divergence in MEqs (3) gives *charges conservation law*:

$$\frac{\partial \rho^E}{\partial t} + divj^E = 0, \quad \frac{\partial \rho^H}{\partial t} + divj^H = 0. \tag{6}$$

Note that vectorial equations (3) of this system aren't connected with scalar equations (4). They are sufficient for determination EM-field, if electric currents are known.

2. Vectorial Maxwell equations for currents. Shock electromagnetic waves and their properties

Matrix form of two vectorial MEqs for electric currents are written in the form:

$$M(\partial_x, \partial_t)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -\varepsilon\partial_t I & \Phi \\ \Phi & \mu\partial_t I \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} j^E \\ j^H \end{pmatrix},$$

$$\Phi = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

From here follow that by given (known) currents this system is closed relative E, H and sufficient for their definition. It is hyperbolic and its characteristic equation has the form

$$\nu_t^2(\nu_t^2/c^2 - \nu_1^2 - \nu_2^2 - \nu_3^2)^2 = 0, \tag{8}$$

where *light speed* $c = \frac{1}{\sqrt{\varepsilon\mu}}$, $(\nu, \nu_t) \triangleq (\nu_1, \nu_2, \nu_3, \nu_t)$ is a normal to characteristic surface F in Minkowski space $\mathbb{M}$. Consequently we have two type of conditions on F:

$$\nu_t^2 = 0 \quad \texttt{or} \quad (\nu_t^2/c^2 - \nu_1^2 - \nu_2^2 - \nu_3^2)^2 = 0. \tag{9}$$

In the first case, any fixed Lyapunov's surface ($F(x) = const$) in R^3 is characteristic surface and Cauchy problem for this system doesn't resolved in its vicinity.

In the second case, it defines a moving surface F_t in R^3 and the solutions of (7) can have jumps of intensities $([E]_F, [H]_F)$ of EM-fields on F . Such waves are called *shock EM-waves*. Vector $\nu = (\nu_1, \nu_2, \nu_3)$ is a normal to a wave front F_t in R^3, which is moving with light speed . Wave vector $m \triangleq \nu / \|\nu\|$ is directed towards the front movement and from (9) it follows

$$c = -\frac{\nu_t}{\|\nu\|}.$$

In the space of distributions the classical decision of (7) (considered as generalized vector-function $\hat{u}$) satisfies the following system [10]:

$$M\,\hat{u} + (\mathrm{c}^2 + 1)^{-0,5} G\,[u]_F\ \delta_F(x,t) = \hat{J}, \tag{10}$$

$$G = \begin{pmatrix} \varepsilon c I & \Gamma \\ \Gamma & -\mu c I \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix},$$

$$u = \begin{pmatrix} E \\ H \end{pmatrix}, \quad J = \begin{pmatrix} j^E \\ j^H \end{pmatrix}, \quad m = \frac{\nu}{\|\nu\|}.$$

Here singular generalized function $[u]_F\delta_F(x,t)$ is a *simple layer* on F (generalized singular function), gap $[u]_F$ is a *vectorial density of simple layer*.

It follows from (10) that $\hat{u}$ is the generalized solution of Eqs.(7) only if this density is equal to 0. Then the next conditions on EM- waves fronts are performed:

$$\sum_{j=1}^{3} G_{ij}\,[u_j]_F = 0, \quad i,j = 1,2,3.$$

From here next corollaries follow.
Corollary 1. *On fronts of shock EM-waves the gaps of intensities satisfy the next conditions:*

$$[E]_{F_t} = \sqrt{\frac{\mu}{\varepsilon}}\,[H]_{F_t} \times m, \quad [H]_{F_t} = -\sqrt{\frac{\varepsilon}{\mu}}\,[E]_{F_t} \times m.$$

Here the sigh " $\times$ " denotes the vector product.
Corollary 2. *Shock electromagnetic waves are transverse:*

$$([E]_{F_t}, m) = 0, ([H]_{F_t}, m) = 0.$$

About construction of solutions of Maxwell Eqs (7) and solving the boundary value problems for them see [10, 11].

3. Scalar Maxwell equation for charges. Coulomb's potential

Maxwell equations for electric charges have the next form:

$$\begin{aligned} \varepsilon \operatorname{div} E &= \rho^E(x,t), \\ \operatorname{div} H &= 0. \end{aligned} \tag{11}$$

From the second equation follow that magnetic charges don't exist.

If to enter potentials of E-field:

$$E = \operatorname{grad} u + \operatorname{rot} U$$

then from the first Eq.(11) follow the elliptic equation for Coulomb's potential u:

$$\triangle u = \rho^E(x,t), \tag{12}$$

where Δ is Laplace operator. If the support over x of $\rho(x, \tau)$ is limited, we get from (12) its damped on infinity solution [9]:

$$u(x,t) = -\frac{1}{4\pi} \int\limits_{\mathrm{supp}\,\rho(y)} \frac{\rho^E(y,t)}{\|y-x\|} dy_1 dy_2 dy_3. \tag{13}$$

But it contradicts to wave nature of EM-wave expansion at alternating electric charges. By this cause this equation is usually used only for description of static electric fields generated by static charges $\rho^E(x)$.

4. Biquuaternions, biwave equation

So, you see that classic system of MEqs is not connected and not hyperbolic. Here we'll give biform of MEqs following to [4]. For this purpose we give some definitions from biquaternions algebra [12].

We consider the functional space of biquaternions (Bqs.) in Hamilton's form of quaternions representation [13]:

$$\mathbb{B}(\mathbb{M}) = \{F = f(\tau, x) + F(\tau, x)\}$$

on Minkowski space $\mathbb{M} = \{(\tau, x) : x = \sum_{j=1}^{3} x_j e_j\}$, $f(\tau, x)$ is a complex functions , $F(\tau, x)$ is a three-dimensional complex vector-function. They are locally integrable and differentiable on $\mathbb{M}$ or, in general case, they are generalized functions [9]; $1, e_1, e_2, e_3$ are the basic elements in biquaternions algebra.

Summation and multiplication on $\mathbb{B}(\mathbb{M})$ have the forms:

$$F + B = (f + F) + (b + B) \triangleq (f + b) + (F + B),$$

$$F \circ B = (f + F) \circ (b + B) \triangleq fb - (F, B) + fB + bF + [F, B],$$

where

$$(F, B) = \sum_{k=1}^{3} F_k B_k, \quad [F, B] = \sum_{j,k,l=1}^{3} \epsilon_{jkl} F_j B_k e_l,$$

are usual scalar and vector products in R^3, ϵ_{jkl} is a symbol of Leavy-Civitta. We denote

$$f = scal F, \quad F = Vect F.$$

This algebra is associative but not commutative. We call
mutual biquaternion $F^- = f - F$,
complexconjugate $\bar{F} = \bar{f} + \bar{F}$,
conjugate $F^* = \bar{f} - \bar{F}$.

4.1 Mutual bigradients and their properties

We will use the differential operators (*mutual bigradients*):

$$\nabla^{\pm} B \triangleq (\partial_\tau \pm i\nabla) \circ (b + B) \triangleq \partial_\tau b \mp i(\nabla, B) \pm i\nabla b \pm \partial_\tau B \pm i[\nabla, B]$$

$\Rightarrow$

$$\nabla^{\pm} B = \partial_\tau b \mp i\,\mathrm{div} B \pm i\, grad\, b \pm \partial_\tau B \pm i\,\mathrm{rot} B.$$

Their composition poses the useful property:

$$\nabla^+ \nabla^- = \nabla^- \nabla^+ = \partial_\tau^2 - \Delta = \Box \quad (dalambertian),$$

which gives possibility easy to construct the solutions of biquaternionic wave equation(*biwave* Eq.)

$$\nabla^+ B = G \quad \Rightarrow \quad \Box B = \nabla^- G. \tag{14}$$

The solutions of (14) are presented in the form

$$B = \nabla^- (\psi * G) + B^0 = \nabla^- \int\limits_{\|y-x\| \le \tau} \frac{G(y, \tau - \|y - x\|)}{4\pi \|y - x\|} dy_1 dy_2 dy_3 + B^0 \tag{15}$$

where the first summand is the convolution $G(x, \tau)$ with the fundamental solution of D'Alembert equation

$$\psi = (4\pi \, \|x\|)^{-1} \delta(\tau - \|x\|).$$

Singular function $\psi(x, \tau)$ is *the simple layer on light cone* ($\tau = \|x\|$):

$$\Box \psi = \delta(\tau)\delta(x),$$

B^0 is arbitrary solution of homogeneous D'Alembert equation.

5. Biquaternionic and hamiltonian form of MEqs

If to introduce the biquaternions of EM-field:
EM-tension

$$A = 0 + A = \sqrt{\varepsilon}E + i\sqrt{\mu}H,$$

charge-current

$$\Theta = (i\rho + J) = i\rho^E/\sqrt{\varepsilon} + \sqrt{\mu}j^E,$$

energy-impuls

$$\Xi = 0,5\, A^* \circ A = 0,5\, (\bar{A}, A) - 0,5\, [A, \bar{A}] = W + iP,$$

where

$$W = \frac{1}{2}\left(\varepsilon\, \|E\|^2 + \mu\, \|H\|^2\right) \texttt{ is } \ adensityofenergyofEM-field,$$

$$P = c^{-1}\,[E, H] \texttt{ is } Pointingvector,$$

then the Maxwell equations can be written in the form of the biwave equation:
Biquaternonic form of Maxwell equations

$$\nabla^+ A = -\Theta. \tag{16}$$

From here it follows that charges and currents are physical manifestation of the bigradient of tension of EM-field! At their absence bigradient of tensions is equal to zero!

If to write the scalar and vector part of (16) separately, we get
Hamiltonian form of Maxwell equations [14]:

$$\begin{aligned} \operatorname{div} A &= \rho, \\ -\partial_\tau A - i\operatorname{rot} A &= J. \end{aligned} \tag{17}$$

If to take mutual bigradients from (16) we get the wave equation for A:

$$\Box A = -\nabla^- \Theta.$$

From here it follows (as $scal A = 0$) well known charges conservation law:

$$\partial_\tau \rho + \operatorname{div} J = 0, \tag{18}$$

and wave equation for tensions:

$$\Box\, A = -\nabla\rho - \partial_\tau J + \mathrm{i}\operatorname{rot} J. \tag{19}$$

6. Generalized solutions of MEqs biform

Solutions of MEqs biform are

$$A = -\nabla^{-}(\psi * \Theta) + A^0, \tag{20}$$

where spinor A^0 is arbitrary solution of homogeneous Maxwell equation:

$$A^0 = \nabla^{-}\psi^0 + i\sum_{j=1}^{3}\nabla^{-}(\psi^j e_j), \tag{21}$$

$$\Box\psi^j = 0, \quad j = 0,1,2,3,4, \tag{22}$$

$$\psi^j = \int_{R^3}\varphi^j(\xi)\exp(-i(\xi,x) - i\,\|\xi\|\,t)\;d\xi_1 d\xi_2 d\xi_3 \quad \forall\varphi^j(\xi)\in L_1(R^3).$$

As for Maxwell equations $scal A = 0$ then for potentials ψ^j we have

Lorentz calibrations:

$$\partial_t\psi^0 = \sum_{j=1}^{3}\partial_j\psi^j.$$

Let consider some examples of spinors.

1. *Plane waves in the direction of a vector* e are generated by the potentials

$$\psi^j(x,\tau) = f((e,x)-\tau), \quad \|e\| = 1,$$

where $f(\tau)$ is arbitrary generalized function.

2. *Elementary harmonic spherical* ω*-spinor*, ω is a frequency of vibration. Its potentials in spherical coordinates (r,θ,φ)) is

$$\psi^j(n,k,l) = j_n(r)P_k^l(\cos\theta)e^{i(l\varphi-\omega\tau)},$$

j_n are spherical Bessel functions, P_k^l are associated Legendre polynomials.

3. *Spinors field*

$$A^0 = \nabla^{-}\left\{\psi^0 + i\sum_{j=1}^{3}\psi^j e_j\right\} * C(\tau,x) \quad \forall C(\tau,x).$$

Here $C(\tau,x)$ are arbitrary biquaternions which admit such convolutions.

So, there are EM-fields without electric charges and currents.

7. Maxwell-Dirac equations and their properties

We introduce *a complex scalar field*

$$\alpha(\tau, x) = \frac{ia_1}{\sqrt{\varepsilon}} + \frac{a_2}{\sqrt{\mu}} \tag{23}$$

in the tension Bq. of EM-field

$$A = \alpha + A \tag{24}$$

and *gravymagnetic charge and currents* ρ^H, j^H in Θ:

$$\Theta = i\rho + J, \tag{25}$$

$$\rho = \frac{\rho^E}{\sqrt{\varepsilon}} + i\frac{\rho^H}{\sqrt{\mu}}, \qquad J = \sqrt{\mu}j^E + i\sqrt{\varepsilon}j^H.$$

Then the biform of MEqs. (16) conserves the kind but its hamiltonian form is changed. It contains scalar α-field which is connected, to the system:

$$\begin{aligned} \partial_\tau \alpha - \operatorname{div} A = -\rho, \\ \partial_\tau A + grad\, \alpha + i\operatorname{rot} A = -J. \end{aligned} \tag{26}$$

It is the hyperbolic system of differential equations and its solutions can be presented in the form:

$$\begin{gathered} \alpha = -i\partial_\tau\left(\psi * \rho\right) + i\sum_{j=1}^{3}\partial_j\psi^j, \\ A = i\nabla\left(\psi * \rho\right) - \left(\psi * J\right)_{,\tau} + irot\left(\psi * J\right) + i\sum_{j=1}^{3}\partial_\tau\psi^j e_j + rot\sum_{j=1}^{3}\psi^j e_j. \end{gathered} \tag{27}$$

From hamiltonian form (25) it's easy to get the connective system of

Maxwell-Dirac equations

$$\begin{aligned} rotH - \varepsilon\frac{\partial E}{\partial t} + c^2 grad\, \alpha_1 = j^E, \\ rotE + \mu\frac{\partial H}{\partial t} - c^2 grad\, \alpha_2 = j^H, \\ \varepsilon divE + \partial_t\alpha_1 = \rho^E, \\ -\mu divH + \partial_t\alpha_2 = \rho^H. \end{aligned} \tag{28}$$

We call this system so because its differential operator coincides with differential part of Dirac equations [15]. The solutions of this system can be obtained from (27) if to write separately their real and imaginary parts.

8. Plane spinors of different polarization

consider some plane waves generated by scalar potentials of the kind:

$$\psi_j = f(\eta), \quad \eta = (k, x) - \omega t, \quad \|k\| = \omega,$$

where f is arbitrary function which describes the plane wave, moving in the direction of wave vector k with speed $c = \omega/\|k\| = 1$. They are the solutions of classic homogeneous wave equation. On its base it's easy to build plane waves of different polarizations.

1. *Longitudanal magnetic wave* in direction H:

$$A^0 = \nabla^- \psi^0 = (\partial_\tau - i\nabla)(f(\eta) + 0) = -\omega f'(\eta) - ikf'(\eta).$$

2. *Longitudanal electric wave* in direction E:

$$A^0 = \nabla^- \psi^0 = (\partial_\tau - i\nabla)(if(\eta) + 0) = -\omega f'(\eta) - kf'(\eta).$$

3. *Tesla's wave* - EM-wave in direction E with torsion component H:

$$A^1 = (\partial_\tau - i\nabla)(0 + f(\eta) e_1) = ik_1 f'(\eta) - \omega e_1 f'(\eta) - ik_3 e_2 f'(\eta) + ik_2 e_3 f'(\eta)$$

as $(E, H) = 0$ and

$$(E, k) = -\frac{\omega k_1}{\sqrt{\varepsilon}} f'(\eta), \quad (H, k) = \left(-\frac{k_3 k_2}{\sqrt{\mu}} + \frac{k_3 k_2}{\sqrt{\mu}}\right) f'(\eta) = 0.$$

4. *Torsion wave* - EM-wave in direction H with torsion component E.

$$A^1 = \nabla^- \psi^1 e_1 = (\partial_\tau - i\nabla)(0 + if(\eta) e_1) = i\omega e_1 f'(\eta) + k_3 e_2 f'(\eta) - k_2 e_3 f'(\eta),$$

$$(H, k) = \frac{\omega k_1}{\sqrt{\mu}} f'(\eta), (E, k) = 0, \quad (E, H) = 0.$$

Here waves, names correspond to the ones in paper [8, Etkin V.A.].

9. Shock EM-wave and condition on wave front

The equation for such waves in the distribution space is as follows:

$$\nabla^+ \hat{A} = \hat{\Theta} + \{\nu_\tau \, [\alpha]_F + (\, [A]_F \, , \nu) - i\nu_\tau \, [A]_F + \nu \times [A]_F + i \, [\alpha]_F \, \nu\} \, \delta_F .$$

To be generalized solutions of (16) the second part in the right side must be equal to 0. By analogy to (9) we get

Conditions on fronts of EM-waves:

$$(\, [A]_{F_\tau} \, , m) = [\alpha]_{F_\tau} \, , \quad [A]_{F_\tau} = im \times [A]_{F_\tau} - [\alpha]_{F_\tau} \, m.$$

EM-waves are transverse only if $[\alpha]_F = 0$, i.e when α-field is continues.

10. Conclusion

Biform of Maxwell equations (16) constructed here is mathematically and physically inherently consistent. But it is unclosed. How to close this system by use the material equations for charges and currents see [16, 17].

In paper [16] it was shown that ϱ_H, j_H describe the gravymagnetic charges and currents and there the biwave equation is constructed for free charges and currents, which encloses the system for their definition.

In paper [17] the system of biwave equations were constructed, which describe electro-gravimagnetic interaction of mass, charges and currents on the base of analogues of Newton laws. There it was shown that a scalar α-field describes property of resistance-absorption of electro-gravimagnetic field to the movement of external electric and gravimagnetic charges and currents.

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Алексеева Л.А. МАКСВЕЛЛ ТЕҢДЕУЛЕРІ, ОЛАРДЫҢ ГАМИЛЬТОНДЫ ЖӘНЕ БИКВАТЕРНИОНДЫ ФОРМАЛАРЫ МЕН ШЕШІМДЕРІНІҢ ҚАСИЕТТЕРІ

Классикалық электродинамиканың Максвелл теңдеулерінің қасиеттері қарастырылады және олардың кейбір кемшіліктері талқыланады. Максвелл теңдеулерінің гамильтонды және биквартениoнды формалары негізіндегі осы теңдеулердің модификациясы ұсынылған, ол аталған кемшіліктер орын алмайтын байланысқан теңдеулер жүйесін береді. Модификацияланған теңдеулердің бірқатар жаңа қасиеттерді иеленген шешімдері тұрғызылған. Атап айтқанда, олар іс жүзінде бақыланатын, бірақ классикалық электродинамикада түсініктемесін таппаған, әралуан полярланған бойлық электромагниттік толқындарды сипаттай алады. Сонымен қатар, толқын бағытында электрлік және магниттік кернеулігінің ақырлы секірісі бар соққылы электромагниттік толқындар қарастырылған. Шешімдердің тұрғызуға жалпыланған функциялар теориясы пайдаланылады.

Алексеева Л.А. УРАВНЕНИЯ МАКСВЕЛЛА, ИХ ГАМИЛЬТОНОВА И БИКВАТЕРНИОННАЯ ФОРМЫ И СВОЙСТВА ИХ РЕШЕНИЙ

Рассматриваются свойства уравнений Максвелла классической электродинамики и обсуждаются их некоторые недостатки. Предложена модификация уравнений Максвелла на основе гамильтоновой и бикватернионной форм этих уравнений, которая дает связанную систему уравнений, лишенную этих недостатков. Построены решения модифицированных уравнений, которые обладают рядом новых свойств. В частности, они могут описывать продольные электромагнитные волны различной поляризации, которые наблюдаются на практике, но не находят объяснения в классической электродинамике. Рассмотрены также ударные электромагнитные волны с конечным скачком электрической и магнитной напряженностей на фронте волны. Для построения решений используется теория обобщенных функций.

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ONE-FORMULAS AND ONE-TYPES IN ORDERED THEORIES

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Dedicated to Askar Dzhumadildaev and Kanat Kudaibergenov on their 60th anniversary

Annotation: The paper considers theories with an $\emptyset$-definable relation of a linear order. Small theories for which any discrete order on convex equivalences is finite were studied. The classification of 1-types based on the properties of 2-formulas, definable sets of 1-formulas and properties of 1-types' interaction was given.

Keywords: 1-formulas in ordered structures, 1-types in ordered theories, nonorthogonality of 1-types

1. Introduction

In this paper we study 1-formulas and 1-types in theories with an $\emptyset$-definable binary relation of linear order (ordered theories). Denote by N a countable saturated model of a given linearly ordered theory. For two formulas $H(x)$ and $\Theta(x)$ such that $H(N) \subset \Theta(N)$ we can introduce in a natural way a *convex equivalence relation*, equivalence classes of which are maximal convex subsets of $H(N)$ separated by a complement of this formula. In the first section we study the properties of linear orders defined on the classes of

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convex equivalence of 1-formula in others 1-formulas. In particular, we give a description of 1-formulas for the class of small ordered theories with the property of finiteness of discrete chains of convex equivalences of two arbitrary 1-formulas (Theorem 2). As a corollary we obtain that restriction of a small ordered theory with few countable models to a signature of a pure linear order is $\aleph_0$-categorical (Corollary 5).

Let A be a subset of a saturated enough model N of T. The analysis of one-types is based on using the properties of A-definable family of 1-formulas (Points A, B), of 2-formulas (Point C) and properties of interactions of one-types (Point D). Taking into consideration the points A, B, C it is possible to show the existence of 18 kinds of one-types in ordered theories. The properties of 1-formulas permits to give an example complete ordered theory with two non-orthogonal 1-types one is definable, second is non-definable (Theorem 6).

2. The Properties of Linear Orders Defined On the Classes of Convex Equivalence of 1-Formula

A complete countable theory T is called *small* if $|\bigcup_{n<\omega} S_n(T)| = \omega$, where $S_n(T)$ is the set of all n-types over $\emptyset$. A complete countable theory T *has a few number of countable models* if the number of countable non-isomorphic models $I(T,\omega)$ is less than 2^ω. It is clear that a theory with few countable models is small.

Notice that for any countable model M of a small theory T, for any finite $A \subset M$, $|S_1(A)| \leq \omega$ there is countable saturated model $N(M \prec N)$. Here, $S_1(A)$ is the set of all one-types over A. Always, in Section N will stand for a countable saturated model of given small theory. We will consider ordered theories and we will suppose that $<$ is an $\emptyset-$definable relation of linear order.

Often we will write first order formulas through relations on definable sets. For example:

$x < \phi(N) \equiv \forall y(\phi(y) \to x < y)$,
$x \in (\beta_1, \beta_2) \equiv \beta_1 < x < \beta_2$,
$\phi(N) \bigcap \theta(N) \neq \emptyset \equiv N \models \exists(\phi(x) \wedge \theta(x))$,
$\phi(N) < \theta(N)^+ \equiv N \models \forall t(\forall y(\theta(y) \to y < t) \to \forall x(\phi(x) \to x < t))$.

For any $A \subset N$ (not necessary definable) denote

$$A^+ := \{\gamma \in N | \forall a \in A : N \models a < \gamma\};$$
$$A^- := \{\gamma \in N | \forall a \in A : N \models \gamma < a\}.$$

Let $A, B \subseteq N$. Then $A \subset B$ means $A \subseteq B$ and $A \neq B$. Let $A \subset B$. A is said to be *is convex in* B, if $\forall x, y \in A(x < y), \forall z \in B(x < z < y \to z \in B)$. If A is convex in N, we say that A is *convex*.

Let B be an ordered set and Γ be a family of convex subsets of B such that $\forall C_1 \neq C_2 \in \Gamma, C_1 \cap C_2 = \emptyset$. Then Γ is an ordered structure with the following ordering $<^c$:

$$\forall C_1, C_2 \in \Gamma \; [C_1 <^c C_2 \iff \forall x \in C_1, \forall y \in C_2 (x < y)].$$

Denote $R_A := \{\phi(x) | \phi(x)$ is an A-definable 1-formula (i.e. 1-formula with parameters from A), such that $N \models \exists y(\phi(N) < y)\}$.

Let $\phi, \psi \in R_A$, then put

$$[\phi <_r \psi \iff \psi(N)^+ \subset \phi(N)^+];$$
$$[\phi \sim_r \psi \iff \phi(N)^+ = \psi(N)^+];$$

and denote $\phi / \sim_r := \{\Theta \in R_A | \Theta \sim_r \phi\}$, $R_A / \sim_r := \{\phi / \sim_r | \phi \in R_A\}$.

Denote $L_A := \{\phi(x) | \phi(x)$ be an A-definable 1-formula, such that $N \models \exists y(y < \phi(N))\}$. Let $\phi, \psi \in L_A$, then put

$$[\phi <_l \psi \iff \phi(N)^- \subset \psi(N)^-];$$
$$[\phi \sim_l \psi \iff \phi(N)^- = \psi(N)^-]$$

and denote $\phi / \sim_l := \{\Theta \in L_A | \Theta \sim_l \phi\}$, $R_A / \sim_l := \{\phi / \sim_l | \phi \in L_A\}$.

Proposition 1. *Let T be a small ordered theory, A be a finite subset of a model of T. Then $\langle R_A / \sim_r ; <_r \rangle$ and $\langle L_A / \sim_l ; <_l \rangle$ do not contain dense intervals.*

Let A be a finite subset of a saturated model N of a small theory T with an $\emptyset$-definable relation of a linear order, $H(x)$ and $\Theta(x)$ be A-definable 1-formulas such that $H(N) \subset \Theta(N)$. Define

$E_{H,\Theta}(x, y) := H(x) \wedge H(y) \wedge (x < y \to \forall z((x < z < y \wedge \Theta(z)) \to H(z))) \wedge (y < x \to \forall z((y < z < x \wedge \Theta(z)) \to H(z)))$.

$E_{H,\Theta}(x, y)$ is an A-definable relation of equivalence on $H(N)$ such that any $E_{H,\Theta}$-class is convex in $\Theta(N)$. This equivalence is said to be the *convex equivalence of $H(x)$ in $\Theta(x)$*. On the set of all $E_{H,\Theta}$-classes $\{E_{H,\Theta}(N, \alpha) | \alpha \in H(N)\}$ there is A-definable linear order $<^c$ such that for any $\alpha, \beta \in H(N), E(N, \alpha) \neq E(N, \beta)$ the following holds:

$E_{H,\Theta}(N,\alpha) <^c E_{H,\Theta}(N,\beta)$ iff $\forall\gamma \in E_{H,\Theta}(N,\alpha), \forall\delta \in E_{H,\Theta}(N,\beta)$ we have $N \models \gamma < \delta$.

Let $\phi(x)$ and $\psi(x), \Theta(x)$ be A-definable formulas such that

$$\phi(N) \cap \psi(N) = \emptyset,\ \phi(N) \cup \psi(N) \subseteq \Theta(N).$$

We say that $\phi(x)$ is a *dense subformula* of $\Theta(x)$ if the A-definable relation of equivalence $E_{\phi,\Theta}(x,y)$ on $\phi(N)$ satisfies the condition of density, i.e. the order on the set of these $E_{\phi,\Theta}$-classes is dense: $\forall a, b \in \phi(N)$

$$(E_{\phi,\Theta}(x,a) <^c E_{\phi,\Theta}(x,b) \rightarrow \exists c \in \phi(N)(E_{\phi,\Theta}(N,a) <^c E_{\phi,\Theta}(N,c) <^c E_{\phi,\Theta}(N,b))).$$

We say that $\phi(x)$ and $\psi(x)$ are *mutually dense in* $\Theta(x)$ if $\phi(x)$ and $\psi(x)$ are dense subformulas of $\Theta(x)$ and for every two $E_{\phi,\Theta}$-classes ($E_{\phi,\Theta}(x,a)$, $E_{\phi,\Theta}(x,b)$, $E_{\phi,\Theta}(N,a) <^c E_{\phi,\Theta}(N,b)$) there is $E_{\psi,\Theta}$-class such that

$$E_{\phi,\Theta}(N,a) <^c E_{\psi,\Theta}(N,c) <^c E_{\phi,\Theta}(N,b)$$

and the same is true for two different arbitrary $E_{\psi,\Theta}$-classes.

Notice, that if $\phi(x)$ is dense subformula of $\Theta(x)$ then $\neg\phi(x) \wedge \Theta(x)$ is dense subformula of $\Theta(x)$ and the pair of formulas $\phi(x)$ and $\neg\phi(x) \wedge \Theta(x)$ are mutually dense in $\Theta(x)$.

Introduce [1] the notion of a *convex hull (closure)* of a one-formula $\phi(x)$:

$$\phi^c(x) := \exists y_1 \exists y_2(\phi(y_1) \wedge \phi(y_2) \wedge y_1 \leq x \leq y_2)).$$

Fact 1. *Let $\phi(x)$ and $\psi(x)$ be two one-formulas which are mutually dense in a formula $\Theta(x)$. Then the formulas ϕ and ψ are mutually dense in the formula $\phi^c(x) \wedge \psi^c(x) \wedge \Theta^c(x)$.*

In this Section for any $A \subset_{\text{finite}} N$, for any A-definable one-formulas $H(x)$, $\Theta(x)$, $(H(N) \subset \Theta(N))$ we will study the properties of a linear order of

$$\langle\{E_{H,\Theta}(N,\alpha) | \alpha \in H(N)\}; =, <^c\rangle.$$

The following A-definable formulas $D_0^k[H,\Theta](x)$, $D_1^k[H,\Theta](x)$ determine the first and last convex subsets of finite discrete chains of fixed length $k < \omega$ of convex $E_{H,\Theta}$-classes.

Let $k(1 < k < \omega)$, then denote

$D_0^k[H,\Theta](x) := H(x) \wedge \exists x_1 \cdots \exists x_{k-1}(\wedge_i H(x_i) \wedge E_{H,\Theta}(N,x) <^c E_{H,\Theta}(N,x_1) <^c \cdots <^c E_{H,\Theta}(N,x_{k-1}) \wedge \forall t((H(t) \wedge E_{H,\Theta}(N,x) <^c E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,x_{k-1}) \to \vee_{i<k-1} E_{H,\Theta}(N,t) = E_{H,\Theta}(N,x_i)) \wedge \forall t[(H(t) \wedge E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,x)) \to \exists z(H(z) \wedge E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,z) <^c E_{H,\Theta}(N,x))] \wedge \forall t[H(t) \wedge E_{H,\Theta}(N,x_{k-1}) <^c E_{H,\Theta}(N,t)) \to \exists z(H(z) \wedge E_{H,\Theta}(N,x_{k-1}) <^c E_{H,\Theta}(N,z) <^c E_{H,\Theta}(N,t)]$;

$D_1^k[H,\Theta](x) := H(x) \wedge \exists x_1 \cdots \exists x_{k-1}(\wedge_i H(x_i) \wedge E_{H,\Theta}(N,x_1) <^c \cdots <^c E_{H,\Theta}(N,x_{k-1}) <^c E_{H,\Theta}(N,x) \wedge \forall t((H(t) \wedge E_{H,\Theta}(N,x_1) <^c E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,x) \to \vee_{1<i<k-1} E_{H,\Theta}(N,t) = E_{H,\Theta}(N,x_i)) \wedge \forall t[(H(t) \wedge E_{H,\Theta}(N,x) <^c E_{H,\Theta}(N,t)) \to \exists z(H(z) \wedge E_{H,\Theta}(N,x) <^c E_{H,\Theta}(N,z) <^c E_{H,\Theta}(N,t))] \wedge \forall t[H(t) \wedge E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,x_1)) \to \exists z(H(z) \wedge E_{H,\Theta}(N,t) <^c E_{H,\Theta}(N,z) <^c E_{H,\Theta}(N,x_1)]$.

The next observation follows from the above definition:

Observation 1. *(i)* $i \in \{0,1\}$, $\forall\ \alpha \in D_i^k[H,\Theta](N)$, $E_{H,\Theta}(N,\alpha) \subseteq D_i^k[H,\Theta](N)$.

(ii) $\langle \{E_{H,\Theta}(N,\alpha) |\ \alpha \in D_0^k[H,\Theta](N)\}; <^c\rangle \cong \langle \{E_{H,\Theta}(N,\alpha) |\ \alpha \in D_1^k[H,\Theta](N)\}; <^c\rangle$ *and this isomorphism is A-definable.*

(iii) $1 < k < \omega$, $D_1^k[H,\Theta](N) \cap D_0^k[H,\Theta](N) = \emptyset$.

(iv) $1 < k \neq k' < \omega$; $i,j \in \{0,1\}$, $D_i^k[H,\Theta](N) \cap D_j^{k'}[H,\Theta](N) = \emptyset$.

Denote for every $l(0 < l < \omega)$, $2^l := \{\tau | \tau = (\tau(1), \tau(2), \ldots, \tau(l)),$ $\tau(1), \tau(2), \ldots, \tau(l) \in \{0,1\}\}$, $2^{<\omega} := \bigcup_{l<\omega} 2^l$ and $2^\omega := \{\tau | \tau = (\tau(1), \ldots, \tau(n), \ldots)_{n<\omega}, \tau(n) \in \{0,1\}\}$.

Define by induction on l $(0 < l < \omega)$, for any k $(1 < k < \omega)$, for any $\sigma \in (\omega \setminus \{0,1\})^l := \{< n_1, \ldots, n_l > | n_i \in \omega \setminus \{0,1\}\}$, for any $\tau \in 2^l$, the family of A-definable 1-formulas.

l=1 $F_0^k[H,\Theta,1](x) := D_0^k[H,\Theta](x)$, $F_1^k[H,\Theta,1](x) := D_1^k[H,\Theta](x)$.

l+1 Suppose for $\sigma \in (\omega \setminus \{0,1\})^l$, for $\tau \in 2^l$, $F_\tau^\sigma[H,\Theta,l](x)$ is defined. Then $\forall k(1 < k < \omega)$, $\forall i \in \{0,1\}$ put

$F_{\tau i}^{\sigma k}[H,\Theta,l+1](x) := D_i^k(F_\tau^\sigma[H,\Theta,l],\Theta)(x)$.

Example. Let $\langle C; =, <\rangle$ be an arbitrary linearly ordered structure. Define $N_C := \langle N; =, <_1\rangle$ in the following way

$N := C \times (\{I, II, III\} \cup Q \cup \{IV, V, VI\}) \cup (C \times C)_D \times Q \times \{I, II, III\}$.

Here, $(C \times C)_D := \{(c_1, c_2)|c_1, c_2 \in C, C \models c_1 < c_2 \wedge \neg\exists y(c_1 < y < c_2)\}$ and Q is the set of all rational numbers.

We assume that $I < II < III < Q < IV < V < VI$ and $<_1$ is a lexicographical linear order on $C \times (\{I, II, III\} \cup Q \cup \{IV, V, VI\})$ as well as on $(C\times C)_D\times Q\times\{I, II, III\}$. To finish the definition of the relation of order on N we define the relation between elements of $C\times(\{I, II, III\}\cup Q\cup\{IV, V, VI\})$ and $(C \times C)_D \times Q \times \{I, II, III\}$.

For every $c \in C, q \in Q, (c_1, c_2) \in (C \times C)_D$ we assume

$N_C \models (c, I) <_1 (c, II) <_1 (c, III) <_1 (c, q) <_1 (c, IV) <_1 (c, V) <_1 (c, VI) \wedge (c_1, VI) <_1 (c_1, c_2, q, I) <_1 (c_1, c_2, q, II) <_1 (c_1, c_2, q, III) <_1 (c_2, I)$.

Consider the following definable sets:

$F^3_0[x = x, x = x, 1](N) = C \times \{I\} \cup C \times \{IV\} \cup (C \times C)_D \times Q \times \{I\}$,

$F^3_1[x = x, x = x, 1](N) = C \times \{III\} \cup C \times \{VI\} \cup (C \times C)_D \times Q \times \{III\}$,

$F^{32}_{00}[x = x, x = x, 2](N) = C \times \{I\}$,

$F^{32}_{01}[x = x, x = x, 2](N) = C \times \{IV\}$,

$F^{32}_{10}[x = x, x = x, 2](N) = C \times \{III\}$,

$F^{32}_{11}[x = x, x = x, 2](N) = C \times \{VI\}$.

It is clear that $\langle F^3_0[x = x, x = x, 1](N); <_1\rangle \cong \langle F^3_1[x = x, x = x, 1](N); <_1\rangle$ and for every $\tau, \tau' \in 2^2$,

$\langle F^{32}_\tau[x = x, x = x, 2](N); <_1\rangle \cong \langle F^{32}_{\tau'}[x = x, x = x, 2](N); <_1\rangle$.

Observation 2. *For any A-definable formulas $H(x)$ and $\Theta(x)$ such that $H(N) \subset \Theta(n), \forall l < \omega$, $\forall\sigma \in (\omega \setminus \{0, 1\})^l$, $\forall\tau \in 2^l$ the following is true:*

(i) $\forall\alpha \in F^\sigma_\tau[H, \Theta, l](N),\ E_{H,\Theta}(N, \alpha) \subset F^\sigma_\tau[H, \Theta, l](N)$

(ii) $\forall k < \omega, \forall i \in \{0, 1\},\ F^{\sigma k}_{\tau i}[H, \Theta, l + 1](N) \subset F^\sigma_\tau[H, \Theta, l](N)$,

$F^{\sigma k}_{\tau 0}[H, \Theta, l + 1](N) \cap F^{\sigma k}_{\tau 1}[H, \Theta, l + 1](N) = \emptyset$

(iii) $\langle\{E_{H,\Theta}(N, \alpha)|\alpha \in F^{\sigma k}_{\tau 0}[H, \Theta, l + 1](N)\}; <^c\rangle \cong \langle\{E_{H,\Theta}(N, \alpha)|\alpha \in F^{\sigma k}_{\tau 1}[H, \Theta, l + 1](N)\}; <^c\rangle$ *and this isomorphism is A-definable.*

(iv) $\forall k \neq k' < \omega, \forall i, j \in \{0, 1\}\ F^{\sigma k}_{\tau i}[H, \Theta, l+1](N)\cap F^{\sigma k'}_{\tau j}[H, \Theta, l+1](N) = \emptyset$.

Proposition 2. *Let T be a small ordered theory. Let $H(x), \Theta(x)$ be A-definable formulas such that $H(N) \subset \Theta(N)$. Then the following holds:*

(i) $\forall l < \omega, \forall\sigma \in (\omega \setminus \{0, 1\})^l, \forall\tau \in 2^l$ *the following is true:*

If $F^\sigma_\tau[H, \Theta, l](N) \neq \emptyset$ *then* $\forall\tau' \in 2^l, F^\sigma_{\tau'}[H, \Theta, l](N) \neq \emptyset$,

$$F_\tau^\sigma[H,\Theta,l](N) \cap F_{\tau'}^\sigma[H,\Theta,l](N)=\emptyset;$$
$$\langle\{E_{H,\Theta}(N,\alpha) | \alpha\in F_\tau^\sigma[H,\Theta,l](N)\};<^c\rangle \cong$$
$$\langle\{E_{H,\Theta}(N,\alpha) | \alpha\in F_{\tau'}^\sigma[H,\Theta,l](N)\};<^c\rangle.$$

(ii) $\exists l_0<\omega$, *such that* $\forall l>l_0, \forall\sigma\in(\omega\setminus\{0,1\})^l, \forall\tau\in 2^l$

$$F_\tau^\sigma[H,\Theta,l](N)=\emptyset.$$

Proof of Proposition 2. (i) It follows from the definition of $F_\tau^\sigma[H,\Theta,l](x)$ and Observation 2.

(ii) If we suppose that such l_0 does not exist then we obtain a contradiction with our assumption that T is small. Indeed, then there is an ω-consequence of natural numbers $\sigma:=<n_1,n_2,\ldots,n_l,\cdots>_{l<\omega}$ $(n_l\in\omega\setminus\{0,1\})$ such that

$$\forall l<\omega, \forall\tau\in 2^l,\ F_\tau^{\sigma(l)}[H,\Theta,l](N)\neq\emptyset,$$

where $\sigma(l)=<n_1,\ldots,n_l>$. This means by (i) that we have 2^ω one-types over A. □

Let $H(x),H_1(x),\ldots,H_n(x),\Theta(x)$ be A-definable 1-formulas such that
$H(N),H_1(N),\ldots,H_n(N)\subset\Theta(N)$, $\forall i\neq j<n+1$,
$H_i(N)\cap H_j(N)=\emptyset,\ H(N)\cap H_i(N)=\emptyset$.
Let $l,l_1,\ldots,l_n\in\omega$; $\sigma\in(\omega\setminus\{0,1\})^l, i<n, \sigma_i\in(\omega\setminus\{0,1\})^{l_i}$; $\tau\in 2^l, i<n, \tau_i\in 2^{l_i}$.

Introduce by induction on $i<n$ the notion of an A-definable 1-formula $F_{\tau\tau_1\ldots\tau_n}^{\sigma\sigma_1\ldots\sigma_n}[H,\Theta;H_1,\ldots,H_n;l,l_1,\ldots,l_n](x)$:
$F_{\tau\tau_1}^{\sigma\sigma_1}[H,\Theta;H_1;l,l_1](x):=F_{\tau_1}^{\sigma_1}[F_\tau^\sigma[H,\Theta,l],H\vee H_1,l_1](x)$,
$F_{\tau\tau_1\ldots\tau_i}^{\sigma\sigma_1\ldots\sigma_i}[H,\Theta;H_1,\ldots,H_i;l,l_1,\ldots,l_i](x):=$
$F_{\tau_i}^{\sigma_i}[F_{\tau\tau_1\ldots\tau_{i-1}}^{\sigma\sigma_1\ldots\sigma_{i-1}}[H,\Theta;H_1,\ldots,H_{i-1};l,l_1,\ldots,l_{i-1}],H\vee H_i,l_i](x)$ for $i<n$.

Proposition 3. *Let T be a small ordered theory. Let $H(x),H_1(x),\ldots,H_n(x),\Theta(x)$ be A-definable 1-formulas such that $H(N),H_1(N),\ldots,H_n(N)\subset\Theta(N)$, $\forall i\neq j$, $H_i(N)\cap H_j(N)=\emptyset$, $H(N)\cap H_i(N)=\emptyset$. Then the following holds:*

(i) $\forall l,l_1,\ldots,l_n<\omega,\forall i<n,\forall\sigma\in(\omega\setminus\{0,1\})^l,\forall\tau\in 2^l$,
$\forall\sigma_i\in(\omega\setminus\{0,1\})^{l_i},\forall\tau_i\in 2^{l_i}$ *the following is true:*

If $F_{\tau\tau_1\ldots\tau_n}^{\sigma\sigma_1\ldots\sigma_n}[H,\Theta;H_1,\ldots,H_n;l,l_1,\ldots,l_n](N)\neq\emptyset$ *then* $\forall\tau'\in 2^l,\forall i<n$, $\forall\tau_i'\in 2^{l_i}$, $F_{\tau'\tau_1'\ldots\tau_n'}^{\sigma\sigma_1\ldots\sigma_n}[H,\Theta;H_1,\ldots,H_n;l,l_1,\ldots,l_n](N)\neq\emptyset$, *and if* $\tau\neq\tau'$ *or there exists* $i<n$ *such that* $\tau_i\neq\tau_i'$ *then*

$$F^{\sigma\sigma_1\dots\sigma_n}_{\tau\tau_1\dots\tau_n}[H,\Theta;H_1,\dots,H_n;l,l_1,\dots,l_n](N)\cap$$
$$F^{\sigma\sigma_1\dots\sigma_n}_{\tau'\tau'_1\dots\tau'_n}[H,\Theta;H_1,\dots,H_n;l,l_1,\dots,l_n](N)=\emptyset.$$

(ii) $\exists l_0<\omega$, *such that* $\forall l,l_1,\dots,l_n<\omega$, *if*

$$l>l_0\vee l_1>l_0\vee\dots\vee l_n>l_0$$

then $\forall i<n,\forall\sigma\in(\omega\setminus\{0,1\})^l$,
$\forall\tau\in 2^l,\forall\sigma_i\in(\omega\setminus\{0,1\})^{l_i},\forall\tau_i\in 2^{l_i}$,

$$F^{\sigma\sigma_1\dots\sigma_n}_{\tau\tau_1\dots\tau_n}[H,\Theta;H_1,\dots,H_n;l,l_1,\dots,l_n](N)=\emptyset.$$

Proof of Proposition 3 (i) It follows from definition of $F^{\sigma}_{\tau}[H,\Theta,l](x)$ and Proposition 2(i)

(ii) It follows from the proof of Proposition 2(ii). □

We say that an ordered theory T has the *property of finiteness of discrete chains of convex equivalences* (*FDCCE*) if for any finite $A\subset N$, for every two one-A-formulas $H(x),\Theta(x)$ such that $H(N)\subset\Theta(N)$, every discrete chain of convex $E_{H,\Theta}$-classes is finite.

It follows from the compactness theorem that if an ordered theory T has the *FDCCE* property then for any two formulas $H(x,\bar y)$ and $\Theta(x,\bar z)$, there is a natural number $k<\omega$ such that for any $\bar a,\bar b\in N$, if $H(N,\bar a)\subset\Theta(N,\bar b)$ then every discrete chain of convex $E_{H(x,\bar a),\Theta(x,\bar b)}$-classes has cardinality less or equal to k ([2]).

Recall, that an ordered structure M is *weakly o-minimal* [3] if for any M-definable 1-formula $H(x)$ the structure $\langle\{E_{H,x=x}(M,a)|a\in H(M)\};<^c\rangle$ is finite and M is *o-minimal* [4] if this structure is finite and every such convex $E_{H,x=x}$-class is an interval (opened, closed, semi-closed) or a singleton. An ordered theory T is *weakly o-minimal* [3], if any model of T is weakly o-minimal.

It follows from the definition that any weakly o-minimal theory has the *FDCCE* property.

In [5] the following notion was introduced:

Definition 1. *An A-definable increasing (decreasing) on B 2-formula $\phi(x,y)$ is a quasi-successor on B, if* $\forall\alpha\in B,\exists\beta\in\phi(\alpha,N)\cap B$,

$$\phi(\beta,N)\setminus\phi(\alpha,N)\neq\emptyset.$$

Theorem 1. *[5] Let A be a finite subset of a countable saturated model N of a small ordered theory T, $p \in S_1(A)$, $\phi(x,y)$ be an A-definable quasi-successor on $p(N)$. Then T has 2^ω countable non-isomorphic models.*

Corollary 1. *If T is a complete ordered theory with a few number of countable models then T has the FDCCE property.*

Proof of Corollary 1. Towards a contradiction suppose that there exist two one-formulas $H(x)$ and $\Theta(x)$ $(H(N) \subset \Theta(N))$ with an infinite chain of convex $E_{H,\Theta}$-classes. Then a formula-quasi-successor can be constructed, and by the Theorem 1 T has 2^ω countable models. □

Corollary 2. *If T is an ordered, non weakly o-minimal theory having weakly o-minimal model, then T has continuum countable models.*

Let $\phi_1(x), \phi_2(x)$ be two A-definable one-formulas. Then for every $k(1 < k < \omega)$, for every $i(0 < i < k+1)$ define an A-definable one-formula $\phi_1(x) \triangleleft_k^i \phi_2(x)$ by induction:

$$\phi_1(x) \triangleleft_k^1 \phi_2(x) := D_0^k[\phi_1(x) \wedge \phi_2(x), \phi_2(x)](x);$$

$0 < i < k+1$, $\phi_1(x) \triangleleft_k^i \phi_2(x) := \exists x_1, \ldots, x_k(\wedge_{j<k}((\phi_1 \wedge \phi_2)(x_j) \wedge \neg E_{\phi_1 \wedge \phi_2, \phi_2}(x_j, x_{j+1}) \wedge x_j < x_{j+1}) \wedge D_0^k[\phi_1 \wedge \phi_2, \phi_2](x_1) \wedge D_1^k[\phi_1 \wedge \phi_2, \phi_2](x_k) \wedge E_{\phi_1 \wedge \phi_2, \phi_2}(x, x_i) \wedge \forall y((\phi_1(y) \wedge \phi_2(y)) \to \vee_{0<j<k+1} y = x_j);$

$$\phi_1(x) \triangleleft_k^k \phi_2(x) := D_1^k[\phi_1(x) \wedge \phi_2(x), \phi_2(x)](x).$$

Let $F_1(x), F_2(x')$ be two 1-formulas, $E_1^2(x,y), E_2^2(x',y')$ be two definable relations of equivalence on $F_1(N)$ and $F_2(N)$ respectively. We say that formulas F_1 and F_2 are *mutually attached*, if there exists 2-formula $L^2(x,x')$ such that $\exists x' L(x,x') \equiv F_1(x)$, $\exists x L(x,x') \equiv F_2(x')$ and the following holds:

$$\models \forall x \forall x'[(F_1(x) \wedge F_2(x') \wedge L^2(x,x')) \to \forall y \forall y'((E_1^2(x,y) \wedge E_2^2(x',y')) \to L^2(y,y'))].$$

It follows from definition that for any $\phi_1(x), \phi_2(x)$, for any $k < \omega$, for any $i, j \le k$ two 1-formulas $\phi_1(x) \triangleleft_k^i \phi_2(x)$ and $\phi_1(x) \triangleleft_k^j \phi_2(x)$ are mutually attached.

We say that the formula $H(\bar{x})$ *divides the formula* $\Theta(\bar{x})$ if the following is true:

$$\models \exists \bar{x}(H(\bar{x}) \wedge \Theta(\bar{x})) \wedge \exists \bar{y}(\neg H(\bar{y}) \wedge \theta(\bar{y})).$$

Suppose that the formula $H(x)$ divides the formula $F(x)$ and E^2 is a relation of equivalence on $F(N)$. We say that the formula H *respects* the relation of equivalence E^2, if the following holds:

$$\models \forall x \forall y((F(x) \wedge F(y) \wedge E^2(x,y)) \to (H(x) \iff H(y))).$$

PROPOSITION 4. *Let $F_1(x)$ and $F_2(x')$ be mutually attached one-A-formulas. If there exists one-A-formula $H(x)$ such that $H(x)$ divides $F_1(x)$ and respects the relation of equivalence E_1^2 on $F_1(N)$ then there exists one-A-formula $H'(x')$ such that H' divides F_2 and respects E_2^2. Moreover, if two mutually attached one-formulas H_1 and H_2 divide F_1 and respect E_1^2 then there exist two mutually attached one-formulas H_1' and H_2' that both (H_1', H_2') divide F_2 and respect E_2^2.*

We will denote the formula H' from Proposition 4 by $Att(F_1^1, E_1^2; F_2^1, E_2^2; L^2, H^1) = H'$. So, the logical operation defined on three one-formulas and three two-formulas with above conditions has as a result a one-formula.

Let B be an ordered structure, $C \subset B$ be a convex subset. There is a partition of all convex subsets into 10 next parts, kinds of convexity. For some $a \neq b \in B$ consider:

1. $C = (a,b)$ is ab open interval. We will write $Kind(C) = (oi, oi)$;
2. $C = [a,b]$ is a closed interval. We will write $Kind(C) = (ci, ci)$;
3. $C = [a,b)$ is a semi-closed interval. We will write $Kind(C) = (ci, oi)$;
4. $C = (a,b]$ is a semi-closed interval. We will write $Kind(C) = (oi, ci)$;
5. $C = [a,.)$ is a convex set with infimum and without supremum. We will write $Kind(C) = (ci, c)$;
6. $C = (.,b]$ is a convex set with supremum and without infimum. We will write $Kind(C) = (c, ci)$;
7. $C = (a,.)$ is a convex set with infimum which does not belong to the set and without supremum. We will write $Kind(C) = (oi, c)$;
8. $C = (.,b)$ is a convex set with supremum which does not belong to the set and without infimum. We will write $Kind(C) = (c, oi)$;
9. $C = (.,.)$ is a convex set without supremum and infimum. We will write $Kind(C) = (c, c)$;

10. $C = [a, a]$ is a singleton. We will write $Kind(C) = (ci, ci, =)$.

Denote by KC a set of all kinds of convexity. Notice that $|KC| = 10$ and if C is definable in Language $\{=, <\}$ then any kind of convexity is definable.The last is also true if B is a family of disjoint convex sets and relation of order is $<^c$.

Let A be a finite subset of a saturated model N of a small theory T with an $\emptyset$-definable relation of a linear order, $H(x)$ and $\Theta(x)$ be one-A-formulas such that $H(N) \subset \Theta(N)$. Consider family of $E_{H,\Theta}$-classes of convex equivalence. Since any kind of convex set is definable, denote by $F(x)$ a one-A-formula such that $F(N)$ is definable set of convex $E_{H,\Theta}$ classes having Kind of convexity (ci, ci). Then denote $F_l(x) := F(x) \wedge \neg\exists y(y < x \wedge F(y) \wedge E_{H,\Theta}(x, y))$, $F_r(x) := F(x) \wedge \neg\exists y(x < y \wedge F(y) \wedge E_{H,\Theta}(x, y))$, $F_m(x) := F(x) \wedge \neg F_l(x) \wedge \neg F_r(x)$. The formulas $F_l(x)$, $F_r(x)$ and $F_m(x)$ are mutually attached because for any element a from $F_l(N)$ there exists only one element $b \in F_r(N)$ such that $\models E_{H,\Theta}(a, b)$. Thus Kind of Convexity (ci, ci) gives three mutually attached one-formulas, other cases of Kind of Convexity containing ci give the pairs of mutually attached formulas. More exact, for any one-formula $F(x)$ containing convex E-classes the same Kind of convexity we have $F(x) \equiv (F_l(x) \vee F_m(x) \vee F_r(x))$. So, we have three logical operations for the formula $F(x)$, such that any two results give mutually attached pair of one-formulas. The same consideration give us three logical operations $F_{c,l}, F_{c,m}, F_{c,r}$ on the formulas containing $<^c$-convex sets of convex E-classes.

Let $H =< H_1, H_2, ..., H_k >$ be a finite sequence of convex 1-formulas (into some formula $K(x)$) such that

$$H_1(N) <^c H_2(N) <^c ... <^c H_k(N)$$

and $\mu \in KC^k$ then we say that H *satisfies* μ if for any $i (1 \leq i \leq k)$ we have $\mu(i) = Kind(H_i(N))$ and denote it by $\mu(H_1, H_2, ..., H_k)$. Notice that $\mu(H_1, H_2, ..., H_k)$ is definable.

Let $\phi_1(x), \phi_2(x)$ be two one-A-formulas. Then for every $k (1 < k < \omega)$, for every $i (0 < i \leq k)$, for any $\mu \in KC^k$ define an one-A-formula as $\phi_1(x) \lhd_k^{i,\mu} \phi_2(x) := \phi_1(x) \lhd_k^i \phi_2(x) \wedge \mu(\phi_1(x) \lhd_k^1 \phi_2(x), ..., \phi_1(x) \lhd_k^k \phi_2(x))$.

It follows from definition that $\phi_1(x) \lhd_k^i \phi_2(x) \equiv \vee_\mu \phi_1(x) \lhd_k^{i,\mu} \phi_2(x)$.

Proposition 5. *Let $\phi_1(x), \phi_2(x)$ be two one-A-formulas, $k < \omega$, $i (1 \leq i \leq k)$. Suppose for some $\mu \in KC^k$ one-A-formula $\phi_1(x) \lhd_k^{i,\mu} \phi_2(x)$ divides one-A-formula $\phi_1(x) \lhd_k^i \phi_2(x)$ then the following holds:*

(i) for any $j (1 \leq j \leq k)$, $\phi_1(x) \lhd_k^{j,\mu} \phi_2(x)$ divides $\phi_1(x) \lhd_k^j \phi_2(x)$.

(ii) there exists $\mu' \in KC^k$, $\mu' \neq \mu$ such that $\phi_1(x) \lhd_k^{i,\mu'} \phi_2(x)$ divides one-A-formula $\phi_1(x) \lhd_k^i \phi_2(x)$.

Proof of Proposition 5

(i) Notice that two one-formulas $\phi_1(x) \lhd_k^j \phi_2(x)$ and $\phi_1(x) \lhd_k^i \phi_2(x)$ are mutually attached.

(ii) Since the set of realization of one-formula $\phi_1(x) \lhd_k^i \phi_2(x) \wedge \neg(\phi_1(x) \lhd_k^{i,\mu} \phi_2(x))$ in N is not empty set and because $\phi_1(x) \lhd_k^i \phi_2(x)$ is equivalent to disjunction by all μ, we have the existence of such μ'. □

We say that the set of A-definable one-formulas $\Gamma \subset F_1(A)$ is a *BH-algebra* if it is closed under the following logical operations: $\wedge, \vee, \neg, \lhd_k^i, \lhd_k^{i,\mu}$ $(0 < i < k, 1 < k < \omega, \mu \in KC^k), Att, F_l, F_m, F_r, F_{c,l}, F_{c,m}, F_{c,r}, \phi^c$.

Notice that if we take finite family of formulas with one free variable and close it under the standard logical operations $\wedge, \vee. \rightarrow, \neg$ and the logical operation ϕ^c we obtain finite family of non equivalent one-formulas and in the condition of a small theory the second part of Proposition 4 do not give the possibility to use infinitely many times the logical operations $\lhd_k^i, \lhd_k^{i,\mu}, Att, F_l, F_m, F_r, F_{c,l}, F_{c,m}, F_{c,r}$.

Thus by using the Proposition 2, Proposition 3, Proposition 4 and Proposition 5 we can formulate the following theorem:

Theorem 2. *Let T be a small ordered theory with the FDCCE property, A be a finite subset of a countable saturated model N of the theory T. Then for every finite set of A-definable one-formulas $\{\phi_1(x), \ldots, \phi_n(x)\}, n < \omega$ the BH-algebra generated by this set is finite.*

We say that an ordered theory T is a theory of a *pure order* if it is a theory of the language $L = \{=, <\}$.

Theorem 3. *Let T be a small theory of a pure order. Then T is ω-categorical if and only if T has the FDCCE property.*

Proof of Theorem 3. If the theory T is ω-categorical, then it has a few number of countable models and consequently, T has the *FDCCE* property.

Let the theory T has the *FDCCE* property. Then there exist natural numbers n_1, n_2, ..., n_k $(1 \leq n_1 < n_2 < ... < n_k < \omega)$ such that any discrete chain of elements lies in discrete chain of elements of a length n_i for some $i \leq k$. Denote by $F^j_{n_i}(x)$ $(1 \leq j \leq n_i)$ a formula which distinguishes j-th element in the chains of length n_i.

Also denote

$$F(x) := \forall y(y < x \to \exists z(y < z < x)) \wedge (x < y \to \exists z(x < z < y)).$$

Consider kind of convexity of $E_{F(x),x=x}(x,y)$-classes. There are 10 definable cases. Suppose that all convex classes of $F(x)$ in $x = x$ are the same kind of convexity. Consider the partition $F(x)$ on three parts (left, middle, right) $F(x) \equiv F_l(x) \vee F_m(x) \vee F_r(x)$.

In any case for arbitrary $\alpha \in F(N)$, if $E_{F_l(x),x=x}(N,\alpha) \neq \emptyset$ then $|E_{F_l(x),x=x}(N,\alpha)| = 1$ and if $E_{F_r(x),x=x}(N,\alpha) \neq \emptyset$ then $|E_{F_r(x),x=x}(N,\alpha)| = 1$. If $E_{F_m(x),x=x}(N,\alpha) \neq \emptyset$ then $E_{F_m(x),x=x}(N,\alpha)$ is infinite dense subset of N. In this case, for any $n < \omega$, for any $\alpha_1 < \alpha_2 < ... < \alpha_n$, for any $\beta_1 < \beta_2 < ... < \beta_n$, for any subset $B \subset N$, if $\{\alpha_1, \alpha_n, \beta_1, \beta_n\} \subset E_{F_m(x),x=x}(N,\alpha)$, $B \cap E_{F_m(x),x=x}(N,\alpha) = \emptyset$, then $tp(\alpha_1, \alpha_2, ..., \alpha_n|B) = tp(\beta_1, \beta, ..., \beta_n|B)$. Indeed, since $E_{F_m(x),x=x}(N,\alpha)$ is countable dense convex subset without ends there is automorphism f of N such that for any $1 \leq j \leq n$ $f(\alpha_j) = \beta_j$ and for any $b \in N \setminus E_{F_m(x),x=x}(N,\alpha)$, $f(b) = b$.

Thus without constants from $E_{F_m(x),x=x}(N,\alpha)$ for any one-N-formula $H(x)$, if $H(x)$ divides $F_m(x)$, then $H(x)$ respects the relation of convex equivalence $E_{F_m(x),x=x}(x,y)$.

Applying to the obtained finite set of formulas the following operations: $\wedge, \vee, \neg, \lhd^i_k, \lhd^{i,\mu}_k$ $(0 < i < k, 1 < k < \omega, \mu \in KC^k), Att, F_l, F_m, F_r, F_{c,l}, F_{c,m}$, $F_{c,r}, \phi^c$; we obtain a BH-algebra which is finite by the Theorem 2. Thus there is a finite set $\{A_1(x)$, ..., $A_n(x)\}$ of atomic formulas of BH-algebra considered as Boolean algebra. For any atom of BH-algebra $A_i(x)$ there are four possibities:

(i) $A_i(N)$ is convex and $|A_i(N)| = 1$.

(ii) $A_i(N)$ is convex and $\langle A_i(N); <\rangle$ is a countable dense order without endpoints.

(iii) There is convex set $\Theta(x)$ such that $A_i(x)$ is a dense subformula of $\Theta(x)$ such that any convex $E_{A_i(x),\Theta(x)}(x,y)$-class is singlton.

(iv) There is convex set $\Theta(x)$ such that $A_i(x)$ is a dense subformula of $\Theta(x)$

such that any convex $E_{A_i(x),\Theta(x)}(x,y)$-class is countable dense order without endpoints.

Notice that in constructed finite BH-algebra there is a partition of N on finite number of definable convex subset. For any definable minimal convex subset there exists partition of this convex set on finite number of atomic formulas $A_i(x)$ mutually attached and mutually dense.

Now take two arbitrary countable models M_1 and M_2 of the theory T. These two models have the same sets $\{A_i\}$ of atomic formulas. Since any two dense linear orders without endpoints are isomorphic we can construct step by step an order-preserving bijection between M_1 and M_2. □

COROLLARY 3. *Let T be an ω-categorical theory of a pure order. Then T is finitely axiomatizable.*

PROOF OF COROLLARY 3 Description of a BH-algebra constructed in the proof of the Theorem 3 constitute axioms of the ω-categorical theory. □

COROLLARY 4. *Let T be a non-ω-categorical small theory of a pure order. Then there is $\emptyset$-definable 1-formula $\phi(x)$ such that for some two elements $\alpha, \beta \in \phi(N)$ ($\alpha < \beta$), $(\alpha, \beta) \cap \phi(N)$ is an infinite discrete chain.*

In the articles [6] and [7] the following was discussed:

COROLLARY 5. *Let T be a small ordered theory with a few number of countable models. Then the restriction $T' := T \upharpoonright \{=, <\}$ of T to a pure linear order is ω-categorical.*

PROOF OF COROLLARY 5 Since T has a few number of countable models, it follows from the Corollary 1 that T has the *FDCCE* property and so do T'. Hence, by the Theorem 3 T' is ω-categorical. □

3. One-Types in Ordered Theories

Let A be a subset of a saturated model N of T. The analysis of one-types is based on the using the properties of A-definable family of 1-formulas (Points A, B), of 2-formulas (Point C) and properties of interactions of one-types (Point D).

Point A Let $\phi(x), \Theta(x)$ be the A-definable formulas. We say that $\phi(x)$ is *convex* in $\Theta(x)$ if $\phi(x)$ is subformula of $\Theta(x)(\phi(N) \subseteq \Theta(N))$ and for every $\alpha, \beta \in \phi(N)$

$$(\alpha < \beta \to \forall \gamma \in \Theta(N)(\alpha < \gamma < \beta \to \gamma \in \phi(N))).$$

We say that $\phi(x)$ is *convex* if $\phi(x)$ is convex in $x = x$.

Let $p \in S_1(A), \Theta(x) \in p$. Then denote

$$\Gamma_{p,\Theta}(x) := \{\phi(x) \in p | \phi(x) \text{ is convex in } \Theta(x)\}.$$

We say that p is *quasirational to the right* if $\Gamma_{p.\Theta}(N)^+ = \phi(N)^+$ for some $\phi(x) \in \Gamma_{p,\Theta}(x)$.

We say that p is *quasirational to the left* if $\Gamma_{p,\Theta}(N)^- = \phi(N)^-$ for some $\phi(x) \in \Gamma_{p,\Theta}(x)$.

Notice, that $\Gamma p, \Theta(N)$ is convex subset in $\Theta(N)$.

Let $p := tp(\gamma|A), \Theta(x) \in p$. Then denote

$$tp^c(\gamma|A, \Theta)(x) := \Gamma_{p,\Theta}(x).$$

Claim 1. *If for some $\Theta(x) \in p, \phi(x) \in \Gamma_{p,\Theta}(x), \Gamma_{p,\Theta}(N)^+ = \phi(N)^+$ then for any $\Theta'(x) \in p$ there exists $\phi'(x) \in \Gamma_{p,\Theta'}$ such that $\Gamma_{p,\Theta'}(N)^+ = \phi'(N)^+ = \phi(N)^+ = p(N)^+$. The same is true for $\Gamma_{p,\Theta}(N)^-$.*

A1. We say that p is *quasirational* if either p is quasirational to the left and non-quasirational to the right or p is quasirational to the right and non-quasirational to the left.

A2. We say that p is *bi-quasirational* if p is quasirational to the right and quasirational to the left.

A3. We say that p is *irrational* if p is non-quasirational and p is non-bi-quasirational.

Notice that in weakly o-minimal theory $\Gamma_{p,x=x}(N) = p(N)$ for every one-type and every bi-quasirational one–type is isolated.

Point B. Let $\Theta(x)$ be an A-definable 1-formula from one-type $p \in S_1(A)$, A be a finite subset of N.

Consider $K_{p,\Theta} := \{\neg\phi(x) | (\Theta(x) \wedge \phi(x)) \in p$ and for $E_{\phi\wedge\Theta,\Theta}(x, y)$ there exists infinite set $\{a_i | a_i \in \Gamma_{p,\Theta}(N), i < \omega\}$ such that $E_{\phi\wedge\Theta,\Theta}(N, a_i) \cap E_{\phi\wedge\Theta}(N, a_j) = \emptyset$ for $i \neq j < \omega\}$,

$K^d_{p,\Theta} := \{\neg\phi(x) \in K_{p,\Theta} | \ \langle\{E_{\phi\wedge\Theta,\Theta}(N, a_i)|i < \omega, a_i$ are from definition of $K_{p,\Theta}\}; <^c\rangle$ is dense order$\}$.

Notice, that for every $\Theta(x), \phi(x)$, $\langle\{E_{\phi\wedge\Theta,\Theta}(N, a)|a \in \Theta(N)\}; <^c\rangle$ is densely ordered structure iff $\langle\{E_{\neg\phi\wedge\Theta,\Theta}(N, a)|a \in \Theta(N)\}; <^c\rangle$ is densely ordered structure.

B1. We say that p is *convex in some formula* $\Theta(x) \in p$ if $K_{p,\Theta} = \emptyset$. In this case $\Gamma_{p,\Theta}(N) = p(N)$ and $\Gamma p, \Theta(N)$ is convex in $\Theta(N)$.

If $p \in S_1(A)$ is convex in some formula, we say that p is *convex*.

B2. We say that p is *non-convex*, if for every $\Theta(x) \in p$ $K_{p,\Theta} \neq \emptyset$.

Consider an arbitrary $\neg\phi(x) \in K_{p,\Theta}$ for non-convex one-type $p \in S_1(A)$ and an arbitrary $\Theta(x) \in p$. Let $\alpha \in (\neg\phi(N) \cap \Gamma_{p,\Theta}(N))$.

Denote $F^r_{\neg\phi\wedge\Theta}(x, \alpha) := E_{\neg\phi\wedge\Theta,\Theta}(N, \alpha) < x \wedge \forall y((E_{\neg\phi\wedge\Theta,\Theta}(N, \alpha) < y < x \wedge \Theta(y)) \to \phi(x))$ and

$F^l_{\neg\phi\wedge\Theta}(x, \alpha) := E_{\neg\phi\wedge\Theta,\Theta}(N, \alpha) > x \wedge \forall y((E_{\neg\phi\wedge\Theta,\Theta}(N, \alpha) > y > x \wedge \Theta(y)) \to \phi(x))$.

B2.1. We say that a non-convex $p \in S_1(A)$ $(A \subset_{finite} N)$ is *strictly non-convex* if there exists $\Theta(x) \in p$ such that for every A-definable $\phi(x)$ such that $\Theta(x) \wedge \phi(x) \in p$, for (some) every $\alpha \in \phi(N) \cap \Gamma_{p,\Theta}(N)$ we have $(*)$ $F^r_{\neg\phi\wedge\Theta}(N, \alpha) = \emptyset \vee F^r_{\neg\phi\wedge\Theta}(N, \alpha) \cap p(N) = \emptyset$ and

$F^l_{\neg\phi\wedge\Theta}(N, \alpha) = \emptyset \vee F^l_{\neg\phi\wedge\Theta}(N, \alpha) \cap p(N) = \emptyset$.

B2.2. We say that non-convex p is *limite of infinite family of mutually dense A-definable 1-formulas* if it is non-strictly non-convex i.e.

for very $\Theta(x) \in p$ there is A-definable $\neg\phi(x) \in K_{p,\Theta}$ such that $\forall\alpha \in \phi(N) \cap \Gamma_{p,\Theta}(N)$, $\neg(*)$.

Claim 2. *Suppose* $\neg(*)$ *then if* $F^r_{\neg\phi\wedge\Theta}(N, \alpha) \cap p(N) \neq \emptyset$,

$F^l_{\neg\phi\wedge\Theta}(N, \alpha) \cap p(N) = \emptyset$

Proof of Claim 2. Consider A-definable 1-formulas $S^r_{\neg\phi\wedge\Theta}(x) := \exists y$ $(F^r_{\neg\phi\wedge\Theta}(x, y) \wedge \Theta_0(y))$ and $S^l_{\neg\phi\wedge\Theta}(x) := \exists y(F^l_{\neg\phi\wedge\Theta}(x, y) \wedge \Theta_0(y))$ with $\Theta_0(y) \in p(y)$. We can find Θ_0 such that $S^r_{\neg\phi\wedge\Theta}(N) \cap S^l_{\neg\phi\wedge\Theta}(N) = \emptyset$. Then we have $S^r_{\neg\phi\wedge\Theta}(x) \in p$ and $S^l_{\neg\phi\wedge\Theta}(x) \notin p$. □

Point C. Notice that $\Gamma_{p,\Theta}(N)(N) \cap K_{p,\Theta}(N) = p(N)$.

We consider the ordered structure $\langle p(N); <\rangle$.

We say that an A-definable 2-formula $\phi(x, y)$ is *convex to the right in* $p(N)$ if $\forall\alpha \in p(N)$, $\forall\beta \in \phi(N, \alpha) \cap p(N)$ we have

$$(\alpha \le \beta \text{ and } \forall \gamma \in p(N)(\alpha \le \gamma < \beta \to \gamma \in \phi(N, \alpha))).$$

It is clear that for every A-definable 2-formula $\phi(x, y)$, if $\phi(x, y)$ is convex to the right in $p(N)$ then $\phi(x, y)$ is convex to the right in some $\Theta_\phi \in p$.

The definition of convex to the left in $p(N)$ A-definable 2-formula is analogous.

Denote $V_p(\alpha) := \{\gamma \in p(N)|$ There exists a convex to the right in $p(N)$, A-definable 2-formula $\phi(x, y)$ such that $\gamma \in \phi(N, \alpha)$ or there exists a convex to the left in $p(N)$, A-definable 2-formula $\phi(x, y)$ such that $\gamma \in \phi(N, \alpha)\}$, $V_p(\alpha)$ is said to be *neighbourhood of α in p*.

C1. We say that $\alpha \in N \setminus A$ is *solitary over A* if for $p := tp(\alpha|A)$ we have $|V_p(\alpha)| = 1$ and in this case we say that $p \in S_1(A)$ is solitary because it is true for any element $\beta \in p(N)$.

We say that $p \in S_1(A)$ is *quasisolitary* if $V_p(\alpha)$ is $(A \cup \{\alpha\})$-definable.

C2. We say that $p \in S_1(A)$ is *semi-quasisolitary* if either $V_p(\alpha)^+$ is $(A \cup \{\alpha\})$-definable and $V_p(\alpha)^-$ is non-definable or $V_p(\alpha)^-$ is $(A \cup \{\alpha\})$-definable and $V_p(\alpha)^+$ is non-definable.

C3. We say that $p \in S_1(A)$ is *social* if $\forall \alpha \in p(N)$, $V_p(\alpha)^+$ and $V_p(\alpha)^-$ are both non-definable.

Thus, taking into consideration the points A, B, C it is possible to show the existence of 3*3*2=18 kinds of one-types in small ordered theories (in further we show non-existence C2 for theories such that any one-type over finite set is convex i.e. of kinds bf B1).

Point D. Let $p, q \in S_1(A)$. The type p is said to be *weakly orthogonal to q* ($p \perp^w q$) if if $p(x) \cup q(y)$ is complete two-type over A [8].

We say that p is *almost orthogonal to q* ($p \perp^a q$) if there is no any A-definable formula $H(x, y)$ such that

$$\forall \alpha \in p(N), \emptyset \ne H(N, \alpha) \subset q(N).$$

We say that p is *densely orthogonal to q* ($p \perp^d q$) if p and q are non mutually dense in some $\Theta(x)$ i.e. there do not exist $\phi_1(x), \phi_2(x)$ such that $\neg\phi_1(x) \in K_{p,\Theta}$, $\neg\phi_2(y) \in K_{q,\Theta}$ and ϕ_1 and ϕ_2 are mutually dense in $\Theta(N)$.

We say that p is *splitely orthogonal to q* ($p \perp^s q$) if there is no A-definable 2-formula $H(x, y)$ such that $\forall \alpha \in p(N)$, $H(N, \alpha) \cap q(N) \ne \emptyset$, $H(N, \alpha)^+ \cap q(N) \ne \emptyset$.

Let $p, q \in S_1(A), p \not\perp^s q$ by A-definable 2-formula $H(x,y)$. Then for some $\alpha \in p(N)$ let $H_0(x,\alpha)$ be an $(A \cup \{\alpha\})$-definable formula such that

$$H_0(N,\alpha) = (H(N,\alpha)^+)^-.$$

Then the following holds: $H(N_0,\alpha) \cap q(N) \neq \emptyset,\ H_0(N,\alpha)^- \cap q(N) \neq \emptyset$.

We say that $\mathrm{H}_0(x)$ is *left frontier of* H. Thus also we can define split e ly orthogonality by left frontier of formula H.

Notice that

1) any one of last three kinds of non-orthogonality of two one-types implies non-weakly orthogonality of these two types;

2) in weakly o-minimal theory the notion of weakly orthogonality of two one-types coincides with notion of split ely orthogonality, because the notion of densely orthogonality without sense;

3) in o-minimal theory the notion of weakly orthogonality coincides with notion of almost orthogonality [4], [9], [10], [11].

Theorem 4. *Let $p, q \in S_1(A), p \not\perp^s q$. Then*
p is bi-quasirational iff q is bi-quasirational.

From the definition of almost orthogonality of one-types it follows

Fact 2. *Let A be a subset of a $|A|^+$-saturated model N of a complete theory T, $p, q \in S_1(A)$ be the one-type over A such that p is isolated and q is non-isolated. Then $p \perp^a q$.*

So, we will use these three kinds of orthogonality to describe the interactions between one-types and during in constructions of models.

Definition 2. *Type p is definable if and only if $\forall\varphi(x,\bar{y})\ \exists\psi_\varphi(\bar{y},\bar{a})$, such that*

$$A \models \psi_\varphi(\bar{b},\bar{a}) \Leftrightarrow \varphi(x,\bar{b}) \in p$$

Fact 3. *Let $p, q \in S_1(A)$. p is weakly orthogonal to q ($p \perp^w q$) if for any $H(x,y,\bar{a}), \bar{a} \in A$, for any $\alpha \in p(M^{'})$ the following holds:*

$$[H(M^{'},\alpha,\bar{a}) \cap q(M^{'}) \neq \emptyset \Rightarrow q(M^{'}) \subseteq H(M^{'},\alpha,\bar{a})].$$

Let T be a weakly o-minimal theory, $p \in S_1(A), A \subset N$, N be $|A|^+$-saturated model T. Any $\bar{a}$-definable set $\phi(N, \bar{a}) \subset N$ is union finite number $\bar{a}$-definable convex sets $\phi(N, \bar{a}) = \phi_1(N, \bar{a}) \cup ... \cup \phi_n(N, \bar{a})$, consequently for any $\phi(x, \bar{y})$, for any $\bar{a} \in A$ the following: $\phi(x, \bar{a}) \in p \iff \exists i(1 \leq i \leq n)\phi_i(x, \bar{a}) \in p$. Thus $p(N) = \cap_{\phi(x,\bar{a}\in p)}\phi(N, \bar{a}) = \cap_{\phi(x,\bar{a}\in p)}\phi^c(N, \bar{a})$ and the set of realizations of p in any model of T is convex set [3], [12]. If two 1-types $q, p \in S_1(A)$ are non-weakly orthogonal there is by Fact 3 2-A-formula $H(x, y)$ such that for any $\alpha \in q(N)$, $H(N, \alpha) \cap p(N) \neq \emptyset$ and $\neg H(N, \alpha) \cap p(N) \neq \emptyset$. We can suppose that for any α, $H(x, \alpha) \in R(A\alpha)$ and $H(x, \alpha) \notin L(A\alpha))$. We say, that $H(x, y)$ is monotone increasing (decreasing) 2-formula on $\Theta(N)$, if $b, c \in \exists x H(x, N)$ and $b < c$ then $H(N, c)^+ \subseteq H(N, b)^+$. By [13], [14], [15], [16], [17] it follows that $H(x, y)$ is monotone 2-formulas on some convex 1-A-formula $\Theta(y) \in q$.

Suppose $q \not\perp^w p$, $H(x, y, \bar{a})$ is monotone increasing on convex $\Theta(y, \bar{a}) \in q$ and q is definable. Consider arbitrary formula $\phi(x, \bar{z})$ and $\bar{b} \in A$. Then $\phi(x, \bar{b}) \in p \iff \exists y_1 \exists y_2 (y_1 < y < y_2 \wedge \Theta(y_) \wedge \Theta(y_2) \wedge (\emptyset \neq (H(N, y_2) \setminus H(N, y_1) \subset \phi(N, \bar{b})) \in q)$. Since q is definable the last is equivalent to $\models \psi(\bar{b}, \bar{a}, \bar{c})$, here $\bar{c} \in A$. Thus if q is definable then p is definable and because the relation $\not\perp^w$ is symmetric we have

Theorem 5. *[14] Let T be an weakly o-minimal theory. For any model M of theory T, $A \subseteq M$, $p, q \in S_1(A)$, if $p \not\perp^w q$ then p is definable if and only if q is definable*

In contrast with weakly o-minimal theory we have

Theorem 6. *There exist a linearly ordered structurel $M = \langle M; =, <, E^2, S^2 \rangle$, infinite set $A \subset N$, two one-types over A, $p, q \in S_1(A)$ such that $p \not\perp^w q$, p is definable and q is non-definable.*

Proof of Theorem 6. Let R be a set of all real numbers, Q be a set of all rational numbers, $B = \{\beta_\alpha \in R \setminus Q | \alpha \in Q\}$ be algebraic (linear) independent set of irrational numbers indexed by all rational numbers. Pose

$$C_\alpha = \{\frac{n}{m} + \beta_\alpha | \frac{n}{m} \in Q\} = Q + \beta_\alpha.$$

Let $M = \cup_{\alpha \in Q} C_\alpha$. Then the following hold:
(i) For any $\alpha \neq \gamma \in Q$, $C_\alpha \cap C_\gamma = \emptyset$.

(ii) For any $a, b \in M$, for any $\alpha \in Q$ if $\langle R; =, < \rangle \models a < b$ then there is $c \in C_\alpha$ such that $\langle R; =, < \rangle \models a < c < b$.

Define structure $< M; =, <, E^2, S^2 >$. For any $a, b \in M$, $M \models a < b \iff R \models a < b$.

$$M \models E^2(a, b) \iff \exists \alpha \in Q, a \in C_\alpha, b \in C_\alpha.$$

$$M \models S^2(a, b) \iff \exists \alpha \in Q, a \in C_\alpha, \exists \gamma \in Q, b \in C_\gamma, Q \models \alpha < \gamma.$$

From definition of the structure $\langle M; =, <, E^2, S^2 > \rangle$ it follows that the binary relation E^2 is a relation of equivalence on M, binary relation S^2 is a relation of linear order on the set of E^2-classes and this order is dense order without ends.

Let A be an infinite subset of M such that A is subset of open interval $(1, 0)$ if it consider as subset of set all real numbers R with next condition: for any $\alpha \in Q$, $|C_\alpha \cap A| = 1$. Denote by a_α the element from this intersection. Consider locally consistent set of A-1-formulas $r = \{a < x | a \in A\}$. Consider for arbitrary $\alpha \in Q$ its extension $p_\alpha = r \cup \{E^2(x, a_\alpha)\}$. For any $\alpha \in Q$, 1-type p_α is definable and if A has maximal element then r and p_α are isolated. If A has not maximal element then p_α is non-isolated.

Let $\delta \in R \setminus Q$. Define 1-type over A, $q_\delta = r \cup \{S^2(a_\alpha, x) \wedge S^2(x, a_\gamma) | R \models \alpha < \delta < \gamma\}$. For any $\delta \in R \setminus Q$ 1-type q_δ is non-definable.

Fix $\delta(0) \in R \setminus Q$. Consider arbitrary realization of $p_{\delta(0)} \models b_{\delta(0)}$. Any p_α has two extensions over $A \cup \{b_{\delta(0)}\}$: $p_\alpha \cup \{x < b_{\delta(0)}\}$ and $p_\alpha \cup \{x > b_{\delta(0)}\}$. This means $q_{\delta(0)} \not\perp^w p_\alpha$. □

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Байжанов Б.С., Байжанов С.С., Саулебаева Т., Замбарная Т.С. РЕТТЕЛГЕН ТЕОРИЯЛАРДАҒЫ БІР-ФОРМУЛАЛАР МЕН БІР-ТИПТЕР

Мақалада желілік реттің $\emptyset$-анықталған қатынасы бар теориялар қаралған. Дөңес эквиваленттіліктегі кез келген дискретті рет саны ақырғы кіші теориялары зерттелген. 1-типтердің берілген жіктеуі 2-формулалардың қасиеттерінде, 1-формулардың анықталған тұқымдастарында және 1-типтердін рекеттесу қасиеттерінде негізделген.

Байжанов Б.С., Байжанов С.С., Саулебаева Т., Замбарная Т.С. ОДИН-ФОРМУЛЫ И ОДИН-ТИПЫ В УПОРЯДОЧЕННЫХ ТЕОРИЯХ

В статье рассмотрены теории с $\emptyset$-определимым отношением линейного порядка. Изучены малые теории, для которых любой дискретный порядок на выпуклых эквивалентностях конечен. Дана классификация 1-типов, основанная на свойствах 2-формул, определимых семейств 1-формул и свойствах взаимодействия 1-типов.

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INVARIANT THEORY OF RELATIVELY FREE RIGHT-SYMMETRIC AND NOVIKOV ALGEBRAS

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Dedicated to Askar Dzhumadil'daev on the occasion of his 60th birthday

Annotation: Algebras with the polynomial identity $(x_1, x_2, x_3) = (x_1, x_3, x_2)$, where $(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3$ is the associator, are called right-symmetric. Novikov algebras are right-symmetric algebras satisfying additionally the polynomial identity $x_1(x_2x_3) = x_2(x_1x_3)$. We consider the free right-symmetric algebra $F_d(\mathfrak{R})$ and the free Novikov algebra $F_d(\mathfrak{N})$ freely generated by $X_d = \{x_1, \ldots, x_d\}$ over a filed K of characteristic 0. The general linear group $GL_d(K)$ with its canonical action on the d-dimensional vector space KX_d acts on $F_d(\mathfrak{R})$ and $F_d(\mathfrak{N})$ as a group of linear automorphisms. For a subgroup G of $GL_d(K)$ we study the algebras of G-invariants $F_d(\mathfrak{R})^G$ and $F_d(\mathfrak{N})^G$. For a large class of groups G we show that the algebras $F_d(\mathfrak{R})^G$ and $F_d(\mathfrak{N})^G$ are never finitely generated. The same result holds for any subvariety of the variety $\mathfrak{R}$ of right-symmetric algebras which contains the subvariety $\mathfrak{L}$ of left-nilpotent of class 3 algebras in $\mathfrak{R}$.

Keywords: Free right-symmetric algebras, free Novikov algebras, noncommutative invariant theory.

1. Introduction

In this paper we fix a field K of characteristic 0 and consider nonassociative K-algebras. An algebra A is called *right-symmetric* if it satisfies the polynomial identity

$$(x_1, x_2, x_3) = (x_1, x_3, x_2), \tag{1}$$

Ключевые слова: *Свободные право-симметрические алгебры, свободные алгебры Новикова, некоммутативная теория инвариантов.*

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where $(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3$ is the associator, i.e.,

$$(a_1, a_2, a_3) = (a_1, a_3, a_2) \text{ for all } a_1, a_2, a_3 \in A.$$

A right-symmetric algebra is *Novikov* if it satisfies additionally the polynomial identity of left-commutativity

$$x_1(x_2x_3) = x_2(x_1x_3). \tag{2}$$

We denote by $\mathfrak{R}$ and $\mathfrak{N}$ the varieties of all right-symmetric algebras and all Novikov algebras, respectively. For details on the history of right-symmetric and Novikov algebras we refer to the introductions of the paper by Dzhumadil'daev and Löfwall [1] and the recent preprint by Bokut, Chen, and Zhang [2]. The origins of the right-symmetric algebras can be traced back till the paper by Cayley [3] in 1857. Translated in modern language, Cayley mentioned an identity which implies the right-symmetric identity for the associators and holds for the right-symmetric Witt algebra in d variables

$$W_d^{\text{rsym}} = \left\{ \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} \mid f_i \in K[X_d] \right\}$$

equipped with the multiplication

$$\left(f_i \frac{\partial}{\partial x_i} \right) * \left(f_j \frac{\partial}{\partial x_j} \right) = \left(f_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Cayley also considered the realization of W_d^{rsym} in terms of rooted trees. Later right-symmetric algebras were studied under different names: Vinberg, Koszul, Gerstenhaber, and pre-Lie algebras, see the references in [1]. The opposite algebras of Novikov algebras (satisfying the left-symmetric identity for the associators and right commutativity) appeared in the paper by Gel'fand and Dorfman [4]. There the authors gave an algebraic approach to the notion of Hamiltonian operator in finite-dimensional mechanics and the formal calculus of variations. Independently, later Novikov algebras were rediscovered by Balinskii and Novikov in the study of equations of hydrodynamics [5], see also the survey article by Novikov [6]. (Due to the contributions in [4, 5] and [6] Bokut, Chen, and Zhang [2] suggest to call these algebras Gel'fand-Dorfman-Novikov algebras. We shall continue to keep the name Novikov algebras.) An example of a Novikov algebra is the right-symmetric Witt algebra W_1^{rsym} in one variable. In a series of papers, see, e.g., [1, 7, 8] Dzhumadil'daev, with coauthors or alone, has studied free right-symmetric and free Novikov algebras, with applications to nonassociative algebras with polynomial identities.

In commutative invariant theory one usually considers the general linear group $GL_d(K)$ with its canonical action on the d-dimensional vector space V_d with basis $\{e_1, \ldots, e_d\}$. This induces an action on the polynomial algebra $K[X_d] = K[x_1, \ldots, x_d]$ in d variables

$$g(f(v)) = f(g^{-1}(v)), \quad g \in GL_d(K), v \in V_d,$$

where the linear functions $x_i : V_d \to K$ are defined by

$$x_i(e_j) = \delta_{ij}, \quad i, j = 1, \ldots, d,$$

and δ_{ij} is the Kronecker symbol. For our noncommutative considerations it is more convenient to suppress one step and, replacing V with its dual space V^*, to assume that $GL_d(K)$ acts canonically on the vector space KX_d with basis $X_d = \{x_1, \ldots, x_d\}$. Then, identifying the polynomial algebra $K[X_d]$ with the symmetric algebra of KX_d, we extend diagonally this action of $GL_d(K)$ on $K[X_d]$:

$$g(f(X_d)) = g(f(x_1, \ldots, x_d)) = f(g(x_1), \ldots, g(x_d)), \tag{3}$$

$g \in GL_d(K)$, $f(X_d) \in K[X_d]$. In this way $GL_d(K)$ acts as the group of linear automorphisms of $K[X_d]$. For a subgroup G of $GL_d(K)$ *the algebra of G-invariants* is

$$K[X_d]^G = \{f \in K[X_d] \mid g(f) = f \text{ for all } g \in G\}.$$

This is a $\mathbb{Z}$-graded vector space and its *Hilbert* (or *Poincaré*) *series* is the formal power series

$$H(K[X_d]^G, z) = \sum_{n \geq 0} \dim(K[X_d]^G)_n z^n,$$

where $(K[X_d]^G)_n$ is the homogeneous component of degree n in $K[X_d]^G$. The following are among the main problems related with the description of the algebra $K[X_d]^G$ for different groups or classes of groups G. For details concerning also computational and algorithmic problems see [9] or [10].

- *Is the algebra $K[X_d]^G$ finitely generated?* This problem was the main motivation for the Hilbert 14th problem in his famous lecture *"Mathematische Probleme"* given at the International Congress of Mathematicians held in 1900 in Paris [11]. It is known that $K[X_d]^G$ is finitely generated for finite groups (the theorem of Emmy Noether [12]), for reductive groups (the Hilbert-Nagata theorem, see e.g., [13]), and for groups close to reductive (see e.g., Grosshans [14] and Hadžiev [15]). The first example of an algebra of invariants $K[X_d]^G$ which is not finitely generated is due to Nagata [16].

- *If $K[X_d]^G$ is finitely generated, describe it in terms of generators and defining relations.* In different degree of generality this problem is solved for classes of groups. For example, the theorem of Emmy Noether [12] gives that for finite groups the algebra $K[X_d]^G$ is generated by invariants of degree $\leq |G|$. Also for finite groups, the Chevalley-Shephard-Todd theorem [17, 18] states that the algebra $K[X_d]^G$ is isomorphic to the polynomial algebra in d variables (i.e., it is generated by a set of d algebraically independent invariants) if and only if G is generated by pseudo-reflections.

- *Calculate the Hilbert series $H(K[X_d]^G, z)$.* For finite groups the answer is given by the Molien formula [19]

$$H(K[X_d]^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gz)}.$$

  The analogue for reductive and close to them groups is the Molien-Weyl integral formula [20], see also [21].

In noncommutative invariant theory one replaces the polynomial algebras $K[X_d]$ with other noncommutative or nonassociative algebras still keeping some of the typical features of polynomial algebras. One such feature is the universal property that for an arbitrary commutative algebra A every mapping $X_d \to A$ is extended to a homomorphism $K[X_d] \to A$. In the noncommutative set-up the class of commutative algebras is replaced by an arbitrary variety of algebras $\mathfrak{V}$ and instead on $K[X_d]^G$ one studies the algebra of G-invariants $F_d(\mathfrak{V})^G$ of the d-generated relatively free algebra $F_d(\mathfrak{V})$ in $\mathfrak{V}$, $d \geq 2$. For a background see the surveys [22, 23]. Comparing with commutative invariant theory, when $K[X_d]^G$ is finitely generated for all "nice" groups, the main difference in the noncommutative case is that $F_d(\mathfrak{V})^G$ is finitely generated quite rarely. For a survey on invariants of finite groups G acting on relatively free associative algebras see [22, 23] and [24]. For finite groups $G \neq \langle 1 \rangle$ and varieties of Lie algebras $F_d(\mathfrak{V})^G$ is finitely generated if and only if $\mathfrak{V}$ is nilpotent, see [25, 26].

Concerning the Hilbert series of $F_d(\mathfrak{V})^G$, for G finite there is an analogue of the Molien formula, see Formanek [22]. Let

$$H(F_d(\mathfrak{V}), z_1, \ldots, z_d) = \sum_{n_i \geq 0} \dim F_d(\mathfrak{V})_{(n_1,\ldots,n_d)} z_1^{n_1} \cdots z_d^{n_d}$$

be the Hilbert series of $F_d(\mathfrak{V})$ as a multigraded vector space. It is equal to the generating function of the dimensions of the vector spaces $F_d(\mathfrak{V})_{(n_1,\ldots,n_d)}$ of the elements in $F_d(\mathfrak{V})$ which are homogeneous of degree n_i in x_i. If $\xi_1(g), \ldots, \xi_d(g)$ are the eigenvalues of $g \in G$, then the Hilbert series of the algebra of invariants $F_d(\mathfrak{V})^G$ is

$$H(F_d(\mathfrak{V})^G; z) = \frac{1}{|G|} \sum_{g \in G} H(F_d(\mathfrak{V}); \xi_1(g)z, \ldots, \xi_d(g)z).$$

There is also an analogue of the Molien-Weyl formula for the Hilbert series of $F_d(\mathfrak{V})^G$ which combines ideas of De Concini, Eisenbud, and Procesi [27] and Almkvist, Dicks, and Formanek [28]. Evaluating the corresponding multiple integral one uses the Hilbert series of $F_d(\mathfrak{V})$ instead of the Hilbert series of $K[X_d]$

$$H(K[X_d], z_1, \ldots, z_d) = \prod_{i=1}^{d} \frac{1}{1 - z_i}.$$

We refer to [29] for other methods for computing the Hilbert series of $F_d(\mathfrak{V})^G$ when G is isomorphic to the special linear group $SL_m(K)$ or to the group $UT_m(K)$ of the $m \times m$ unitriangular matrices.

In this paper we study invariant theory of relatively free right-symmetric and Novikov algebras. Let $\mathfrak{L}$ be the variety of right-symmetric algebras which are left-nilpotent of class 3, i.e., $\mathfrak{L}$ is the subvariety of $\mathfrak{R}$ satisfying the polynomial identity

$$x_1(x_2x_3) = 0. \tag{4}$$

For a large class of subgroups G of $GL_d(K)$, $G \neq \langle 1\rangle$, $d > 1$, we show that $F_d(\mathfrak{V})^G$ is not finitely generated for any variety $\mathfrak{V}$ containing $\mathfrak{L}$. More precisely, let $A_d = K[X_d]_+$ be the algebra of polynomials without constant term and let $(A_d)_1^G = (KX_d)^G$ be the vector space of linear polynomials fixed by G. Clearly, $(A_d^2)^G$ is a $K[(A_d)_1^G]$-module. If $(A_d^2)^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, then $F_d(\mathfrak{V})^G$ is not finitely generated for any $\mathfrak{V}$ containing $\mathfrak{L}$. The class of such groups G contains all finite groups. It contains also the classical and close to them groups under some natural restrictions on the embedding into $GL_d(K)$. In particular, if $(A_d)_1^G = 0$ and $(A_d^2)^G \neq 0$, then $F_d(\mathfrak{V})^G$ is not finitely generated. Results in the same spirit hold if we replace the polynomial algebra $K[X_d]$ with the free metabelian Lie algebra $F_d(\mathfrak{A}^2) = L_d/L_d''$, where L_d is the free Lie algebra freely generated by X_d and $\mathfrak{A}^2$ is the variety of all metabelian (solvable of class 2) Lie algebras. If $(KX_d)^G = (A_d)_1^G = 0$ and $\dim F_d(\mathfrak{A}^2)^G = \infty$, then again $F_d(\mathfrak{V})^G$ is not finitely generated.

2. Preliminaries

We fix a field K of characteristic 0. All vector spaces and algebras will be over K. Let

$$F(X) = K\{X\} = K\{x_1, x_2, \ldots\}$$

be the (absolutely) free nonassociative algebra freely generated by the countable set $X = \{x_1, x_2, \ldots\}$. Recall that the polynomial $f(x_1, \ldots, x_m) \in K\{X\}$ is a polynomial identity for the algebra A if $f(a_1, \ldots, a_m) = 0$ for all $a_1, \ldots, a_m \in A$. The class of all algebras satisfying a given set $U \subset K\{X\}$ of polynomial identities is called the variety of associative algebras defined by the system U. If $\mathfrak{V}$ is a variety, then $T(\mathfrak{V})$ is the ideal of $K\{X\}$ consisting of all polynomial identities of $\mathfrak{V}$. Let $X_d = \{x_1, \ldots, x_d\} \subset X$. Then the algebra

$$F_d(\mathfrak{V}) = K\{x_1, \ldots, x_d\}/(K\{x_1, \ldots, x_d\} \cap T(\mathfrak{V})) = K\{X_d\}/(K\{X_d\} \cap T(\mathfrak{V}))$$

is the relatively free algebra of rank d in $\mathfrak{V}$. We shall denote the generators of $F_d(\mathfrak{V})$ with the same symbols X_d. The ideals $K\{X_d\} \cap T(\mathfrak{V})$ of $K\{X_d\}$ are preserved by all endomorphisms φ of $K\{X_d\}$, i.e., $\varphi(K\{X_d\} \cap T(\mathfrak{V})) \subseteq K\{X_d\} \cap T(\mathfrak{V})$. In particular, $GL_d(K)(K\{X_d\} \cap T(\mathfrak{V})) = K\{X_d\} \cap T(\mathfrak{V})$. Here the general linear group $GL_d(K)$ acts canonically on the vector space KX_d with basis X_d and this action is extended diagonally on the whole $F_d(\mathfrak{V})$ as in (3). Hence $F_d(\mathfrak{V})$ has a natural structure of a $GL_d(K)$-module. For a background on representation theory of $GL_d(K)$ see, e.g., [30, 21]. Since $\text{char}(K) = 0$, the algebra $F_d(\mathfrak{V})$ is a direct sum of irreducible $GL_d(K)$-modules and

$$F_d(\mathfrak{V}) = \sum m_\lambda(\mathfrak{V}) W_d(\lambda),$$

where $W_d(\lambda)$ is the irreducible polynomial $GL_d(K)$-module corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$, and $m_\lambda(\mathfrak{V})$ is the multiplicity of $W_d(\lambda)$ in the decomposition of $F_d(\mathfrak{V})$. Then the Hilbert series of $F_d(\mathfrak{V})$ is

$$H(F_d(\mathfrak{V}), z_1, \ldots, z_d) = \sum m_\lambda(\mathfrak{V}) S_\lambda(z_1, \ldots, z_d),$$

where $S_\lambda(z_1, \ldots, z_d)$ is the Schur function corresponding to λ. Since the Schur functions form a basis of the vector space $K[X_d]^{S_n}$ of symmetric polynomials in d variables, the Hilbert series $H(F_d(\mathfrak{V}), z_1, \ldots, z_d)$ determines the $GL_d(K)$-module structure of $F_d(\mathfrak{V})$.

In the sequel we shall need some well known information for two relatively free algebras: the polynomial algebra $K[X_d]$ and the free metabelian Lie algebra $F_d(\mathfrak{A}^2) = L_d/L_d''$.

Lemma 1. (i) *The $GL_d(K)$-module structure of the polynomial algebra $K[X_d]$ is*

$$K[X_d] = \sum_{n \geq 0} W_d(n).$$

(ii) *The free metabelian Lie algebra $F_d(\mathfrak{A}^2)$ has a basis*

$$\{x_i, [[\cdots[x_{i_1}, x_{i_2}], \ldots], x_{i_n}] \mid i, i_j = 1, \ldots, d, i_1 > i_2 \leq \cdots \leq i_n\}.$$

The $GL_d(K)$-module structure of $F_d(\mathfrak{A}^2)$ is

$$F_d(\mathfrak{A}^2) = W_d(1) + \sum_{n \geq 2} W_d(n-1, 1).$$

Part (i) of the lemma is well known. Part (ii) is also well known, see e.g., [31, §52, pp. 274-276 of the English translation] for the basis of $F_d(\mathfrak{A}^2)$ and [32, the proof of Lemma 2.5] for its $GL_d(K)$-module structure.

The product of two Schur functions $S_\lambda(z_1,\dots,z_d)S_\mu(z_1,\dots,z_d)$ can be expressed as a sum of Schur functions using the Littlewood-Richardson rule. A very special case of this rule is the Branching theorem, when $\mu=(1)$. It states that

$$S_\lambda(z_1,\dots,z_d)S_{(1)}(z_1,\dots,z_d)=\sum S_\nu(z_1,\dots,z_d), \tag{5}$$

where the sum runs on all partitions $\nu=(\nu_1,\dots,\nu_d)$ obtained by adding 1 to one of the components λ_i of $\lambda=(\lambda_1,\dots,\lambda_d)$. In other words, the Young diagram of ν is obtained by adding a box to the diagram of λ. Since the product of two Schur functions corresponds to the tensor product of the corresponding irreducible $GL_d(K)$-modules, we obtain equivalently

$$W_d(\lambda)\otimes_K W_d(1)=\sum W_d(\nu), \tag{6}$$

with the same summation on ν as in (5).

If G is a subgroup of $GL_d(K)$, then the $GL_d(K)$-action on the irreducible $GL_d(K)$-module $W_d(\lambda)$ induces a G-action on $W_d(\lambda)$. Let $W_d(\lambda)^G$ be the vector space of the elements of $W_d(\lambda)$ fixed by G, i.e., of the G-invariants of $W_d(\lambda)$. If W is a graded $GL_d(K)$-module with polynomial homogeneous components,

$$W=\bigoplus_{k\geq 0}W_k,\quad W_k=\sum_\lambda m_\lambda(k)W_d(\lambda), \tag{7}$$

then its Hilbert series is

$$\begin{aligned}H(W,z_1,\dots,z_d,z)&=\sum_{k\geq 0}\left(\sum_{n_i\geq 0}\dim(W_k)_{(n_1,\dots,n_d)}z_1^{n_1}\cdots z_d^{n_d}\right)z^k\\&=\sum_{k\geq 0}\sum_\lambda m_\lambda(k)S_\lambda(z_1,\dots,z_d)z^k.\end{aligned} \tag{8}$$

Lemma 2. *Let W be a graded $GL_d(K)$-module with polynomial homogeneous components, as in (7), and let G be a subgroup of $GL_d(K)$. Then the Hilbert series of the G-invariants of W*

$$H(W^G,z)=\sum_{k\geq 0}\dim W_k^G z^k$$

is determined from the Hilbert series (8) of W.

Proof. We follow the main ideas of the recent preprint [33] which contains more applications in the spirit of the lemma. Since the dimension of $W_d(\lambda)^G$ depends on $W_d(\lambda)$ only, and the Schur functions $S_\lambda(z_1,\ldots,z_d)$ are in 1-1 correspondence with the modules $W_d(\lambda)$, we conclude that $\dim W_d(\lambda)^G$ is a function of $S_\lambda(z_1,\ldots,z_d)$. This immediately completes the proof because

$$H(W^G,z)=\sum_{k\geq 0}\left(\sum_{\lambda} m_\lambda(k)\dim W_d(\lambda)^G\right)z^k.$$

The proof of the next statement can be found in [34, Proposition 4.2] in the case of homomorphic images of the free associative algebra $K\langle X_d\rangle$. The proof in the case below is exactly the same.

Proposition 1. *Let I be an ideal of the relatively free algebra $F_d(\mathfrak{W})$ of the variety $\mathfrak{W}$ and let I be preserved under the $GL_d(K)$-action on $F_d(\mathfrak{W})$. If G is a subgroup of $GL_d(K)$, then every G-invariant of the factor algebra $F_d(\mathfrak{W})/I$ can be lifted to a G-invariant of $F_d(\mathfrak{W})$, i.e., under the canonical homomorphism*

$$\pi : F_d(\mathfrak{W})\to F_d(\mathfrak{W})/I$$

$F_d(\mathfrak{W})^G$ maps onto $(F_d(\mathfrak{W})/I)^G$. In particular, if $\mathfrak{V}$ is a subvariety of the variety $\mathfrak{W}$ and $\pi : F_d(\mathfrak{W})\to F_d(\mathfrak{V})$, then

$$\pi(F_d(\mathfrak{W})^G)=F_d(\mathfrak{V})^G.$$

For more details on varieties of algebras (in the associative case) and the applications of representation theory of $GL_d(K)$ to PI-algebras we refer to the book [35].

3. The main result

Let $\mathfrak{L}$ be the subvariety of the variety of right-symmetric algebras $\mathfrak{R}$ defined by the identity (4) of left-nilpotency of class 3. Since the identity (2) of left-commutativity is a consequence of (4), $\mathfrak{L}$ is also a subvariety of the variety $\mathfrak{N}$ of Novikov algebras. Working in $\mathfrak{L}$, the only nonzero products are left-normed. We shall omit the parentheses and shall write $a_1a_2\cdots a_n$ instead of $(\cdots(a_1a_2)\cdots)a_n$ and $a_1a_2^k$ instead of $a_1\underbrace{a_2\cdots a_2}_{k\text{ times}}$.

Lemma 3. (i) *The relatively free algebra* $F_d(\mathfrak{L})$ *has a basis*

$$\{x_{i_1}x_{i_2}\cdots x_{i_n} \mid i_1=1,\ldots,d, \quad 1\leq i_2\leq\cdots\leq i_n\leq d\}. \tag{9}$$

(ii) *The* $GL_d(K)$*-module structure of* $F_d(\mathfrak{L})$ *is*

$$F_d(\mathfrak{L})=W_d(1)+\sum_{n\geq 2}(W_d(n)+W_d(n-1,1)). \tag{10}$$

Proof. (i) Modulo the identity (4) the right-symmetric identity (1) reduces to

$$x_1x_2x_3=x_1x_3x_2. \tag{11}$$

Hence $\mathfrak{L}$ satisfies the identity

$$x_1x_2\cdots x_n=x_1x_{\sigma(2)}\cdots x_{\sigma(n)}, \quad \sigma\in S_n, \sigma(1)=1,$$

and the algebra $F_d(\mathfrak{L})$ is spanned as a vector space on the elements (9). In order to show that (9) is a basis of $F_d(\mathfrak{L})$ it is sufficient to construct an algebra A in $\mathfrak{L}$ which is generated by $a_1,\ldots,a_d$ and has a basis

$$\{a_ia_1^{n_1}\cdots a_d^{n_d} \mid i_1=1,\ldots,d, \quad n_j\geq 0\}. \tag{12}$$

Since A is a homomorphic image of $F_d(\mathfrak{L})$, this would imply that (9) is a basis of $F_d(\mathfrak{L})$. Consider the vector space A with basis (12) and define a multiplication there by the rule

$$(a_ia_1^{n_1}\cdots a_d^{n_d})*a_j=a_ia_1^{n_1}\cdots a_j^{n_j+1}\cdots a_d^{n_d},$$

$$(a_ia_1^{n_1}\cdots a_d^{n_d})*(a_ia_1^{m_1}\cdots a_d^{m_d})=0, \text{ if } m_j>0 \text{ for some } j.$$

Obviously A satisfies the identities (4) and (11), and hence belongs to $\mathfrak{L}$.

(ii) For $n\geq 2$ we divide the basis elements from (9) in two groups. The first group contains the monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$ with $i_1\leq i_2$ and the second group the monomials with $i_1>i_2$. Obviously, the monomials in the first group are in 1-1 correspondence with the monomials of degree ≥ 2 in $K[X_d]$. By Lemma 1 (ii), the same holds for the monomials from the second group and the elements of degree ≥ 2 in $F_d(\mathfrak{A}^2)$. Hence the Hilbert series of $F_d(\mathfrak{L})$ is a sum of the Hilbert series of the algebra of polynomials without constant term and the commutator ideal of the Lie algebra $F_d(\mathfrak{A}^2)$. Now the proof follows from Lemma 1.

The construction in the proof of Lemma 3 (i) suggests that the algebra $F_d(\mathfrak{L})$ has the structure of a right $K[X_d]$-module with action defined by

$$(x_p x_1^{n_1}\cdots x_d n_d)\circ(x_1^{m_1}\cdots x_d^{m_d}) = x_p x_1^{n_1+m_1}\cdots x_d^{n_d+m_d},\quad p=1,\ldots,d, n_j, m_j\geq 0.$$

Clearly, the ideal $F_d^2(\mathfrak{L})$ of the elements in $F_d(\mathfrak{L})$ without linear term is a $K[X_d]$-submodule. We shall denote by $(A_d)_1$ the vector space KX_d and shall identify $K[X_d]$ and $K[(A_d)_1]$.

The following theorem and its consequences together with the examples in the next section are the main results of the paper.

Theorem 1. *Let $\mathfrak{V}$ be a subvariety of the variety $\mathfrak{R}$ of all right-symmetric algebras and let $\mathfrak{V}$ contain the variety $\mathfrak{L}$ of left-nilpotent of class 3 algebras in $\mathfrak{R}$. If $G\neq\langle 1\rangle$ is a subgroup of $GL_d(K)$ such that the ideal $F_d^2(\mathfrak{L})^G$ of the algebra of invariants $F_d(\mathfrak{L})^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, then the algebra of G-invariants $F_d(\mathfrak{V})^G$ is not finitely generated.*

Proof. By Proposition 1 the canonical homomorphism $F_d(\mathfrak{V})\to F_d(\mathfrak{L})$ maps $F_d(\mathfrak{V})^G$ onto $F_d(\mathfrak{L})^G$ and if $F_d(\mathfrak{V})^G$ is finitely generated, the same is $F_d(\mathfrak{L})^G$. Hence it is sufficient to show that $F_d(\mathfrak{L})^G$ is not finitely generated. Therefore we may work in $F_d(\mathfrak{L})$ and assume that $F_d(\mathfrak{L})^G$ is finitely generated. As a vector space $F_d(\mathfrak{L})^G$ is a direct sum of the invariants of first degree $(KX_d)^G = (A_d)_1^G$ and the invariants $F_d^2(\mathfrak{L})^G$ without linear term. We may assume that $F_d(\mathfrak{L})^G$ is generated by $U=\{u_1,\ldots,u_k\}\subset(A_d)_1^G$ and $W=\{w_1,\ldots,w_l\}\subset F_d^2(\mathfrak{L})^G$. Since $F_d(\mathfrak{L})^G F_d^2(\mathfrak{L})^G=0$, the only nonzero products of the generators of $F_d(\mathfrak{L})^G$ are $u_p u_{i_1}\cdots u_{u_m}$, and $w_q u_{i_1}\cdots u_{u_m}$, $m\geq 0$. Hence $KU=(A_d)_1^G$,

$$F_d^2(\mathfrak{L})^G=\sum_{i=1}^{k} u_{p_1}u_{p_2}\circ K[(A_d)_1^G]+\sum_{j=1}^{l} w_q\circ K[(A_d)_1^G]$$

and $F_d^2(\mathfrak{L})^G$ is a finitely generated $K[(A_d)_1^G]$-module which is a contradiction.

Corollary 1. *Let $A_d=K[X_d]_+$ be the algebra of polynomials without constant term and let G be a subgroup of $GL_d(K)$. If $(A_d^2)^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, then $F_d(\mathfrak{V})^G$ is not finitely generated for any variety $\mathfrak{V}$ containing $\mathfrak{L}$.*

PROOF. By the Branching theorem (6)

$$W_d(n-1,1)\otimes_K W_d(1) = W_d(n,1)\oplus W_d(n-1,2)\oplus W_d(n-1,1,1). \quad (13)$$

Consider the $GL_d(K)$-module decomposition of $F_d(\mathfrak{L})$ given in Lemma 3 (ii). Since $F_d(\mathfrak{L})F_d^2(\mathfrak{L}) = 0$, the only nonzero products $W_d(\lambda)W_d(\mu)$ with λ or μ equal to $(n-1,1)$, $n\geq 2$, come from

$$W_d(n-1,1)W_d(1) = W_d(n-1,1)F_d(\mathfrak{L}) = W_d(n-1,1)(KX_d) = W_d(n-1,1)(A_d)_1.$$

This is a homomorphic image in $F_d(\mathfrak{L})$ of $W_d(n-1,1)\otimes_K W_d(1)$. By (13) we derive that $W_d(n-1,1)F_d(\mathfrak{L})\subset W_d(n,1)$. This implies that

$$I = \sum_{n\geq 2} W_d(n-1,1)\subset F_d(\mathfrak{L})$$

is an ideal of $F_d(\mathfrak{L})$ and the $GL_d(K)$-module structure of the factor algebra is

$$F_d(\mathfrak{L})/I = \sum_{n\geq 1} W_d(n)\cong A_d.$$

Hence the algebras $F_d(\mathfrak{L})/I$ and A_d have the same Hilbert series and by Lemma 2 the same holds for their algebras of invariants. Since $(A_d^2)^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, the same is true for the $K[(A_d)_1^G]$-module $F_d^2(\mathfrak{L})/I$. By Proposition 1, the $K[(A_d)_1^G]$-module $F_d^2(\mathfrak{L})$ is not finitely generated and the application of Theorem 1 completes the proof.

COROLLARY 2. *Let $F_d(\mathfrak{A}^2)$ be the free metabelian Lie algebra and let G be a subgroup of $GL_d(K)$. If $(KX_d)^G = (A_d)_1^G = 0$ and $\dim F_d(\mathfrak{A}^2)^G = \infty$, then $F_d(\mathfrak{V})^G$ is not finitely generated for any variety $\mathfrak{V}$ containing $\mathfrak{L}$.*

PROOF. Since $(KX_d)^G = (A_d)_1^G = 0$ we obtain that $F_d(\mathfrak{L})^G = F_d^2(\mathfrak{L})^G$. Hence the algebra $F_d(\mathfrak{L})^G$ is with trivial multiplication and the finite generation is equivalent to the finite dimensionality. As a $GL_d(K)$-module $F_d(\mathfrak{A}^2)$ is a homomorphic image of $F_d(\mathfrak{L})$. Hence the vector space $F_d(\mathfrak{A}^2)^G$ is a homomorphic image of $F_d(\mathfrak{L})^G$. This implies that $\dim F_d^2(\mathfrak{L})^G = \infty$, i.e., both the algebras $F_d(\mathfrak{L})^G$ and $F_d(\mathfrak{V})^G$ are not finitely generated.

Remark 3.1. *In Corollary 2 we cannot remove directly the restriction* $(KX_d)^G = 0$, *as in Corollary 1, because the* $GL_d(K)$*-submodule* $I = \sum_{n\geq 2} W_d(n)$ *of* $F_d(\mathfrak{L})$ *is not an ideal. For example, one can show that* $W_d(2)(KX_d) = W_d(3) \oplus W_d(2,1)$. *Hence we cannot use the property that the Lie algebra* $F_d(\mathfrak{A}^2)^G$ *is not finitely generated to show that the algebra* $F_d(\mathfrak{L})$ *is also not finitely generated. On the other hand, we do not know examples of groups* G *when* $(KX_d)^G = 0$, $K[X_d]^G = K$, *and* $\dim F_d(\mathfrak{A}^2)^G = \infty$. *Such an example would show that we may apply Corollary 2 when we cannot apply Corollary 1.*

4. Examples

All examples in this section use the following statement which is a consequence of Corollary 1.

Proposition 2. *If for a subgroup* G *of* $GL_d(K)$

$$\text{transcend.deg}(K[X_d]^G) > \dim(KX_d)^G,$$

then the algebra $F_d(\mathfrak{V})^G$ *is not finitely generated for any variety* $\mathfrak{V}$ *containing* $\mathfrak{L}$.

Proof. Let $t = \text{transcend.deg}(K[X_d]^G)$. Since $K[X_d]^G$ is graded, we may choose t algebraically independent homogeneous elements in $A_d^G = (K[X_d]_+)^G$. If $m = \dim(KX_d)^G$, changing linearly the variables X_d we assume that $(KX_d)^G$ has a basis $X_m = \{x_1, \dots, x_m\}$ and $K[(A_d)_1^G] = K[X_m]$. Since $t > m$, we obtain that $((A_d)^2)^G$ contains an element $f(X_d)$, such that the system $X_m \cup \{f(X_d)\}$ is algebraically independent. Hence the $K[X_m]$-module generated by the powers $f^k(X_d)$, $k = 1, 2, \dots$, is not finitely generated. Now the proof follows from Corollary 1.

3.1. Finite groups.

Theorem 2. *Let* G *be a finite subgroup of* $GL_d(K)$ *and* $G \neq \langle 1 \rangle$. *Then the algebra* $F_d(\mathfrak{V})^G$ *is not finitely generated for any variety* $\mathfrak{V}$ *containing* $\mathfrak{L}$.

Proof. It is well known that for a finite group G

$$\text{transcend.deg}(K[X_d]^G) = \text{transcend.deg}(K[X_d]) = d. \qquad (14)$$

For self-containedness of the exposition, every element $f(X_d) \in K[X_d]$ satisfies the equation

$$u_f(z) = \prod_{g \in G}(z - g(f(X_d))) = z^{|G|} - c_1 z^{|G|-1} + c_2 z^{|G|-2} - \cdots \pm c_{|G|}$$

where the coefficients c_k are equal to the elementary symmetric polynomials in $\{g(f(X_d)) \mid g \in G\}$. Hence $c_k \in K[X_d]^G$ and as a $K[X_d]^G$-module $K[X_d]$ is generated by

$$x_1^{a_1} \cdots x_d^{a_d}, \quad 0 \leq a_i < |G|.$$

The finite generation of the $K[X_d]^G$-module $K[X_d]$ implies (14) and the theorem follows from Proposition 2.

3.2. Reductive groups. If $G \subset GL_d(K)$ is a reductive group then there exists a G-submodule W of KX_d such that $KX_d = (KX_d)^G \oplus W$.

Proposition 3. *In the above notation, if $K[W]^G \neq K$, then $F_d(\mathfrak{V})^G$ is not finitely generated for all $\mathfrak{V}$ containing $\mathfrak{L}$.*

Proof. Since the elements of $K[W]$ cannot be expressed as polynomials in $(KX_d)^G$, the condition $K[W]^G \neq K$ implies that

$$\text{transcend.deg}(K[X_d]^G) = \dim(KX_d)^G + \text{transcend.deg}(K[W]^G) > \dim(KX_d)^G$$

and this completes the proof in virtue of Proposition 2.

Example. *For each $k \geq 1$ there is a unique irreducible rational k-dimensional $SL_2(K)$-module W_k. Let the subgroup G of $GL_d(K)$ be isomorphic to $SL_2(K)$ and*

$$KX_d \cong W_{k_1} \oplus \cdots \oplus W_{k_p}$$

as an $SL_2(K)$-module. It is well known that if $k_1 \geq 3$, then $K[W_{k_1}]$ contains nontrivial $SL_2(K)$-invariants. Similarly, $K[W_2 \oplus W_2]^{SL_2(K)} \neq K$. Hence the only cases when $K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}]$ are $k_1 = 2$, $k_2 = \cdots = k_p = 1$ when $K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}] \cong K[X_{d-1}]$ and $k_1 = \cdots = k_p = 1$ with the trivial action of $SL_2(K)$ on KX_d (and the latter case is impossible because $G \cong SL_2(K)$ is a nontrivial subgroup of $GL_d(K)$).

3.3. Weitzenböck derivations. A linear operator δ of an algebra A is a *derivation* if

$$\delta(uv) = \delta(u)v + u\delta(v), \quad u, v \in A.$$

If $\mathfrak{V}$ is a variety of algebras, then every mapping $\delta : X_d \to F_d(\mathfrak{V})$ can be uniquely extended to a derivation of $F_d(\mathfrak{V})$ which we shall denote by the same symbol δ. If δ is a nilpotent linear operator on KX_d, then the induced derivation is called a *Weitzenböck derivation*. Weitzenböck [36] proved that in the case of polynomial algebras the *algebra of constants*

$$K[X_d]^\delta = \{f(X_d) \in K[X_d] \mid \delta(f(X_d)) = 0\}$$

is finitely generated. Details on the algebra of constants $K[X_d]^\delta$ can be found in the book by Nowicki [37]. For varieties $\mathfrak{V}$ of unitary associative algebras (and $\delta \neq 0$) the algebra $F_d(\mathfrak{V})^\delta$ is finitely generated if and only if $\mathfrak{V}$ does not contain the algebra $T_2(K)$ of 2×2 upper triangular matrices, see [38, 34]. Up to a change of the basis of KX_d the Weitzenböck derivation δ is determined by the Jordan normal form $J(\delta)$ of the linear operator δ acting on KX_d. Since δ acts nilpotently on KX_d, the matrix $J(\delta)$ consists of Jordan blocks with zero diagonals.

Proposition 4. *If $d > 2$ and the Jordan normal form $J(\delta)$ of the Weitzenböck derivation consists of less than $d - 1$ blocks, then the algebra $F_d(\mathfrak{V})^\delta$ is not finitely generated for any variety $\mathfrak{V}$ containing the variety $\mathfrak{L}$.*

Proof. Since $\alpha\delta$, $\alpha \in K$, is nilpotent on KX_d, it is a *locally nilpotent derivation* of $F_d(\mathfrak{V})$, i.e., for every $f(X_d) \in F_d(\mathfrak{V})$ there exists an $n \geq 1$ such that $(\alpha\delta)^n(f(X_d)) = 0$. Hence

$$\exp(\alpha\delta) = 1 + \frac{\alpha\delta}{1!} + \frac{(\alpha\delta)^2}{2!} + \cdots$$

is a well defined linear automorphism of $F_d(\mathfrak{V})$. It is well known that the group

$$\{\exp(\alpha\delta) \mid \alpha \in K\}$$

is isomorphic to the unipotent group $UT_2(K)$ and

$$F_d(\mathfrak{V})^\delta = F_d(\mathfrak{V})^{UT_2(K)}.$$

If the matrix $J(\delta)$ consists of p blocks, then the dimension of the vector space $(KX_d)^\delta$ of the linear constants is equal to the number of the blocks p. Reading carefully [37, Proposition 6.5.1, p. 65] we can see that

$$\text{transcend.deg}(K[X_d]^\delta) = d - 1$$

which is larger than $p = \dim(KX_d)^\delta$. Now the proof follows from Proposition 2 applied for $UT_2(K) \subset GL_d(K)$.

Remark 4.2. *If in Proposition 4 the Jordan normal form of δ consists of $d-1$ blocks, then the algebra of constants $K[X_d]^\delta$ is generated by linear constants. In this case we may assume that $\delta(x_1) = x_2$ and $\delta(x_i) = 0$ for $i = 2, \ldots, d$. It is easy to see that $F_d(\mathfrak{L})^\delta$ is generated by $x_1x_2 - x_2x_1, x_2, \ldots, x_d$. We do not know how far can be lifted to $F_d(\mathfrak{V})$ the finite generation property of the algebra of constants and do not have a description of the varieties $\mathfrak{V}$ containing $\mathfrak{L}$ such that the algebra $F_d(\mathfrak{V})^\delta$ is finitely generated.*

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Дренски В.С. ЕРКIН ОҢ-СИММЕТРИЯЛЫ АЛГЕБРАЛАР МЕН НОВИКОВ АЛГЕБРАЛАРЫНА ҚАТЫСТЫ ИНВАРИАНТТАР ТЕОРИЯСЫ

(x1; x2; x3) = (x1;x3; x2) полиномиалды тепе-теңдiгi бар алгебралар оң-симметриялы алгебралар деп аталады, мұнда (x1; x2; x3) = x1(x2x3)..(x1x2)x3 - ассоциатор болып табылады. Новиков алгебралары - бұл қосымша x1(x2x3) = x2(x1x3) полиномиалды тепе-теңдiгiн қанағаттандыратын оң-симметриялы алгебралар болып табылады. Бiз 0 сипаттамалы K өрiсiнiң көптеген еркiн Xd = fx1; : : : ; xdg туындаушылары бар еркін оң-симметриялы Fd(R) алгебрасын және еркін Fd(N) Новиков алгебрасын қарастырамыз. d-өлшемді сызықты KXd кеңістігіне канондық әсері бар толық сызықты GLd(K) тобы Fd(R) мен Fd(N) сызықты автоизиморфизмдер тобы ретінде әсер етеді. GLd(K) тобының G ішкі тобы үшін біз Fd(R)G және Fd(N)G G- инварианттар алгебрасын зерттейміз. Для большого класса групп G топтарының көптеген кластары үшін біз мы показываем, что алгебры Fd(R)G және Fd(N)G алгебралары ешқашан ақырлы туындаған алгебралар екенін көрсетеміз. Осындай нәтиже оң-симметриялы алгебралардың R көпбейнелегінің R-дегі алгебралардың 3 сол-нильпотентті класының L ішкікөпбейнелігін қамтитын әрбір ішкікөпбейнелік үшін де орынды болады.

Дренски В.С. ТЕОРИЯ ИНВАРИАНТОВ ОТНОСИТЕЛЬНО СВОБОДНЫХ ПРАВО-СИММЕТРИЧЕСКИХ АЛГЕБР И АЛГЕБР НОВИКОВА

Алгебры с полиномиальным тождеством $(x_1,x_2,x_3) = (x_1,x_3,x_2)$, где $(x_1,x_2,x_3) = x_1(x_2x_3) - (x_1x_2)x_3$ – ассоциатор, называются право-симметрическими. Алгебры Новикова – это право-симметрические алгебры, дополнительно удовлетворяющие полиномиальному тождеству $x_1(x_2x_3) = x_2(x_1x_3)$. Мы рассматриваем свободную право-симметрическую алгебру $F_d(\mathfrak{R})$ и свободную алгебру Новикова $F_d(\mathfrak{N})$ с множеством свободных порождающих $X_d = \{x_1,\ldots,x_d\}$ над полем K характеристики 0. Полная линейная группа $GL_d(K)$ с ее каноническим действием на d-мерное линейное пространство KX_d действует на $F_d(\mathfrak{R})$ и $F_d(\mathfrak{N})$ как группа линейных автоморфизмов. Для подгруппы G группы $GL_d(K)$ мы изучаем алгебру G-инвариантов $F_d(\mathfrak{R})^G$ и $F_d(\mathfrak{N})^G$. Для большого класса групп G мы показываем, что алгебры $F_d(\mathfrak{R})^G$ и $F_d(\mathfrak{N})^G$ никогда не являются конечно порожденными. Такой-же результат верен для каждого подмногообразия многообразия $\mathfrak{R}$ право-симметрических алгебр, которое содержит подмногообразие $\mathfrak{L}$ лево-нильпотентных класса 3 алгебр из $\mathfrak{R}$.

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COMMUTATOR ALGEBRAS OF PRE-COMMUTATIVE ALGEBRAS

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Annotation: We consider a class of anti-commutative algebras embeddable into pre-commutative algebras (Zinbiel algebras) relative to the commutator product. It is proved that this class is not a variety. An analogous approach shows that the class of commutative algebras embeddable into pre-commutative algebras relative to the anti-commutator is not a variety as well.

Keywords: Novikov algebra, Zinbiel algebra, Rota–Baxter operator.

1. Introduction

One of the most common problems in algebra is a description of the class $\mathcal{M}^{(\omega)}$ of algebraic systems obtained from a given class $\mathcal{M}$ by means of new operations on the same base set. These operations are assumed to be expressed in some way (denoted as ω) via the initial operations of the class $\mathcal{M}$. In what follows, we will only consider linear algebras, i.e., linear spaces equipped with a family of polylinear operations in arbitrary number of arguments. An algebra of the form $A^{(\omega)}$, $A \in \mathcal{M}$, is supposed to be the same linear space A equipped with new family of operations.

For example, if $\mathcal{M} = \mathrm{As}$ is the variety of all associative algebras with multiplication $\mu : (x, y) \to xy$ then the new commutator operation $[x, y] = xy - yx$ on the space of algebra $A \in \mathrm{As}$ turns A into a Lie algebra $A^{(-)}$.

Moreover, every Lie algebra may be embedded into an appropriate $A^{(-)}$, $A \in \mathrm{As}$, therefore, $S(\mathrm{As}^{(-)})$ is a variety (equal to *Lie*). Here $S(\mathcal{K})$ stands for the class of all subalgebras of all algebras from the class $\mathcal{K}$. In the sequel, we will denote by $H(\mathcal{K})$ the class of all homomorphic images of algebras from $\mathcal{K}$.

Keywords: *Novikov algebra, Zinbiel algebra, Rota–Baxter operator.*

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On the other hand, it is well known that the anti-commutator operation $x \circ y = xy + yx$ turns an algebra $A \in \mathrm{As}$ into a Jordan algebra $A^{(+)}$, but $S(\mathrm{As}^{(+)})$ is not a variety: this class is not closed with respect to homomorphic images [1].

Generally speaking, let $\mathcal{M}$ and $\mathcal{N}$ be two varieties of linear algebras defined by polylinear identities. Then an algebra $A \in \mathcal{M}$ may be represented as a morphism from the operad $\mathcal{C}_{\mathcal{M}}$ governing the variety $\mathcal{M}$ (see [2]) to the multicategory of Vec of linear spaces [3]:

$$A : \mathcal{C}_{\mathcal{M}} \to \mathrm{Vec}.$$

Every morphism of operads $\omega : \mathcal{C}_{\mathcal{N}} \to \mathcal{C}_{\mathcal{M}}$ naturally defines a functor $\mathcal{M} \to \mathcal{N}$:

$$A^{(\omega)} : \mathcal{C}_{\mathcal{N}} \xrightarrow{\omega} \mathcal{C}_{\mathcal{M}} \xrightarrow{A} \mathrm{Vec}.$$

The kernel of ω in $\mathcal{C}_{\mathcal{N}}$ is the collection of those polylinear identities in the variety $\mathcal{N}$ that hold on all algebras of the form $A^{(\omega)}$, $A \in \mathcal{M}$.

Algebras of the class $S(\mathcal{M}^{(\omega)})$ are called *special* $\mathcal{N}$-algebras (relative to the morphism ω). Denote the closure of this class with respect to homomorphic images by $\mathcal{S}^{(\omega)}\mathcal{N}$.

For every triple $(\mathcal{M}, \mathcal{N}, \omega)$ as above, one may pose the following questions.

- Find the kernel of ω in $\mathcal{C}_{\mathcal{N}}$, or at least determine whether it is trivial. The operad $\mathcal{C}_{\mathcal{N}}/\mathrm{Ker}\,\omega$ governs the variety $\mathcal{S}^{(\omega)}\mathcal{N}$.

- Whether $S(\mathcal{M}^{(\omega)})$ coincides with $\mathcal{N}$?

- If the previous question meets negative answer, decide whether $S(\mathcal{M}^{(\omega)}) \subseteq \mathcal{N}$ is a variety.

The first two questions are known as the problem of finding special identities and speciality problem. A study of functors of this type as well as a study of speciality problems was performed, in particular, in [4]. To solve the last question it is enough to check whether we have $\mathcal{S}^{(\omega)}\mathcal{N} = S(\mathcal{M}^{(\omega)})$ (since the Cartesian products preserve speciality). A general method of counterexample construction (non-special homomorphic images of special $\mathcal{N}$-algebras) is given by the following analogue of P.M. Cohn's lemma [1].

Let X be a set, and let $\mathcal{M}\langle X\rangle$, $\mathcal{N}\langle X\rangle$ be the free algebras (generated by X) in the varieties $\mathcal{M}$, $\mathcal{N}$, respectively. Denote $\mathcal{N}_\omega\langle X\rangle$ the subalgebra in the $\mathcal{N}$-algebra $\mathcal{M}\langle X\rangle^{(\omega)}$ generated by X. For a subset $B\subset \mathcal{N}_\omega\langle X\rangle$ denote $I_\omega(B)$ and $I(B)$ the ideals of $\mathcal{N}_\omega\langle X\rangle$ and $\mathcal{M}\langle X\rangle$ generated by B.

Lemma 1. *Algebra $\mathcal{N}_\omega\langle X\rangle/I_\omega(B)$ is special relative to the morphism ω if and only if $I_\omega(B)=I(B)\cap\mathcal{N}_\omega\langle X\rangle$.*

The proof is completely similar to what is stated in [5, Ch. 3].

If $S(\mathcal{M}^{(\omega)})$ and $\mathcal{N}$ coincide then there is a more precise question on the properties of the triple $(\mathcal{M},\mathcal{N},\omega)$: whether this triple has *PBW-property* [6], i.e., whether an analogue of the Poicaré—Birkhoff—Witt Theorem holds for the universal enveloping algebra $U_\omega(L)\in\mathcal{M}$, $L\in\mathcal{N}$. As shown in [6], the PBW-property allows to transfer many combinatorial properties from $\mathcal{M}$ to $\mathcal{N}$.

In the present work we consider a series of varieties obtained from classical varieties As, Com, and Lie of associative, associative-commutative, and Lie algebras by means of *splitting* procedure. The classes obtained are related with Novikov algebras and with Tortken- and Tortkara-algebras introduced by A.S. Dzhumadil'daev.

2. Splitting of operads and corresponding morphisms

Denote by Perm the variety of associative algebras satisfying the identity $xyz=yxz$ [7].

Let $\mathcal{M}$ be a variety of algebras with one bilinear operation of multiplication $\mu:(x,y)\mapsto xy$ defined by a family of polylinear identities. Denote by $\mathcal{C}_\mathcal{M}$ the operad governing the variety $\mathcal{M}$.

Definition 1 ([8]). *Denote by* pre$\mathcal{M}$ *the class of linear spaces A equipped with two bilinear operations $\mu_\succ:(x,y)\mapsto x\succ y$ and $\mu_\prec:(x,y)\mapsto x\prec y$ such that for every algebra $P\in Perm$ the space $P\otimes A$ relative to the operation*

$$(p\otimes a)(q\otimes b)=pq\otimes a\succ b+qp\otimes a\prec b,\quad p,q\in P,\ a,b\in A, \tag{1}$$

belongs to the variety $\mathcal{M}$.

It follows immediately from definition that pre$\mathcal{M}$ is a variety, its defining identities may be obtained from the defining identities of $\mathcal{M}$ by comparing similar terms in the expression

$$t(p_1 \otimes a_1, \dots, p_n \otimes a_n) = 0, \quad p_i \in P,\ a_i \in A,$$

where $t(x_1, \dots, x_n)$ is a defining identity of $\mathcal{M}$. Here it is enough to choose P equal to the free Perm-algebra generated by countable set $\{p_1, p_2, \dots\}$. The same identities are obtained as a result of the splitting procedure described in [9], [10].

Example 1 (Pre-associative algebras). For $\mathcal{M} = \mathrm{As}$, the class of pre-associative algebras consists of linear spaces equipped with two operations $\succ$ and $\prec$ such that

$$\begin{aligned} x \succ (y \succ z) &= (x \succ y) \succ z + (x \prec y) \succ z, \\ x \succ (y \prec z) &= (x \succ y) \prec z, \\ x \prec (y \succ z) + x \prec (y \prec z) &= (x \prec y) \prec z. \end{aligned} \tag{2}$$

This is exactly the system of defining identities of dendriform di-algebras [11].

Example 2 (Pre-commutative algebras). For $\mathcal{M} = \mathrm{Com}$, the class of pre-commutative algebras is defined by the identities 2 and

$$x \succ y = y \prec x.$$

These identities, being rewritten in terms of one product $x \circ y = x \succ y$ ($x \prec y = y \circ x$), turn into single expression

$$x \circ (y \circ z) = (x \circ y) \circ z + (y \circ x) \circ z. \tag{3}$$

This is the definig identity of the class of Zinbiel algebras [11].

Note that (3) implies

$$x \circ [y, z] + y \circ [z, x] + z \circ [x, y] = 0, \tag{4}$$

where $[x, y] = x \circ y - y \circ x$

Example 3 (Pre-Lie algebras). For $\mathcal{M} = \mathrm{Lie}$, the class of pre-Lie algebras is defined by identities

$$\begin{gathered} x \succ y = -y \prec x, \\ x \succ (y \succ z) - y \succ (x \succ z) = (x \succ y + x \prec y) z, \\ x \succ (y \prec z) - y \prec (x \succ z + x \prec z) = (x \succ y) \prec z, \\ x \prec (y \prec z + y \succ z) - y \succ (x \prec z) = (x \prec y) \prec z. \end{gathered}$$

Relative to the operation $x \circ y = x \prec y$ these identities turn into single expression

$$x \circ (y \circ z) - (x \circ y) \circ z = x \circ (z \circ y) - (x \circ z) \circ y.$$

This relation defines the variety of right-symmetric algebras [12]–[14].

The study of splitting varieties is essentially motivated by their relation to Rota—Baxter operators. The latter represent constant solutions of the classical Yang—Baxter equation (CYBE) on semisimple finite-dimensional Lie algebras [15].

Namely, a Rota—Baxter operator on an algebra U is a linear operator $R : U \to U$ such that

$$R(u)R(v) = R(R(u)v) + R(uR(v)), \quad u, v \in U.$$

Theorem 1 ([10]). *1. An algebra $A \in \mathcal{M}$ with a Rota—Baxter operator R is a system of the class* pre $\mathcal{M}$ *relative to the operations*

$$a \succ b = R(a)b, \quad a \prec b = aR(b), \quad a, b \in A.$$

2. Every algebra $B \in$ pre $\mathcal{M}$ embeds into an appropriate algebra $A \in \mathcal{M}$ with a Rota—Baxter operator.

This theorem implies every algebra $A \in$ pre $\mathcal{M}$ to be embedded into its universal enveloping $\mathcal{M}$-algebra with a Rota—Baxter operator. Standard universal algebra reasonings allow to derive the main result of [16] (where it was proved for $\mathcal{M} = \mathrm{As}$).

Corollary 1 [c.f. with [16]]. The universal enveloping Rota—Baxter $\mathcal{M}$-algebra of the free pre $\mathcal{M}$-algebra is isomorphic to the free system in the variety of $\mathcal{M}$-algebras with Rota—Baxter operator.

Corollary 2. For a binary quadratic operad $\mathcal{C}_{\mathcal{M}}$ we have $\mathcal{C}_{\mathrm{pre}\ \mathcal{M}} = \mathcal{C}_{\mathrm{pre\ Lie}} \bullet \mathcal{C}_{\mathcal{M}}$, where $\bullet$ stands for the black Manin product of operads [2], [17].

Definition 1 implies that for every algebra $A \in$ pre $\mathcal{M}$ with operations $\succ$, $\prec$ the same space A is an algebra of the variety $\mathcal{M}$ relative to the product

$$a * b = a \succ b + a \prec b.$$

Indeed, it is enough to consider $P = \Bbbk \in \mathrm{Com} \subset \mathrm{Perm}$. The $\mathcal{M}$-algebra obtained is denoted by $A^{(*)}$.

Therefore, we have a morphism of operads $\mathcal{F}_{\mathcal{M}} \to \mathcal{F}_{\mathrm{pre}\ \mathcal{M}}$ inducing the functor $(*) : \mathrm{pre}\ \mathcal{M} \to \mathcal{M}$, $A \mapsto A^{(*)}$.

Theorem 2. *Let* $\mathcal{M} = \mathrm{As}, \mathrm{Lie}$. *Then the triple* $(\mathrm{pre}\ \mathcal{M}, \mathcal{M}, (*))$ *has the PBW-property.*

For $\mathcal{M} = \mathrm{Lie}$, this statement was proved in [18], where the Gröbner—Shirshov bases method was developed for the class pre Lie: there is a basis of the universal enveloping Lie algebra of a pre-Lie algebra L that does not depend on the multiplication table of L.

The same approach was developed in pre-associative case in [19], where the Gröbner—Shirshov bases theory for pre-associative algebras was developed. Calculation of bases of universal envelopes shows the triple $(\mathrm{pre}\ \mathrm{As}, \mathrm{As},\ (*))$ has the PBW-property.

3. Splitting morphisms of operads

Let $\mathcal{M}$, $\mathcal{N}$ be two varieties of linear algebras, and let $\omega : \mathcal{C}_{\mathcal{N}} \to \mathcal{C}_{\mathcal{M}}$ be a morphism of the corresponding operads; ω induces the functor $\mathcal{M} \to \mathcal{N}$, $A \mapsto A^{(\omega)}$. Then there exists a morphism $\mathrm{pre}\ \omega : \mathcal{C}_{\mathrm{pre}\ \mathcal{N}} \to \mathcal{C}_{\mathrm{pre}\ \mathcal{M}}$ inducing the functor $\mathrm{pre}\ \mathcal{M} \to \mathrm{pre}\ \mathcal{N}$, $A \mapsto A^{(\mathrm{pre}\ \omega)}$ compatible with the splitting procedure:

$$(P \otimes A)^{(\omega)} = P \otimes A^{(\mathrm{pre}\ \omega)}$$

for $A \in \mathrm{pre}\ \mathcal{M}$, $P \in \mathrm{Perm}$.

Example 4. Let $\mathcal{N} = \mathrm{Lie}$, $\mathcal{M} = \mathrm{As}$, and let ω be the morphism sending x_1x_2 to $[x_1, x_2] = x_1x_2 - x_2x_1$. The induced functor turns an associative algebra into its commutator Lie algebra. Consider $P = \mathrm{Perm}\langle p, q, \dots\rangle$, $A \in \mathrm{pre}\ \mathrm{As}$, and compute

$$[p \otimes a, q \otimes b] = pq \otimes (a \prec b - b \succ a) + qp \otimes (a \succ b - b \prec a)$$

for $a, b \in A$. Obviously, the morphism $\mathrm{pre}\,\omega$ maps $x_1 \prec x_2$ to $[x_1 \prec x_2] = x_1 \prec x_2 - x_2 \succ x_1$ and $x_1 \succ x_2$ to $[x_1 \succ x_2] = x_1 \succ x_2 - x_2 \prec x_1$. These operations are related to each other: $[x_1 \prec x_2] = -[x_2 \succ x_1]$.

Example 5. Let Diff be the variety of differential algebras, i.e., associative and commutative algebras with a derivation ∂. Consider the functor sending $A \in \mathrm{Diff}$ to a system with new binary operation $x \cdot y = \partial(x)y$. It is well-known that this new operation satisfies *Novikov algebra* axioms:

$$(x, y, z) = (x, z, y), \quad x(yz) = y(xz). \tag{5}$$

Splitting of the corresponding morphism of operads allows to construct a functor from the variety of pre-commutative algebras with a derivation ∂ to the variety pre Nov, whose defining identities may be obtained from (5) in a procedure described by Definition 1. This functor is described by the following relations:

$$x \succ y = \partial(x)y, \quad x \prec y = y\partial(x).$$

Splitting morphisms of operads are compatible with embeddings into algebras with Rota—Baxter operators: if $A \in \mathrm{pre}\,\mathcal{M}$, $A \subseteq B$, where B is a $\mathcal{M}$-algebra with a Rota—Baxter operator R then R is a Rota—Baxter operator on $B^{(\omega)} \in \mathcal{N}$ and $A^{(\mathrm{pre}\ \omega)} \subseteq B^{(\omega)}$.

On the other hand, the properties of pre ω may differ from the properties of ω. Even if $\omega : \mathcal{C}_{\mathcal{N}} \to \mathcal{C}_{\mathcal{M}}$ is a “nice” morphism in a sense that $S(\mathcal{M}^{(\omega)})$ coincides with $\mathcal{N}$, the splitting morphism pre ω may have a nonzero kernel.

For example, consider the variety $\mathcal{D}_2$ of associative algebras with a derivation ∂ such that $\partial^2 = 0$, over a field of characteristic $\neq 2$. The morphism $\omega : x_1x_2 \mapsto x_1 \circ x_2 = \partial(x_1x_2)$ induces a functor from $\mathcal{D}_2$ to the variety $\mathcal{N}_3$ of all 3-nilpotent algebras. Indeed, it is easy to see that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in A \in \mathcal{D}_2$.

Lemma 2. *For every algebra $N \in \mathcal{N}_3$ there exists its envelope from $\mathcal{D}_2$.*

Proof. Let X be a basis of N, and let Y be a basis of N^2 such that $Y \subset X$. Denote by $x \circ y \in \Bbbk Y$, $x, y \in X$, linear forms determining the multiplication in N. Add a new variable $p \notin X$ and consider associative algebra defined by generators and relations as follows:

$$A = As\langle X, p \mid pxy - xyp - x \circ y,\ pup,\ u \in X^*, x, y \in X\rangle.$$

Here X^* denotes the set of all (including empty) words in the alphabet X. The algebra A may be considered as a system in $\mathcal{D}_2$ assuming $\partial(a) = pa - ap$.

Moreover, $N \to A^{(\omega)}$. To prove injectivity of this homomorphism it is enough to find a Gröbner—Shirshov basis (see, e.g., [20]) of the ideal I of $As\langle X, p\rangle$ generated by $pxy - xyp - x \circ y$, pup. It is not hard to see that the confluent system of relations is obtained by adding puy and yup, $y \in Y$, $u \in X^*$. Hence, $I \cap \Bbbk X = \{0\}$, i.e., N embeds into A.

Consider the splitting morphism. The variety pre $\mathcal{N}_3$ is defined by a family of six identities obtained from $(x \circ y) \circ z$, $x \circ (y \circ z)$ as shown in Definition 1:

$$
\begin{gathered}
(x \prec_\circ y + x \succ_\circ y) \succ_\circ z,\ (x \succ_\circ y) \prec_\circ z,\ (x \prec_\circ y) \prec_\circ z, \\
x \succ_\circ (y \succ_\circ z),\ x \succ_\circ (y \prec_\circ z),\ x \succ_\circ (y \prec_\circ z + y \succ_\circ z).
\end{gathered} \tag{6}
$$

Variety pre $\mathcal{D}_2$ consists of systems with two binary operations satisfying (2) and with a derivation ∂ (relative to the both operations), such that $\partial^2 = 0$. The functor induced by morphism pre ω turns $A \in$ pre $\mathcal{D}_2$ into $A^{(\mathrm{pre}\ \omega)}$ with operations

$$
x \succ_\circ y = \partial(x \succ y), \quad x \prec_\circ y = \partial(x \prec y).
$$

Here, in particular, $x \prec_\circ (y \succ_\circ z) = 0$, but this identity does not follow from (6). Therefore, $0 \neq x_1 \prec (x_2 \succ x_3) \in \mathcal{C}_{\mathcal{N}_3}$ belongs to the kernel of pre ω.

4. Commutator pre-commutative algebras

In this section, we consider commutator algebras of pre-commutative (Zinbiel) algebras. Recall that the variety pre Com consists of linear spaces equipped with one bilinear operation satisfying the identity

$$
x(yz) = (xy)z + (yx)z.
$$

A basis of the free algebra $Z_n = \mathrm{pre\,Com}\langle x_1, x_2, \dots, x_n\rangle$ is given by all monomials of the form $(\dots((x_{i_1}x_{i_2})x_{i_3})\dots x_{i_k})$, $k \geq 1$, $i_1, \dots, i_k \in \{1, \dots, n\}$.

As above, let $A^{(-)}$ stands for the commutator algebra of $A \in$ pre Com. Denote by $\mathcal{K}$ the class of all subalgebras of all algebras $A^{(-)}$. It was mentioned in [21] that all algebras in $\mathcal{K}$ satisfy (besides anti-commutativity) the identity

$$
[[t_1, t_2], [t_3, t_4]] + [[t_1, t_4], [t_3, t_2]] - [J(t_1, t_2, t_3), t_4] - [J(t_1, t_4, t_3), t_2] = 0, \tag{7}
$$

where $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$ is the Jacobian of elements in an anti-commutative algebra.

Following [21], let us call *tortkara-algebra* an anti-commutative algebra with operation $[\cdot,\cdot]$ satisfying (7). Denote by Tort the class of all tortkara-algebras. Then $\mathcal{K}$ is the class of all special Tort-algebras relative to the morphism $(-)$: Tortkara $= S(\mathrm{pre\ Com}(-))$.

THEOREM 3. *The algebra $Z_4^{(-)} \in \mathcal{K}$ has a homomorphic image $A \notin \mathcal{K}$.*

PROOF. Use notations from Lemma (1). Consider $B = \{f_1, \dots, f_4\}$, where

$$f_1 = [x_1, [x_3, x_4]],\ f_2 = [x_2, [x_3, x_4]],\ f_3 = [x_3, [x_1, x_2]],\ f_4 = [x_4, [x_1, x_2]].$$

Relation (4) for $x = [x_1, x_2]$, $y = x_3$, $z = x_4$ implies $[x_1, x_2][x_3, x_4] \in I(B)$. Analogously, for $x = [x_3, x_4]$, $y = x_1$, $z = x_2$ we obtain $[x_3, x_4][x_1, x_2] \in I(B)$. Therefore, $f = [[x_1, x_2], [x_3, x_4]] \in I(B) \cap Tort_{(-)}\langle x_1, x_2, x_3, x_4\rangle$.

On the other hand, if f belongs to the ideal $I_{(-)}(B)$ then

$$f = \alpha_1[x_2, f_1] + \alpha_2[x_1, f_2] + \alpha_3[x_4, f_3] + \alpha_4[x_3, f_4] \tag{8}$$

in $Z_4^{(-)}$ for some $\alpha_i \in \Bbbk$ (it follows from the homogeneity of all elements). However, it is not hard to show that f, $[x_2, f_1]$, $[x_1, f_2]$, $[x_4, f_3]$, $[x_3, f_4]$ are linearly independent in Z_4. Indeed, if they were linearly dependent then so are their images in every homomorphic image of Z_4. Consider, for example, the commutative algebra $t\Bbbk[t]$ with Rota—Baxter operator $t^n \mapsto \frac{1}{n}t^n$ and associated pre-commutative algebra with operation $t^n t^m = \frac{t}{n}t^{n+m}$. Evaluating (8) with $x_i = t^{m_i}$ for arbitrary $m_i \geq 1$, $i = 1, \dots, 4$, we obtain a set of conditions on the constants $\alpha_1, \dots, \alpha_4$. To obtain $\alpha_1 = \dots = \alpha_4 = 0$, it is enough to consider the following values $(m_1, \dots, m_4)$: $(1, 1, 1, m)$, $(1, m, 1, 1)$, $(1, m, 1, m)$, $(2, m, 2, m)$, where $m \geq 1$.

COROLLARY 3. The class of all special Tort-algebras is not a variety.

It worths mentioning that an analogous approach shows the class of subalgebras of anti-commutator algebras for pre-commutative algebras is not a variety. As above, let Z_1 stand for the free pre-commutative algebra pre $\mathrm{Com}\langle x\rangle$ in one generator. Anti-commutator algebra $Z_1^{(+)}$ is also generated by x relative to the operation $a \circ b = ab + ba$. It is not hard to note that $Z_1^{(+)}$ as a commutative algebra is isomorphic to the ideal $t\Bbbk[t]$ of the polynomial algebra. Denote by $\mathcal{A}$ the class $S(\mathrm{pre\ Com}^{(+)})$.

Theorem 4. *Algebra $Z_1^{(+)}$ has a homomorphic image $H \notin \mathcal{A}$.*

Proof. Use notations of Lemma (1). Consider the set $B = \{f\}$ in $Z_1^{(+)}$, where

$$f = x \circ x - x = 2xx - x.$$

It follows from (3) that

$$xf = 2x(xx) - xx = 4(xx)x - xx = 2fx + \frac{1}{2}f + \frac{1}{2}x,$$

i.e., $x \in I(B)$ и $I(B) = Z_1$.

On the other hand, the image of ideal $I_{(+)}(B)$ in $t\Bbbk[t]$ is generated by $t^2 - t$, so it is a proper ideal and $I_{(+)}(B) \neq I(B)$.

Remark. The kernel of the morphism (+) is obviously trivial (otherwise, all algebras of the form pre $\mathrm{Com}^{(+)}$ belong to a proper subvariety of Com, but all such varieties are nilpotent).

5. Open questions

1. Let $\omega : \mathcal{C}_{\mathcal{N}} \to \mathcal{C}_{\mathcal{N}}$ be a morphism of operads governing varieties $\mathcal{N}$ and $\mathcal{M}$. Example 3 shows that the variety generated by special (relative to pre ω) pre $\mathcal{N}$-algebras may not coincide with the splitting class of the variety generated by special (relative to ω) $\mathcal{N}$-algebras. Note that in Example 3 the triple $(\mathcal{M}, \mathcal{N}, \omega)$ does not meet the PBW-property in the sense of [6]. This circumstance causes the following questions.

- Whether
$$\mathcal{S}^{(\mathrm{pre}\ \omega)}\mathrm{pre}\ \mathcal{N} = \mathrm{pre}\ \mathcal{S}^{(\omega)}\mathcal{N}$$
provided that $(\mathcal{M}, \mathcal{N}, \omega)$ is a PBW-triple?

- Whether $(\mathrm{pre}\ \mathcal{M}, \mathrm{pre}\ \mathcal{N}, \mathrm{pre}\ \omega)$ is a PBW-triple provided that so is $(\mathcal{M}, \mathcal{N}, \omega)$?

2. Consider the morphism $(-)$ and the corresponding functor pre Com $\to$ Tortkara. The following questions remain open.

- Wheher the kernel of this morphism is trivial? In other words, we are interested in polylinear identities that hold on all algebras in pre $\mathrm{Com}^{(-)}$ but does not follow from anti-commutativity and tortkara-identity (7) (an analogue of the Glennie identity on Jordan algebras)?

- Does there exist a simple non-special Tort-algebra (an analogue of the simple exceptional Jordan algebra, Albert algebra)?

- What is the maximal number of generators n for which all homomorphic images of $\mathrm{Tort}_{(-)}\langle x_1, \dots, x_n\rangle$ are special? Theorem 3 states $n < 4$.

3. Recall that the variety of Novikov algebras Nov is defined by identities (5). A basis of the free Novikov algebra was found in [22]. For $A \in \mathrm{Nov}$, let $A^{(+)}$ denote the anti-commutator algebra of A, $x \circ y = xy + yx$. Denote by $\mathcal{A}$ the class of all subalgebras of all algebras $A^{(+)}$, $A \in \mathrm{Nov}$.

It was shown in [23] that all algebras in $\mathcal{A}$ satisfy (besides commutativity) the identity *tortken*:

$$(t_1, t_2, t_3) \circ t_4 + (t_1 \circ t_2) \circ (t_3 \circ t_4) - (t_1 \circ t_4) \circ (t_2 \circ t_3) - (t_1, t_4, t_3) \circ t_2 = 0. \quad (9)$$

Following [23], call commutative algebras with identity (9) by *Novikov—Jordan algebras* (NJ). Consider the morphism (+) and the corresponding functor Nov $\to$ NJ. Special NJ-algebras relative to (+) are subalgebras of anti-commutator algebras of Novikov algebras.

The following questions remain open.

- What are the generators (as of operadic ideal) of the kernel of (+)? In other words, we are interested in those polylinear identities of NJ-algebras that imply all other identities holding on the class $\mathrm{Nov}^{(+)}$. One of such identities (*besken*) was found in [24].

- Whether the class of all special NJ-algebras is a variety?

- Does there exist a simple non-special NJ-algebra?

- What is the maximal number of generators n, for which all homomorphic images of $\mathrm{NJ}_{(+)}\langle x_1, \dots, x_n\rangle$ are special?

4. Since the class of Novikov algebras is embedded into the variety pre Lie, the commutator morphism $(-)$ determines a functor Nov $\to$ Lie. As it was mentioned in [24], all algebras in the class $\mathrm{Nov}^{(-)}$ satisfy the identity

$$\sum_{\sigma\in S_4}(-1)^\sigma[x_{\sigma(1)},[x_{\sigma(2)},[x_{\sigma(3)},[x_{\sigma(4)},x_5]]]]=0.$$

This causes the following questions.

- Describe the kernel of $(-):\mathcal{C}_{\mathrm{Lie}}\to\mathcal{C}_{\mathrm{Nov}}$.
- Whether $S(\mathrm{Nov}^{(-)})$ is a variety?

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Колесников П.С. АЛДЫҢҒЫ-КОММУТАТИВТІ АЛГЕБРАЛАРДЫҢ КОММУТАТОРЛЫ АЛГЕБРАЛАРЫ

Бұл жұмыста коммутатор көбейтіндісі арқылы пре-коммутативті алгебраларға (Цинбиль алгебраларына) енгізілетін антикоммутативті алгебралар класы қарастырылады. Бұл кластың көпбейне болмайтындығы дәлелденді. Дәл солай, анти-коммутатор көбейтіндісі арқылы пре-коммутативті алгебраларға (Цинбиль алгебраларына) енгізілетін коммутативті алгебралар класы көпбейне болмайтындығы дәлелденді.

Колесников П.С. КОММУТАТОРНЫЕ АЛГЕБРЫ ПРЕ-КОММУТАТИВНЫХ АЛГЕБР

Рассматривается класс антикоммутативных алгебр, вкладывающихся в пре-коммутативные алгебры (алгебры Цинбиля) относительно коммутатора. Доказано, что этот класс не является многообразием. Аналогичным образом показывается, что класс коммутативных алгебр, вкладывающихся в пре-коммутативные алгебры относительно анти-коммутатора, тоже не является многообразием.

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THE DZHUMADILDAEV BRACKETS: A HIDDEN SUPERSYMMETRY OF COMMUTATORS AND THE AMITSUR-LEVITZKI-TYPE IDENTITIES

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Abstract: The Amitsur-Levitzki identity for matrices was generalized in several directions: by Kostant for simple finite-dimensional Lie algebras, by Kirillov (with Kontsevich, Molev, Ovsienko, and Udalova) for simple vectorial Lie algebras with polynomial coefficients, and by Gie, Pinczon, and Ushirobira for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|n)$. Dzhumadildaev switched the focus of attention in these results by considering the algebra formed by antisymmetrizors and discovered a hidden supersymmetry of commutators. We overview these results and their possible generalizations (open problems).

Keywords: Lie superalgebra, commutator, Amitsur-Levitzki identity.

1. Introduction

Hereafter, the ground field is $\mathbb{C}$, although several statements are true over fields $\mathbb{K}$ of characteristic $p > 2$. We are thankful to A. Dzhumadildaev for help and inspiring comments.

1.1 On an experience of superizing

Consciously superizing various notions and statements since 1971, people observed that there are, usually, several ways and results of superizations: a straightforward one (usually, not a breath-catching one) and one or several other, often quite amazing, ways bringing up totally new notions (examples: the Poisson and anti-brackets, the supertrace and the queer trace

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on supermatrices, and the "quasi-classical limit" of these traces, and the corresponding superdeterminants, see [1] and [2], 476 p.).

A difficulty to be able to superize something by at least one method (to say nothing of several) usually indicates that we do not understand, actually, even the allegedly well-understood "nonsuper" situation. A prime example is the integration theory on supermanifolds, which is still far from being completely constructed, see [1] and [3]. Other examples are two somewhat related topics expressed by the following two theorems:

Theorem 1.2 (Cayley-Hamilton). *Every $n \times n$-matrix X satisfies its characteristic polynomial*

$$\det(X - \lambda 1_n) = 0. \tag{1}$$

Its first superization is due to Yastrebov [4]. For various (seemingly completely unrelated) super versions of the Cayley-Hamilton Theorem, see [5, 6, 7, 8, 9].

Theorem 1.3 (Amitsur-Levitzki). *Let C be a commutative and associative algebra. For any $X_1, \dots, X_r \in Mat(n; C)$, define antisymmetrizors a_r by setting*

$$a_r(X_1, \dots, X_r) := \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{sign }\sigma} X_{\sigma(1)} \dots X_{\sigma(r)}. \tag{2}$$

Then the Amitsur-Levitzki Identity (ALI) takes place:

$$a_r(X_1, \dots, X_r) = 0 \ \text{ for any } r \geq 2n. \tag{3}$$

An interesting paper [10] was allegedly the final word concerning superization of ALI, but later a no less interesting paper [11] appeared. In this note, we also discuss superizations of ALI; for the proof of the classical ALI with the help of a Grassmann superalgebra, see §5.

1.3.1 Amitsur-Levitzki type theorem for vectorial Lie algebras

A.A. Kirillov formulated the following analog of the Amitsur-Levitzki theorem, for its proof, see Preprints of Keldysh Inst. of Applied Math. in 1980s; for a translation of one such preprint, see [12]; the other preprints with related results by Kirillov, Kontsevich and Molev had not been translated; Molev reviewed them in [13].

Theorem ([14]). *Let $\mathfrak{g}$ be a simple Lie algebra of vector fields over a field of characteristic* 0. *Let*

$$A_k(x_1,\dots,x_k) := \sum_{\sigma\in S_k} (-1)^{\text{sign }\sigma} ad_{x_{\sigma(1)}}\dots ad_{x_{\sigma(k)}}. \tag{4}$$

For any $x_1,\dots,x_k \in \mathfrak{g}$, the identity $A_k(x_1,\dots,x_k) \equiv 0$ holds
a) for $k \geq (n+1)^2$ if $\mathfrak{g} = \mathfrak{vect}(n)$,
b) for $k \geq n(2n+5)$ if $\mathfrak{g} = \mathfrak{h}(2n)$,
c) for $k \geq 2n^2+5n+5$ if $\mathfrak{g} = \mathfrak{k}(2n+1)$.

1.4. Facts that inspired us

Let $\mathfrak{vect}(n)$ be the Lie algebra of vector fields (for simplicity, with polynomial coefficients).

Fact. *The product of two vector fields is not a vector field (unless is equal to* 0*), but their commutator always is.*

(5)

In [16], Dzhumadildaev revealed a hidden supersymmetry of this well-known Fact(5) and posed a problem natural from this super point of view: quest for "higher" supersymmetries on the good old Lie algebras. Let us recall the less popular definitions and Dzhumadildaev's construction.

Dzhumadildaev called the antisymmetrizor (2) of vector fields $X_1,\dots,X_N \in \mathfrak{vect}(n)$ an *N-commutator* if $a_N(X_1,\dots,X_N) \in \mathfrak{vect}(n)$ for any $X_1,\dots,X_N \in \mathfrak{vect}(n)$ and a_N does not vanish identically. If $a_N(X_1,\dots,X_N)$ is an *N-commutator*, the number $N = N(n)$ is said to be *critical*.

The N-commutator is *subcritical* if $A_N(X_1,\dots,X_N) := a_N(ad_{X_1},\dots$ $\dots,ad_{X_N})$ is multiplication by a function for any $X_1,\dots,X_N \in \mathfrak{vect}(n)$. For example, in [12], it is shown that for $\mathfrak{vect}(1)$, the antisymmetrizor a_3 acts as an operator of multiplication by a function:

$$a_3(ad_{X_1},\dots,ad_{X_3})(Y) = -2\det\begin{pmatrix} x_1 & x_2 & x_3\\ x_1' & x_2' & x_3'\\ x_1'' & x_2'' & x_3''\end{pmatrix}\cdot Y, \tag{6}$$

where $X_i = x_i(t)\frac{d}{dt}$ for $i=1,2,3$, $f' := \frac{df}{dt}$, and $Y = y(t)\frac{d}{dt}$.

1.4.1. Problems

1) Is the following analog to the case for $n = 1$ true?

$$\text{If } A_N(X_1, \dots, X_N) = 0, \text{ then } \deg A_{N-1}(X_1, \dots, X_{N-1}) = 0.$$

2) The number $N(n) = 2$ is always critical for any n; we will call it the *standard* critical number. In [16], Dzhumadildaev conjectured that the numbers $N(n) = (n+1)^2 - 3$ are also critical for $n > 1$, proved the conjecture for $n = 2$ and 3, and raised a natural problem: *List all critical numbers*. The problem is open, except $n = 3$, where Dzhumadildaev established that $N = 10$ is also critical, and there are no more critical numbers.

Before we start considering this problem, let us discuss one more of Dzhumadildaev's results. To present it, we need one more fact. Although we are sure that this fact was known since at least 1960 s (for example, to I. Kantor and/or M. Gerstenhaber), the first reference we know is due to Dzhumadildaev [17]:

Fact. *The antisymmetrizors form an algebra with respect to the product defined to be*

$$(a_k * a_l)(X_1, \dots, X_{k+l-1}) := \sum_{\substack{\sigma \in S_{k+l-1} \text{ such that} \\ \sigma(1) < \dots < \sigma(l) \text{ and} \\ \sigma(l+1) < \dots < \sigma(k+l-1)}} \operatorname{sign}(\sigma)\, a_k(a_l(X_{\sigma(1)} \dots X_{\sigma(l)}), X_{\sigma(l+1)} \dots X_{\sigma(k+l-1)}). \quad (7)$$

More precisely, we have ([17])

$$a_k * a_l = \begin{cases} 0 & \text{if } k, l \text{ are even,} \\ k a_{k+l-1} & \text{if } l \text{ is odd,} \\ a_{k+l-1} & \text{if } l \text{ is even and } k \text{ is odd.} \end{cases} \quad (8)$$

Thus, the antisymmetrizors define a $\mathbb{Z}$-graded superring $A = \oplus A_i$, where $A_i = \operatorname{Span}(a_i)$, such that $A_{\bar 0} = \underset{i \equiv 1 \mod 2}{\oplus} A_i$ and the product of any two odd elements of $A_{\bar 1} = \underset{i \equiv 0 \mod 2}{\oplus} A_i$ is zero. Clearly, A can be considered as a superalgebra over any field. *What is the meaning of the superring or superalgebra A?*

1.5. Dzhumadildaev's approach to antisymmetrizors

In a series of papers, Dzhumadildaev changed the emphasis of the interpretation of the result by Amitsur and Levitzki from the search of the identity of the least order to the description of the superalgebra or the superring constructed from the antisymmetrizors in the classical Lie algebras. This approach revealed a hidden relation of the commutators with a certain universal odd superderivation. We overview various possible generalizations of Dzhumadildaev's result.

Let $\mathcal{F}$ be an associative commutative algebra, $\mathcal{A} = \operatorname{End} \mathcal{F}$ the associative algebra of its endomorphisms, and $\mathcal{A}_L$ the Lie algebra constructed by replacing the associative product by the bracket. If $\mathcal{F} = \mathbb{K}[x_1, ..., x_n]$, one can identify the elements of End $\mathcal{F}$ with differential operators. If End $\mathcal{F}$ is considered as associative algebra, its elements satisfy no identity except associativity. The Lie algebra $L = \operatorname{Der} \mathcal{F}$ is a Lie subalgebra of $(\operatorname{End} \mathcal{F})_L$ naturally identified with the Lie algebra $\mathfrak{vect}(n)$ of vector fields with polynomial coefficients.

Among numerous irreducible representations of L (for their overview, super setting including, see [15]), there are two "smallest" ones: in the space of functions (or, more generally, λ-densities) and the adjoint representation.

Initially, people were interested in polynomial identities in the adjoint representations, see Theorem 1. Dzhumadildaev considered polynomial identities in the "smallest" representation, which for $\mathfrak{vect}(n)$ is the representation in the space of functions $\mathcal{F}$. It is very interesting to generalize Dzhumadildaev's approach on the representations in the space of λ-densities, which is a rank 1 module over the algebra $\mathcal{F}$ generated by the λ-th power of the volume element with the following $\mathfrak{vect}(n)$-action (here $\lambda \in \mathbb{C}$ is fixed):

$$X(f \operatorname{vol}^\lambda) = (X(f) + f\lambda \operatorname{div}(X)) \operatorname{vol}^\lambda \quad \text{for any} \quad f \in \mathcal{F} \text{ and } X \in \mathfrak{vect}(n).$$

It seems that this approach is more natural than the initial one for the following reasons:

1) If one knows identities in the "natural" representation (of the smallest dimension or – for infinite-dimensional algebras – its analog), then it is easy to construct identities in other representations, in particular in the adjoint

In Geometry, $\mathcal{F}$ is the algebra of functions on an n-dimensional manifold; it is interesting to generalize Dzhumadildaev's approach to such cases, e.g., to the algebra of Laurent polynomials, i.e., the algebra of functions on the torus.

representation. For example, $a_{n^2+2n-1} = 0$ is identity in the space of functions $\mathcal{F}$, and since $\mathrm{ad}_X = r_X - l_X$, where r_X and l_X are right and left actions in $\mathcal{F}$, it is easy to deduce that $a_{n^2+2n+1} = 0$ is an identity in the adjoint representation of $\mathfrak{vect}(n)$.

2) If $a_N = 0$ is identity, then one can ask if the "pre-identity" a_{N-1} is a new operation on $\mathfrak{vect}(n)$.

To consider a_{N-1} as a multi-operation on $\mathfrak{vect}(n)$ is meaningless: a_{N-1} maps $\wedge^{N-1}\mathfrak{vect}(n)$ to the whole $\mathcal{A} =$ End $\mathcal{F}$, not just to $\mathfrak{vect}(n)$.

Dzhumadildaev suggested to consider a_N on the space of differential operators, making the question "is the pre-identity a_{N-1} a new operation on $\mathfrak{vect}(n)$?" meaningful: in some special cases a_{N-1} maps $\wedge^{N-1}\mathfrak{vect}(n)$ to $\mathfrak{vect}(n)$ once again!

Now consider eq. (6). It means that 3-antisymmetric sum of the adjoint derivations on $\mathfrak{vect}(1)$ is a multiplication operator (not the adjoint operator). Certainly, it is an interesting observation, but it is another topic. It has no connection with N-commutators: in this setting to speak about N-commutator is meaningless. Under the natural action

$$a_3(X_1, X_2, X_3) = 0 \quad \text{is an identity.}$$

Let us retell Dzhumadildaev's comments on observations due to Kirillov, Molev, Razmuslov, Bergman, and others on identities in $\mathfrak{vect}(n)$. The identities

$$\begin{array}{ll} a_N \equiv 0 & \text{if } N \geq (n+1)^2 \text{ for } \mathfrak{vect}(n) \\ a_N \equiv 0 & \text{if } N \geq n(2n+5) \text{ for } \mathfrak{h}(2n) \end{array}$$

are not of the smallest degree. Moreover, these are "easy" identities. For example, for the Lie algebra $\mathfrak{h}(2)$ of Hamiltonian vector fields in two indeterminates, there are two identities in degree 7. Kirillov's identity is not minimal and it is a consequence of these two identities. A similar situation with $\mathfrak{vect}(n)$. Dzhumadildaev conjectured that the minimal identity for representation of $\mathfrak{vect}(n)$ in the space of functions is of degree $(n+1)^2 - 2$ whereas the degree of Kirillov's identity is $(n+1)^2$.

1.6. Antisymmetrizors for simple finite dimensional Lie algebras. Exponents

The classical Amitsur-Levitzki theorem states that $a_{2n} = 0$ is the minimal identity for $\mathfrak{gl}(n)$. For $\mathfrak{o}(2n+1)$ and $\mathfrak{sp}(2n)$, the minimal identity is $a_{4n} = 0$;

for $\mathfrak{o}(2n)$, the minimal identity is $a_{4n-2} = 0$ (see [18, 19, 20]). Dzhumadildaev formulated the following theorem (known for the serial algebras) and gave explicit formulas for 10-antisymmetrisors in terms of the Chevalley basis for the 7-dimensional representation of $\mathfrak{g}_2$.

Theorem([21]). *Let $A(\mathfrak{g})$ be the algebra with respect to* (8). *Then*

$$\begin{aligned}
A(\mathfrak{sl}(n)) &= \operatorname{Span}\{a_{2k} \mid k = 1, 2, \ldots, n-1\},\\
&\text{in particular, } a_{2k+1} \equiv 0 \text{ for any } k;\\
A(\mathfrak{o}(2n+1)) &= \operatorname{Span}(\{a_{4k+1} \mid k = 1, 2, \ldots, n-1\}\cup\\
&\cup\{a_{4k+2} \mid k = 0, 1, 2, \ldots, n-1\}),\\
A(\mathfrak{sp}(2n)) &= \operatorname{Span}(\{a_{4k+1} \mid k = 1, 2, \ldots, n-1\}\cup\\
&\cup\{a_{4k+2} \mid k = 0, 1, 2, \ldots, n-1\}),\\
A(\mathfrak{o}(2n)) &= \operatorname{Span}(\{a_{4k+1} \mid k = 1, 2, \ldots, n-1\}\cup\\
&\cup\{a_{4k+2} \mid k = 0, 1, 2, \ldots, n-2\} \cup \{a_{4n-2}\}),\\
A(\mathfrak{g}_2) &= \operatorname{Span}(\{a_2;\ \ a_{10}\}).
\end{aligned} \tag{9}$$

1.6.1. Problem

1) *The indices of the antisymmetrizors are doubled exponents of the respective Lie algebras in the cases $\mathfrak{sl}(n)$ and $\mathfrak{g}_2$, but not for $\mathfrak{o}$ or $\mathfrak{sp}$:*

The Coxeter group or Lie algebra	*its exponents m_i*
A_n or $\mathfrak{sl}(n+1)$	$1, 2, 3, \ldots, n$
B_n or $\mathfrak{o}(2n+1)$, C_n or $\mathfrak{sp}(2n)$ for $n \geq 2$	$1, 3, \ldots, 2n-1$
D_n or $\mathfrak{o}(2n)$	$1, 3, \ldots, 2n-3;\ n-1$
$I_2^{(4)}$ or $\mathfrak{g}_2$	$1, 5$
F_4 or $\mathfrak{f}_4$	$1, 5, 7, 11$
E_6 or $\mathfrak{e}_6$	$1, 4, 5, 7, 8, 11$
E_7 or $\mathfrak{e}_7$	$1, 5, 7, 9, 11, 13, 17$
E_8 or $\mathfrak{e}_8$	$1, 7, 11, 13, 17, 19, 23, 29$

(10)

What is precisely the relation between the indices of the nonvanishing identically operations a_i and the exponents?

2) For the matrix realizations in the irreducible module $R(\pi_1)$ of the least dimension (see the right column in table (11)*), is the following conjectural left column in table* (11) *correct?*

$A(\mathfrak{f}_4) = \mathrm{Span}(\{a_2,\ a_{10},\ a_{14},\ a_{22}\})$	$\dim R(\pi_1) = 26$	(11)
$A(\mathfrak{e}_6) = \mathrm{Span}(\{a_2,\ a_8,\ a_{10},\ a_{14},\ a_{16},\ a_{22}\})$	$\dim R(\pi_1) = 27$	
$A(\mathfrak{e}_7) = \mathrm{Span}(\{a_2,\ a_{10},\ a_{14},\ a_{18},\ a_{22},\ a_{26},\ a_{34}\})$	$\dim R(\pi_1) = 56$	
$A(\mathfrak{e}_8) = \mathrm{Span}(\{a_2,\ a_{14},\ a_{22},\ a_{26},\ a_{34},\ a_{38},\ a_{46},\ a_{58}\})$	$\dim R(\pi_1) = 248$	

3) Clearly, the algebras $A(\mathfrak{g})$ may depend on the realization of $\mathfrak{g}$, i.e., on the representation. And this does happen: the algebras $A(\mathfrak{sl}(4))$ (corresponding to $R(\pi_1)$) and $A(\mathfrak{o}(6))$ (corresponding to $R(\pi_2)$) are different. Theorem 1 *corresponds to matrix realizations of the Lie algebras $\mathfrak{g}$ in the irreducible module of the least (except for $\mathfrak{o}(6)$) dimension.*

1.6.1a. Conjecture. *For the Lie algebras with the natural matrix realization, the above approach is reasonable. However, it seems no less reasonable to consider Lie algebra $\mathfrak{g}$ embedded into their universal enveloping algebras and look for k-commutators on $\mathfrak{g}$ inside $U(\mathfrak{g})$, not inside a particular representation. For the finite-dimensional simple Lie algebras, only $k = 2$ remains.*

The proof of Theorem 1 is based on the particular cases of Lemma 2 and [19, 20].

2. Superizations of Theorem 1

First, let us superize the notions involved. For details of superization, see [1]; we only recall here some basics. The supermatrices are considered in the standard format. The associative algebra $\mathrm{Mat}(n)$ of $n \times n$ matrices has two super analogs: $\mathrm{Mat}(n|m)$ and

$$Q(n) = \begin{array}{l}\{X \in \mathrm{Mat}(n|n) \mid [X, J] = 0 \text{ for the odd invertible operator } J\} = \\ \left\{\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in \mathrm{Mat}(n)\right\}.\end{array} \tag{12}$$

Accordingly, the general linear Lie algebra $\mathfrak{gl}(n)$ has two superanalogs: the Lie superalgebras $\mathfrak{gl}(n|m)$ and $\mathfrak{q}(n)$ obtained from the associative superalgebras $\mathrm{Mat}(n|m)$ and $Q(n)$, respectively, by replacing the dot product by the super-bracket.

On $\mathfrak{q}(n)$, the queer trace is defined:

$$qtr : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \longmapsto trB. \tag{13}$$

The Lie superalgebra $\mathfrak{sq}(n)$ is the subalgebra of $\mathfrak{q}(n)$ consisting of queertraceless supermatrices.

The Lie superalgebras $\mathfrak{osp}(n|2m)$ and $\mathfrak{pe}(n)$ preserve the nondegenerate symmetric bilinear form (even and odd, respectively) whose Gram matrices are $\operatorname{diag}(1_n, J_{2m})$, where $J_{2m} = \operatorname{antidiag}(1_m, -1_m)$, and $J_{n|n} = \operatorname{antidiag}(1_n, -1_n)$, respectively (i.e., $J_{n|n}$ coincides with J_{2n} but is odd). The same Lie superalgebras preserve antisymmetric nondegenerate bilinear forms. The supermatrix X is said to preserve the bilinear form B if

$$BX + (-1)^{p(X)p(B)} X^{st} B = 0,$$

where the *supertransposition* st describing the matrix of the dual operator, see [1], is defined as follows (in the standard format):

$$st : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}.$$

Thanks to linearity, it suffices to consider only homogeneous with respect to parity elements.

For the composition $X_1 \dots X_k$ of any k operators $X_1, \dots, X_k$ (supermatrices or vector fields, or whatever) of parities $P = (p_1, \dots, p_k) \in (\mathbb{Z}/2\mathbb{Z})^k$, define its *antisymmetrizor* to be

$$a_N(X_1, \dots, X_N) := \sum_{s \in S_N} \operatorname{sign}(s, P)\, X_{s(1)} \dots X_{s(N)}, \tag{14}$$

where $\operatorname{sign}(s, P) = \operatorname{sign}(s)\operatorname{sign}(s')$ and s' is the permutation induced by s on the ordered subset of odd elements among $X_1, \dots, X_k$. In other words, if $x_1, \dots, x_k$ are elements of a supercommutative superalgebra whose respective parities are $p_1 + \bar{1}, \dots, p_k + \bar{1}$, then $x_{s(1)} \dots x_{s(k)} = \operatorname{sign}(s, P) x_1 \dots x_k$. One can express $\operatorname{sign}(s, P)$ in another form, more convenient for computations. We define

$$\operatorname{sign}(s, P) = \prod_{1 \le i < j \le k, \quad s(i) > s(j)} (-1)^{p_i p_j}. \tag{15}$$

Define the composition of permutations by setting

$$s_1 \circ s_2 = (s_1(s_2(1)), \dots, s_1(s_2(k))).$$

The function $\mathrm{sign}(s, P)$ is a 1-cocycle on S_k ([22]):

$$\mathrm{sign}(s_1 \circ s_2, P) = \mathrm{sign}(s_1, P)\mathrm{sign}(s_2, s_1(P)), \text{ where } s_1(P) = (p_{s_1(1)}, \dots, p_{s_1(k)}).$$

Lemma 2.1. *The Lie superalgebra $\mathfrak{sl}(m|n)$ is closed under the a_{2l} for any $m, n \geq 0$ and $l > 0$. Moreover, $a_{2l}(X_1, \dots, X_{2l}) \in \mathfrak{sl}(m|n)$ for any $X_1, \dots, X_{2l} \in \mathrm{Mat}(m|n)$. For $mn = 0$, the nonvanishing identically operations a_k are listed in Theorem 1.*

Proof. We need to prove that $\mathrm{str}\, a_{2l}(X_1, \dots, X_{2l}) = 0$. Let $P = (p_1, \dots, p_{2l})$ be the vector of parities of $X_1, \dots, X_{2l}$.

For $s \in S_{2l}$, set $s' = (s(2), \dots, s(2l), s(1))$ (i.e., $s' = s \circ s_0$, where $s_0 = (2, \dots, 2l, 1)$. Then the terms in the sum (14) corresponding to s and s' have opposite supertraces:

$$\begin{aligned}
&\mathrm{sign}(s')\mathrm{sign}(s', P)\mathrm{str}(X_{s'(1)} \dots X_{s'(2l)}) = \\
&\mathrm{sign}(s)\mathrm{sign}(s_0)\mathrm{sign}(s, P)\mathrm{sign}(s_0, s(P))\mathrm{str}(X_{s(2)} \dots X_{s(2l)} X_{s(1)}) = \\
&-(-1)^{p_1(p_2+\dots+p_{2l})}\mathrm{sign}(s)\mathrm{sign}(s, P) \times \\
&(-1)^{p_{s(1)}(p_{s(2)}+\dots+p_{s(2l)})}\mathrm{str}((-1)^{p_{s(1)}(p_{s(2)}+\dots+p_{s(2l)})} X_{s(1)} \dots X_{s(2l)}) = \\
&-\mathrm{sign}(s)\mathrm{sign}(s, P)\mathrm{str}(X_{s(1)} \dots X_{s(2l)}).
\end{aligned}$$

Since $2l$ – the order of s_0 – is even, S_{2l} can be represented as the disjoint union of two sets of equal cardinalities; and the set of elements of S_{2l} can be divided in pairs of the form $(s, s \circ s_0)$. Thus, the total supertrace of the sum (14) is equal to 0.

Lemma 2.2. *The Lie superalgebras $\mathfrak{osp}(m|2n)$ and $\mathfrak{pe}(n)$ are closed under a_k for $k = 4l + 1$ and $4l + 2$ for any $m, n, l \geq 0$.*

For $\mathfrak{osp}(m|2n)$ and $mn = 0$, the nonvanishing identically operations a_k are listed in Theorem 1; for $\mathfrak{osp}(1|2n)$, we have $a_{4n} = 0$ ([10]). For $\mathfrak{osp}(m|2n)$ and $mn \neq 0$ but not $\mathfrak{osp}(1|2n)$, and for $\mathfrak{pe}(n)$, the a_k for $k = 4l+1$ and $4l+2$ never vanish identically.

Proof. Let B be the Gram matrix of the bilinear form. Let $X_1, \dots, X_k \in \mathfrak{aut}(B)$ be of parities $p_1, \dots, p_k$. Then $BX_i + (-1)^{p_i} X_i^{st} B = 0$, and we need to show that

$$Ba_k(X_1, \dots, X_k) + (-1)^{p_1+\dots+p_k} a_k(X_1, \dots, X_k)^{st} B = 0.$$

Set $s^I = (k, k-1, \dots, 1)$. Then we can rewrite (14) as

$$a_k(X_1, \dots, X_k) = \sum_{s\in S_k} \operatorname{sign}(s\circ s^I)\operatorname{sign}(s\circ s^I, P) X_{(s\circ s^I)(1)} \dots X_{(s\circ s^I)(k)} =$$
$$\sum_{s\in S_k} \operatorname{sign}(s)\operatorname{sign}(s^I)\operatorname{sign}(s,P)\operatorname{sign}(s^I, s(P)) X_{s(k)} \dots X_{s(1)}.$$

Since

$$BX_{s(k)} \dots X_{s(1)} = -(-1)^{p_{s(k)}} X_{s(k)}^{st} B X_{s(k-1)} \dots X_{s(1)} = \dots$$
$$\dots = (-1)^{k+p_1+\dots+p_k} X_{s(k)}^{st} \dots X_{s(1)}^{st} B,$$

we have

$$Ba_k(X_1, \dots, X_k) =$$
$$\sum_{s\in S_k} \operatorname{sign}(s)\operatorname{sign}(s^I)\operatorname{sign}(s,P)\operatorname{sign}(s^I, s(P))(-1)^{k+p_1+\dots+p_k} X_{s(k)}^{st} \dots X_{s(1)}^{st} B.$$

On the other hand,

$$(X_{s(1)} \dots X_{s(k)})^{st} = \operatorname{sign}(s^I, s(P)) X_{s(k)}^{st} \dots X_{s(1)}^{st},$$

so

$$a_k(X_1, \dots, X_k)^{st} B =$$
$$= \sum_{s\in S_k} \operatorname{sign}(s)\operatorname{sign}(s,P)\operatorname{sign}(s^I, s(P))(-1)^{p_1+\dots+p_k} X_{s(k)}^{st} \dots X_{s(1)}^{st} B.$$

The two sums are opposite if $\operatorname{sign}(s^I)(-1)^k = -1$, and then

$$Ba_k(X_1, \dots, X_k) + (-1)^{p_1+\dots+p_k} a_k(X_1, \dots, X_k)^{st} B = 0.$$

Since $\operatorname{sign}(s^I) = (-1)^{[k/2]}$, this is true for $k = 4l+1, 4l+2$.

Problem 2.2.1. *What is the analog of Lemma 2 for $\mathfrak{spe}(n)$?*

Lemma 2.3. *The Lie superalgebra* $\mathfrak{q}(n)$ *is closed under* a_k *and* $\mathfrak{sq}(n)$ *is closed under* a_{2k} *for any* n *and* k.

Proof. The associative algebra $Q(n)$ is closed with respect to the dot product; hence the result about $\mathfrak{q}$.

Since $qtr(XY) = (-1)^{p(X)p(Y)} qtr(YX)$ $(= qtr(YX)$, since X and Y should be of different parities in order to have $qtr(XY) \neq 0)$ and so the same arguments as for $\mathfrak{sl}$ are applicable.

Questions 2.4. What is the super analog of eq. (8) for the super-antisymmetrizor (14)?

3. Vectorial Lie algebras

3.1. $\mathfrak{vect}(n)$. In [23], Feigin and Fuchs proved, among other things, that for $n = 1$, the only critical pair is the standard one: $(1, 2)$.

In [16], Dzhumadildaev showed that for $n = 2$, the complete list consists only of the standard pairs $(2, 2)$ and $(2, 6)$. The precise expression of the 6-commutator is as follows. The 6-tuple $(X_1, X_2, X_3, X_4, X_5, X_6)$, where $X_i = u_{i,1}\partial_1 + u_{i,2}\partial_2$ for $i = 1, \dots, 6$, goes to

$$\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_2^2 u_{1,2} & \partial_2^2 u_{2,2} & \partial_2^2 u_{3,2} & \partial_2^2 u_{4,2} & \partial_2^2 u_{5,2} & \partial_2^2 u_{6,2} \end{vmatrix} \partial_1 - (\dots)\partial_2 +$$

$$+ \begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_1^2 u_{1,1} & \partial_1^2 u_{2,1} & \partial_1^2 u_{3,1} & \partial_1^2 u_{4,1} & \partial_1^2 u_{5,1} & \partial_1^2 u_{6,1} \end{vmatrix} \partial_1 - (\dots)\partial_2 +$$

$$+ \begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2^2 u_{1,2} & \partial_2^2 u_{2,2} & \partial_2^2 u_{3,2} & \partial_2^2 u_{4,2} & \partial_2^2 u_{5,2} & \partial_2^2 u_{6,2} \end{vmatrix} \partial_1 - (\dots)\partial_2 -$$

$$-2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_{12} u_{1,1} & \partial_{12} u_{2,1} & \partial_{12} u_{3,1} & \partial_{12} u_{4,1} & \partial_{12} u_{5,1} & \partial_{12} u_{6,1} \end{vmatrix} \partial_1 - (\ldots)\partial_2 -$$

$$-2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_{12} u_{1,1} & \partial_{12} u_{2,1} & \partial_{12} u_{3,1} & \partial_{12} u_{4,1} & \partial_{12} u_{5,1} & \partial_{12} u_{6,1} \end{vmatrix} \partial_1 - (\ldots)\partial_2 +$$

$$+3\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_2^2 u_{1,1} & \partial_2^2 u_{2,1} & \partial_2^2 u_{3,1} & \partial_2^2 u_{4,1} & \partial_2^2 u_{5,1} & \partial_2^2 u_{6,1} \end{vmatrix} \partial_1 - (\ldots)\partial_2 -$$

$$-2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_{12} u_{1,2} & \partial_{12} u_{2,2} & \partial_{12} u_{3,2} & \partial_{12} u_{4,2} & \partial_{12} u_{5,2} & \partial_{12} u_{6,2} \end{vmatrix} \partial_1 - (\ldots)\partial_2,$$

where the coefficient of ∂_2 is obtained from that of ∂_1 by interchanging the subscripts 1 and 2 if there is only one subscript, only second subscripts 1 and 2 when dealing with u_{ij}.

3.2. How to write the k-commutator for any n?

Let $X_1, ..., X_k \in \mathfrak{vect}(n)$ with coefficients $u_{i,j}$ (i.e., $X_i = \sum\limits_{1\leq j\leq n} u_{i,j}\partial_j$). Let $a = (a_1, \ldots, a_k)$, where the a_i are integers 1 to n, let (b_{ij}) be a $k\times n$ matrix with elements in $\mathbb{Z}_{\geq 0}$; let $D((a_i),(b_{ij}))$ be the determinant of the $k\times k$ matrix whose (i,j)-th slot is occupied by $\partial_1^{b_{i1}}...\partial_n^{b_{in}} u_{j,a_i}$. Considering the k-commutator of the fields $X_1, ..., X_k$ as a differential operator, its 1-st degree component is equal to

$$\sum_{a_1=1}^{n}\cdots\sum_{a_k=1}^{n}\sum_{s_1=2}^{k}\sum_{s_2=3}^{k}\cdots\sum_{s_{k-1}=k}^{k} D\left((a_i), \sum_{1\leq i\leq k-1} E^{s_i,a_i}\right)\partial_{a_k}, \qquad (16)$$

where the $E^{i,j}$ are matrix units.

Accordingly, if the k-commutator is a first order operator, then (16) is its expression. Unfortunately, this expression is not user-friendly: first, it is longish ($n^k \times (k-1)!$ summands) which even for $n = 2, k = 6$ is > 7000), second, it is very redundant: some of the summands vanish, some are equal to each other, some are equal in absolute value but are of different signs (so there are just 14 distinct types of summands for $n = 2$, not > 7000).

In [24], Dzhumadildaev showed that for $n = 3$, in addition to the standard pairs $(3, 2)$ and $(3, 13)$, there is exactly one more critical pair, $(3, 10)$.

3.3. The other series of simple vectorial Lie algebras with polynomial coefficients

It is equally natural to list all critical pairs for the other types of simple vectorial Lie algebras. For these Lie algebras, only the pairs $(n, 2)$ will be called *standard*.

For the Lie algebras $\mathfrak{svect}(n)$ of divergence-free vector fields, Dzhumadildaev proved [16, 24] that the only nonstandard critical pairs are $(2, 5)$ and $(3, 10)$ (for $n = 2$ and 3, respectively). Since $\mathfrak{svect}(2) \simeq \mathfrak{h}(2)$ the result for this Lie algebra might be pertaining to the Hamiltonian series, rather than to the divergence-free one.

For the Lie algebras $\mathfrak{h}(2n)$ of Hamiltonian vector fields, Dzhumadildaev proved [16] that the only nonstandard critical pair for $n = 1$ is $(2, 5)$. In terms of generating functions in p and q, the 5-commutator is proportional to the following beautiful map

$$(f_1, f_2, f_3, f_4, f_5) \mapsto \det \begin{pmatrix} \partial_q(f_1) & \partial_q(f_2) & \partial_q(f_3) & \partial_q(f_4) & \partial_q(f_5) \\ \partial_p(f_1) & \partial_p(f_2) & \partial_p(f_3) & \partial_p(f_4) & \partial_p(f_5) \\ \partial_p^2(f_1) & \partial_p^2(f_2) & \partial_p^2(f_3) & \partial_p^2(f_4) & \partial_p^2(f_5) \\ \partial_q^2(f_1) & \partial_q^2(f_2) & \partial_q^2(f_3) & \partial_q^2(f_4) & \partial_q^2(f_5) \\ \partial_p\partial_q(f_1) & \partial_p\partial_q(f_2) & \partial_p\partial_q(f_3) & \partial_p\partial_q(f_4) & \partial_p\partial_q(f_5) \end{pmatrix}. \quad (17)$$

Dzhumadildaev's arguments are somewhat involved and lengthy, and the reader is sometimes mystified by typos, but the underlying idea is very simple and brought to the title of [24]: it is a certain odd derivation of a certain superalgebra associated with the problem, which is in the heart of this matter.

Problem 3.3.1. *What are the N-commutators for the Lie algebra of contact* *vector fields* $\mathfrak{k}(2n+1)$?

4. The universal odd derivation and N-commutators

Let L be a Lie (super)algebra, $U(L)$ its enveloping algebra, Π the change of parity functor. Take the associative supercommutative superalgebra $K = S^{\cdot}(\Pi(L))$; in particular, if L is purely even, then K is a Grassmann superalgebra. In L, select an arbitrary basis B and set

$$D = \sum_{b\in B} \Pi(b)\otimes b \in K\otimes L \subset K\otimes U(L).$$

Lemma 4.1. *The N-commutator on L yields an element of L if and only if $D^N \in K\otimes L$. The N-commutator does not vanish identically if and only if $D^N \neq 0$.*

In particular, for $L\subset \mathfrak{vect}(n) = \mathfrak{der}(\mathbb{K}[x])$, we clearly have

$$D\in K\otimes L\subset \mathfrak{vect}(n|n) = \mathfrak{der}(\mathbb{K}[x,\Pi(x)]).$$

The N-commutator on L yields an element of L if and only if $D^N \in \mathfrak{vect}(n|n) = \mathfrak{der}(\mathbb{K}[x,\Pi(x)])$.

Comment. For superspaces, the following modification of Fact (5) takes place:

Fact. *The product of two nonproportional odd vector fields is usually not a vector field, but the square of any odd field is always a vector field.* (18)

Fact (5) is, therefore, a corollary of Fact (18) for $N = 2$. *This is the hidden supersymmetry of the anticommutator mentioned in the title of the paper.*

Conjecture 4.1.1. *We only considered Lie superalgebras of vector fields with polynomial coefficients. We conjecture that the answer will be same for any type of coefficients (at least, if polynomials are dense in the space of coefficients).*

4.2. Discussion and setting of the problem

Usually, attempts to superize a problem or a notion reveal two roads: a straightforward one (not of much interest) and a totally unexpected one. Let us, therefore, consider Dzhumadildaev's problem (describe all critical pairs) for all simple Lie superalgebras of vector fields (with polynomial coefficients to begin with) and see where it will lead us.

To begin with, let us recall steps of Dzhumadildaev's proof.

Let l be the length function on Diff_n defined by Dzhumadildaev, namely:

$$l((\eta_{i_1,\,\alpha_1}\mathrm{del}^{\beta_1})\dots(\eta_{i_k,\,\alpha_k}\mathrm{del}^{\beta_k})) := k.$$

Let us extend l to a grading (Dzhumadildaev's definition is slightly different but equivalent). Note that the possibility of such extension is a little less evident than in the case of $\mathcal{L}_n$ because the elements $\eta_{i_j,\alpha_j}\mathrm{del}_j^{\beta}$ do not supercommute.

Let $X_1,\dots,X_k$ be some abstract vector fields (considered as variables here) of n indeterminates. Define the following map F from $\mathrm{Diff}_n^{[k]}$ to the algebra of differential operators (of arbitrary degree) in n indeterminates:

$$F((\eta_{i_1,\,\alpha_1}\mathrm{del}_1^{\beta})\dots(\eta_{i_k,\,\alpha_k}\mathrm{del}_k^{\beta})) =$$

$$= \sum_{s\in S_k}(-1)^{\mathrm{sign}(s)}((\mathrm{del}^{\alpha_1}X_{s(1),\,i_1})\mathrm{del}^{\beta_1})\dots((\mathrm{del}^{\alpha_k}X_{s(k),\,i_k})\mathrm{del}^{\beta_k}).$$

Here $(\mathrm{del}^{\alpha_j}X_{s(j),\,i_j})$ is a function, $(\mathrm{del}^{\alpha_j}X_{s(j),\,i_j})\mathrm{del}^{\beta_j}$ is a differential operator (possibly of zero degree, if $\beta_j=0$), and the whole term is the composition of differential operators.

Statement 4.3. *The map F is faithful.*

The idea of a proof: the map preserves commutation relations. Note that

a) $F(D^k)$ is just the k-commutator of the X_j (considered as a differential operator of arbitrary degree);

b) the map F preserves the degree of the differential operator (for generic X_i).

So the k-commutator is of degree 1 for any X_j if and only if $\deg D^k = 1$.

5. Appendix: A proof of the classical Amitsur–Levitzki identity

Let A be a supercommutative superalgebra and $X\in\mathrm{Mat}(n|0;\,A)_{\bar 1}$. It is clear that $X^r=0$ for any $r>n^2$. It turns out that r may be diminished considerably.

Proposition 5.1. $X^{2n}=0$ *for* $X\in\mathrm{Mat}(n|0;\,A)_{\bar 1}$.

First of all, let us discuss what does this identity mean from the "ordinary", i.e., nonsuper, algebra point of view. Let C be commutative algebra and $X_1, \dots, X_r \in \mathrm{Mat}(n; C)$. Set $A = C[\xi_1, \dots, \xi_r]$, where the ξ_i are odd and let $X := \sum \xi_i X_i \in \mathrm{Mat}(n|0;\ A)_{\bar{1}}$. Clearly,

$$X^r = a_r(X_1, \dots, X_r)\xi_1 \dots \xi_r, \tag{19}$$

where $a_r(X_1, \dots, X_r) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}\ \sigma} X_{\sigma(1)} \dots X_{\sigma(r)}$.

Hence, Proposition 5 implies the Amitsur–Levitzki identity (3).

Exercise. The Amitsur–Levitzki identity implies Proposition 5.

Proof of Proposition 5. Set $Y = X^2$. The elements of Y belong to the commutative algebra $A_{\bar{0}}$, and therefore, we may consider the characteristic polynomial $P(\lambda) = \det(\lambda 1_n - Y)$ with coefficients in $A_{\bar{0}}$. Let us prove that $P(\lambda) = \lambda^n$. Since the Cayley–Hamilton theorem implies $P(Y) = 0$, we have $Y^n = 0$, i.e., $X^{2n} = 0$. We will prove that $P(\lambda) = \lambda^n$ by three different methods.

1) If char $k = 0$, then the coefficients of $P(\lambda)$ can be expressed in terms of tr Y^r for $r = 1, 2, \dots$. Therefore, it suffices to verify that tr $Y^r = 0$. Indeed,

$$\mathrm{tr}\ Y^r = \mathrm{str}\ \ X^{2r} = \mathrm{str}\ X \cdot X^{2r-1} = -\mathrm{str}\ X^{2r-1} \cdot X = -\mathrm{tr}\ Y^r. \tag{20}$$

Hence, tr $Y^r = 0$ for $r = 1, 2, \dots$.

2) Let us show that $P(\lambda)^2 = \lambda^{2n}$. If 2 is invertible in A, we see that $P(\lambda) = \lambda^n$. We have to show that $\det^2(1_n - \lambda X^2) = 1$. This follows from a more general statement.

Lemma. *Let $U \in \mathrm{Mat}\ (p \times q;\ A)$ and $V \in \mathrm{Mat}\ (q \times p;\ A)$ be matrices whose entries are odd elements of A. Then*

$$\det\ (1_p - UV) = \det\ (1_q - VU)^{-1}. \tag{21}$$

This proof is due to J. Bernstein, 1975. At about the same time V. Drinfeld also noticed the equivalence proved here.

Proof. Let $Z = \begin{pmatrix} 1_p & U \\ V & 1_q \end{pmatrix} \in GL(p|q;\, A)$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\Pi} := \begin{pmatrix} D & C \\ B & A \end{pmatrix}$. From [1] we know that Ber $Z^{\Pi} = (\text{Ber } Z)^{-1}$, so Ber $Z^{\Pi} = \det(1_q - VU)$ because Ber $Z = \det\,(1_p - UV)$.

3) Let $Z = \begin{pmatrix} 1_n & \lambda X \\ \lambda X & 1_n \end{pmatrix} \subset GQ(n;\, A[\lambda])$. From [1] we know that Ber $Z =$ 1. But Ber $Z = \det\,(1_n - \lambda^2 X^2)$, hence, $\det\,(1_n - \lambda^2 Y) = 1$, and we have $\det\,(1_n - \lambda Y) = 1$. Thus, $\det\,(\lambda 1_n - Y) = \lambda^n$.

5.2. How to superize the Cayley-Hamilton theorem?

The degree of the polynomial equation (which) a given $n \times n$ matrix satisfies can be diminished even more compared to what is given by the Amitsur-Levitzki identity (*Cayley–Hamilton theorem*, see (1)).

Problem 5.2.1. *The analog of the Cayley-Hamilton theorem for supermatrices was unknown, except for small values of* n *(equal to* 2 *or* $1|1$*), until recently. Now we have a conjectural formula suggested by the study of quantum algebras and passage to the appropriate "super" limit [9].*

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Алексей Лебедев, Дмитрий Лейтес СКОБКИ ДЖУМАДИЛЬДАЕВА: СКРЫТАЯ СУПЕРСИММЕТРИЯ КОММУТАТОРОВ И ТОЖДЕСТВА АМИЦУРА-ЛЕВИЦКОГО

Тождество Амицура-Левицкого для матриц было обобщено в нескольких направлениях: Костантом для простых конечномерных алгебр Ли, Кирилловым (с Концевичем, Молевым, Овсиенко и Удаловой) для простых алгебр Ли векторных полей с полиномиальными коэффициентами и Гие, Пинкзоном и Ушировирой для ортосимплектической супералгебры Ли $\mathfrak{osp}(1|n)$. Джумадильдаев сдвинул центр интереса в этих результатах, рассмотрев алгебру, образованную антисимметризаторами, и открыл скрытую суперсимметрию коммутаторов. Мы даем обзор этих результатов и рассматриваем их возможные обобщения (открытые проблемы).

Алексей Лебедев, Дмитрий Лейтес ЖҰМАДІЛДАЕВТІҢ ЖАҚШАЛАРЫ: КОММУТАТОРЛАРДЫҢ ЖАСЫРЫН СУПЕРСИММЕТРИЯСЫ ЖӘНЕ АМИЦУР-ЛЕВИЦКИЙДІҢ ТЕПЕ-ТЕҢДІКТЕРІ

Матрицалар үшін Амицур-Левицкийдің тепе-теңдігі бірнеше бағыттарда жалпыланған болатын: жәй ақырлыөлшемді Ли алгебралары үшін Костант, полиномиалды коэффициенттері бар векторлық өрістердің жәй Ли алгебралары үшін Кириллов (Концевич, Молев, Овсиенко және Удаловамен бірге), ал $\mathfrak{osp}(1|n)$ ортосимплексті Ли супералгебрасы үшін Гие, Пинкзон және Уширобира жасады. Жұмаділдаев антисимметризаторлардан құрылған алгебраны қарастырып бұл нәтижелердегі мүдде орталығын жылжытты және коммутаторлардың жасырын суперсимметриясын ашты. Біз осы нәтижелерге шолу жасаймыз және олардың мүмкін жалпылауларын (ашық мәселелерді) қарастырамыз.

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TIME BETWEEN REAL AND IMAGINARY: WHAT GEOMETRIES DESCRIBE UNIVERSE NEAR BIG BANG?

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Annotation: For about a century, a great challenge for theoretical physics consisted in understanding the role of quantum mode of description of our Universe ("quantum gravity"). Einstein space–times on the scale of observable Universe do not easily submit to any naive quantization scheme. There are better chances to concoct a satisfying quantum picture of the very early space–time, near the Big Bang, where natural scales of events like inflation extrapolated from current observations resist any purely classical description and rather require quantum input. Many physicists and mathematicians tried to understand the quantum early Universe, sometimes unaware of input of the other community. One of the goals of this article is to contribute to the communication of the two communities. In the main text, I present some ideas and results contained in the recent survey/research papers [1] (physicists) and [2], [3] (mathematicians).

Keywords: Big Bang, Bianchi IX cosmology, geodesic billiards.

Introduction and survey

0.1. Relativistic models of space–time: Minkowski signature. Most modern mathematical models in cosmology start with description of space-time as a 4-dimensional *pseudo-Riemannian manifold* M endowed with metric

$$ds^2 = \sum g_{ik} dx^i dx^k$$

of signature $(+,-,-,-)$ where $+$ refers to time-like tangent vectors, whereas the infinitesimal light-cone consists of null-directions. Each such manifold is a point in the infinite-dimensional configuration space of cosmological models.

Ключевые слова: *Большой Взрыв, космологические модели Бьянки IX, геодезический биллиард.*

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Basic cosmological models are constrained by *Einstein equations*

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = 8\pi G T_{ik}$$

and/or additional *symmetry postulates*, of which the most essential for us here are the so called *Bianchi IX space-times*, here with symmetry group $SO(3)$, cf. [4] and [5] for a recent context.

In this model, the space–time is fibered over the semi-axis of a global ("cosmological") time t. Fibres are homogeneous spaces over $SO(3)$, and the negative Einstein metric $-ds^2$ induces on them a metric of constant curvature. In order to write ds^2 in convenient coordinates, we choose a fixed time-like geodesic ("observer's history") along which ds^2 is dt^2, and coordinatize each space section at the time t by the invariant distance r from the observer and two natural angle coordinates θ, ϕ on the sphere of radius r. By rescaling the radial coordinate, we may assume that the curvature constant k takes one of three values: $k = \pm 1$ or 0.

This rescaling produces the natural unit of length, when $k \neq 0$, and the respective unit of time is always chosen so that the speed of light is $c = 1$.

The Friedman-Robertson-Walker (FRW) metric is then given by the formula

$$ds^2 := dt^2 - R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + sin^2\theta \, d\phi^2) \right] \tag{0.1}$$

0.2. Input of observations. One of the most counter-intuitive discoveries of the XX-th century cosmology was the "observability" of cosmological time t and possibility to estimate its natural scale ("age of our Universe"). We now know that it is about $14 \cdot 10^9$ years, or five million times longer than the age of human civilisation. Together with considerable homogeneity of the observable space section (local metric disturbances caused by galaxies are counted as negligible) this gives considerable weight to the results of mathematical studies of Bianchi IX $SO(3)$-models.

A robust version of observable global time is the inverse temperature $1/kT$ of the cosmic microwave background (CMB) radiation. It is accepted that the current value of it measures the global age of our Universe starting from the time when it stopped to be opaque for light, about $38 \cdot 10^4$ years after the Big

Bang. Near the Big Bang our Universe was extremely hot, and its evolution is measured by its cooling.

Another version of time is furnished by measurements of the redshift of "standard candles" in observable galaxies, thus putting their current appearance on various cosmological time sections of our Universe (Hubble's Law).

Remarkably, generally accepted physical pictures of the Universe involve also unimaginably small periods of cosmological time: between 10^{-40} and 10^{-30} seconds after the Big Bang the radius of space sections has grown 10^{30} times ("inflation era"), with speed many orders of magnitude exceeding the speed of light. The inflation period is postulated in order to explain the homogeneity of space-time sections of observable Universe (on the scale where galaxies are negligible perturbations).

Last but not least: dynamical equations which must be satisfied by metrics of space-time are defined by the choice of Lagrangian (or Hamiltonian as soon as cosmological time variable is introduced). Besides the metric curvature, this Lagrangian may contain contributions from (models of) massive matter, electro-magnetic field etc. Observations led to the picture of the so called "dark matter" and "dark energy" participating only in gravitational interaction. Their cosmological influence far exceeds that of usual matter, say, content of galaxies. In particular, non–vanishing Einstein's cosmological constant Λ responsible for the "dark energy" effect must explain the observable *accelerating expansion* of the Universe.

For more details, see [6], [7].

0.3. Primeval chaos: going backwards in time. As we have already stressed, in mathematical models of general relativity, the notion of time is local: along each oriented geodesic whose tangent vectors lie inside respective light cones, the differential of its time function dt is ds restricted to this geodesic. Applying this prescription formally, we see that even in a flat space-time, along space-like geodesics time becomes purely imaginary, whereas light-like geodesics along which time "stays still", form a wall. The respective wall-crossing in the space of geodesics produces the Wick rotation of time, from real axis to the pure imaginary axis. Along any light–like geodesic, "real" time stops, however "pure imaginary time flow" makes perfect sense appearing e. g., as a variable in wave-functions of photons.

In the main text, we will describe models (suggested in [2]–[3]) in which *cosmological time becomes imaginary also at the past boundary of the universe* $t = 0$. However, in these models the reverse Wick rotation does not happen instantly. Instead, it includes the movement of time along a random geodesic curve in the complex half plane endowed with its standard hyperbolic metric.

Moreover, the set of all such geodesics (modulo a subgroup of $PSL(2, Z)$) is endowed with much studied invariant measure, and we regard the resulting classical statistical system as an approximation to an (unknown) quantum description of the early Universe.

Our primary motivation (cf. [2]) was the desire to explain the pure formal coincidence of the dynamics of two very different systems:

A. *Mixmaster Universe.* In this model, one studies Bianchi IX $SO(3)$ with metric that in appropriate coordinates takes form $ds^2 = dt^2 - a(t)dx^2 - b(t)dy^2 - c(t)dz^2$, $t > 0$. It turns out that the respective Einstein equations have a family of Kasner's exact solutions $a(t) = t^{p_a}, b(t) = t^{p_b}, c(t) = t^{p_c}$. Moreover, mathematical methods of qualitative studies of dynamical systems suggest that *a generic solution* of the relevant Einstein equations, traced backwards in time towards the Big Bang moment $t = 0$, can be approximated by an infinite sequence of Kasner's solutions.

B. *Hyperbolic billiard.* The relevant dynamical system is the hyperbolic billiard on a standard fundamental domain for $PSL(2, Z)$ (or a finite index subgroup), encoded in the Poincaré return map with respect to the boards of this billiard: see [1], [3], [8], [9].

However, accommodating Mixmaster Universe in the hyperbolic billiard picture seems to require an analytic continuation of Kasner's solutions. It is not known, and according to some computer assisted studies, time in Kasner's models does not admit the necessary analytic continuation involving space–like coordinates as well, cf. [10].

In [3], we avoided this obstacle by looking at the geometry of space–times from the perspective of imaginary time axis. This means that we start with space-times with metrics of the Euclidean signature $(+, +, +, +)$. In the framework of cosmology, they correspond to Bianchi IX $SU(2)$-symmetric space–times, where all coordinates generally can take complex values, so that it makes sense to trace time flow along the relevant geodesics.

0.4. RELATIVISTIC MODELS OF SPACE–TIME: EUCLIDEAN SIGNATURE. In these models, space–times satisfying a complexified version of Einstein equations are Bianchi IX four-dimensional manifolds, fibered over domains of complex plane of time, whose fibres are $SU(2)$-homogeneous spaces (rather than $SO(3)$-homogeneous spaces in the cases of Minkowski signature). By analogy with Yang-Mills instantons, they are sometimes called *gravitational instantons.*

More precisely, consider the $SU(2)$ Bianchi IX model with metric of the form

$$g = F\left(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2}\right). \tag{0.2}$$

Here μ is the relevant version of the cosmological time, (σ_j) are $SU(2)$-invariant forms along space-sections with $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations of $(1, 2, 3)$, and F is a conformal factor.

By analogy with the $SO(3)$ case and metric $dt^2 - a(t)^2dx^2 - b(t)^2dy^2 - c(t)^2dz^2$, in the main text we will treat W_i (as well as some natural monomials in W_i and F) as *$SU(2)$-scaling factors.*

It is important that, contrary to the $SO(3)$-case, generic anti-self-dual Einstein metrics (solutions of Einstein equations) in the $SU(2)$-case can be written explicitly in terms of elliptic modular functions whereas their chaotic behaviour along geodesics in the complex half-plane of time becomes only a reflection of the chaotic behaviour of the respective billiard ball trajectories.

A natural quantisation scheme of gravitational instantons involves non-commutative deformations of their toric space sections. Focussing on this quantisation scheme, in [3] we gave additional arguments about relationship between Mixmaster chaos and quantum mechanics of the Big Bang, but this time not involving Kasner's solutions at all: see section 2 of the main text.

0.5. BOUNDARIES OF SPACE-TIMES. The statement invoked above that the generic $SO(3)$ space-times traced back to $t \to 0$ can be approximated by an infinite sequence of Kasner's solutions is mathematically formulated and proved by considering a partial compactification of the respective phase-spaces and studying the geometry of separatrices on the boundary of a partial compactification of these phase spaces: see [11], [12].

Another type of boundaries was considered in [2], where we tried to produce algebraic–geometric models of Roger Penrose's "aeons": see [13] and [14]–[16]. According to his scheme, the moment $t = 0$ of our cosmological time might have been preceded by evolution of another Universe, the cold death of which was a prequel of our Big Bang. According to Penrose, conformal classes of the respective metrics furnish a continuous transition from the previous aeon to the next one.

Since a conformal change of the metric does not change the relevant light cone in the tangent space at any point of space-time, we suggested in [2] matching pairs of boundaries between aeons, in which the projective compactification of cold Minkowski space-time of previous aeon matches the blown up divisor over the Big Bang point of the next aeon.

***

Cosmology has its own singular place in the body of scientific knowledge: the same quest for the meaning of Universe influences philosophy, poetry, faith (cf. two remarkable books [17], [18] about life, faith and research of Canon Georges Lemaitre, the first discoverer of Hubble's Law and Big Bang picture). I will therefore close this introduction quoting the wonderful lines by Steven Weinberg ([19]):

As I write this I happen to be in an airplane at 30,000 feet, flying over Wyoming en route home from San Francisco to Boston. Below, the earth looks very soft and comfortable–fluffy clouds here and there, snow turning pink as the sun sets, roads stretching straight across the country from one town to another. It is very hard to realize that this is just a tiny part of an overwhelmingly hostile universe. It is even harder to realise that this present universe has evolved from an unspeakably unfamiliar early condition, and faces a future extinction of endless cold or intolerable heat. The more the universe seems comprehensible, the more it also seems pointless.

But if there is no solace in the fruits of our research, there is at least some consolation in the research itself. Men and women are not content to comfort themselves with tales of gods and giants, or to confine their thoughts to the daily affairs of life; they also build telescopes and satellites and accelerators, and sit at their desks for endless hours working out the meaning of the data they gather. The effort to understand the universe is one of the very few things *that lifts human life a little above the level of farce, and gives it some of the grace of tragedy.*

Steven Weinberg. “The first three minutes.”

1. Cosmological time, elliptic integrals, and upper complex half-plane

1.1. Minkowski signature: late Universe. Following [4] and [5], we consider the cosmological time at the late stage of the FRW model (0.1).

It is convenient to replace r in (0.1) by the third dimensionless “angle” coordinate $\chi := r/R(t)$. Then (0.1) becomes

$$ds^2 := dt^2 - R(t)^2 \left[d\chi^2 + S_k^2(\chi)(d\theta^2 + sin^2\theta\, d\phi^2) \right], \tag{1.1}$$

where $S_k(\chi) = sin\,\chi$ for $k = 1$; χ for $k = 0$; and $sinh\,\chi$ for $k = -1$.

This rescaling produces the natural unit of length, when $k \neq 0$, and the respective unit of time is always chosen so that the speed of light is $c = 1$.

Dynamic in this model is described by one real function $R(t)$: it increases from zero at the Big Bang of one aeon to infinity.

We scale $R(t)$ by putting $R = 1$ “now”, as in [4]. Notations in [4] slightly differ from ours. In his formula for metric (2), r is our χ, and $f_k(r)$ is our $S_k(\chi)$.

This function is constrained by the Einstein-Friedman equations (here with cosmological constant $\Lambda = 3$), which leads to the introduction of the elliptic curve given by the equation in the (Y, R)–plane

$$Y^2 = R^4 + aR + b \tag{1.2}$$

(see [4], equation (3), and [5], eq. (9), where their S is the same as our R).

Besides the proper time t, and the scale factor $R(t)$, global time may be measured by its conformal version τ, which according to, formula (3), may be given as the Abelian integral along a real curve on the complex torus, Riemann surface of the elliptic curve (1.2):

$$\tau \cong \int_0^{R(t)} \frac{dR}{Y}. \tag{1.3}$$

Physical interpretation of the coefficients a, b as characterising matter and radiation sources in Einstein equations for this model for which we refer the reader to [4] and [20], shows that in principle a, b also depend on time. Then (1.2) describes a family of elliptic curves parametrized in a way that is classic and well known to algebraic geometers. In particular, cosmological time variable moves along one of the versions of base families of elliptic curves.

Universal families of elliptic curves are parametrized by upper complex half–plane and its quotients (modular curves), and we see now that a family of elliptic curves (1.2) naturally emerges in the description of a late stage of evolution of the FRW model. In a pure mathematical context, the reader is invited to compare our suggestion with the treatment of the Painlevé VI equation in [21] and the whole hierarchy of Painlevé equations in [22].

Now we will discuss a totally different way in which the chaotic evolution in Mixmaster early Universe leads to the appearance of modular curves as well.

1.2. Minkowski signature: early Universe and Mixmaster chaos. As a model of the early universe emerging after the Big Bang we take here the Bianchi IX space–time, admitting $SO(3)$-symmetry of its space-like sections. We will choose coordinates in which its metric takes the following form:

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2, \tag{1.4}$$

where the coefficients $a(t)$, $b(t)$, $c(t)$ are called scaling factors. A family of such metrics satisfying Einstein equations is given by *Kasner solutions*,

$$a(t) = t^{p_1},\ b(t) = t^{p_2},\ c(t) = t^{p_3} \tag{1.5}$$

in which p_i are points on the real algebraic curve

$$\sum p_i = \sum p_i^2 = 1. \tag{1.6}$$

These metrics become singular at $t = 0$ which is the Big Bang moment.

Around 1970, V. Belinskii, I. M. Khalatnikov, E. M. Lifshitz and I. M. Lifshitz argued that almost every solution of the Einstein equations for (1.4) *traced backwards in time* $t \to +0$ can be approximately described by a sequence of solutions (1.5) or equivalently, of points (1.6): see [23] for a later and more comprehensive study. The n–th point of this sequence begins the respective n–th *Kasner era*, at the end of which a jump to the next point occurs, see below.

A mathematically careful treatment of this discovery in [11] has shown that this encoding is certainly applicable to *another dynamical system* which is defined on the boundary of a certain compactification of the phase space of this Bianchi IX model and in a sense is its limit.

Construction of this boundary involves a nontrivial real blow up at the $t = 0$, see details in [12]. The resulting boundary is an attractor, it supports an array of fixed points and separatrices, and the jumps between separatrices which result from subtle instabilities account for jumps between successive Kasner's regimes, corresponding to different points of (1.6).

In what sense this picture approximates the actual trajectories, is a not quite trivial question: cf. the last three paragraphs of the section 2 of [23], where it is explained that among these trajectories there can exist "anomalous" cases when the description in terms of Kasner eras does not make sense, but that they are, in a sense, infinitely rare. See also the recent critical discussion in [10].

Here are some details of the classical description.

(a) Continued fractions. We denote by Z, resp. Z_+, the set of integers, resp. positive integers; Q, resp. R is the field of rational, resp. real numbers. For $x \in R$, we put $[x] := rmax\,\{m \in Z\,|\,m \le x\}$. Irrational numbers $x > 1$ admit the canonical infinite continued fraction representation

$$x = k_0 + \cfrac{1}{k_1 + \frac{1}{k_2 + \dots}} =: [k_0, k_1, k_2, \dots],\ k_s \in Z_+ \tag{1.7}$$

in which $k_0 := [x]$, $k_1 = [1/(x - k_0)]$ etc. Notice that our convention differs from that of [23]: their $[k_1, k_2, \dots]$ means our $[0, k_1, k_2, \dots]$.

(b) Transformation T. The (partial) map $\widetilde{T}:\ [0,1]^2 \to [0,1]^2$ is defined by

$$\widetilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y + [1/x]} \right), \tag{1.8}$$

If both coordinates $(x, y) \in [0,1]^2$ are irrational (the complement is a subset of measure zero), we have for uniquely defined $k_s \in Z_+$:

$$x = [0, k_0, k_1, k_2, \dots],\ y = [0, k_{-1}, k_{-2}, \dots].$$

Then

$$\frac{1}{x} - \left[\frac{1}{x}\right] = [0, k_1, k_2, \dots], \quad \frac{1}{y + [1/x]} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \dots].$$

On this subset, $\widetilde{T}$ is bijective and has invariant density

$$\frac{dx\,dy}{rln\,2 \cdot (1 + xy)^2}$$

(cf. [24]). Thus we may and will bijectively encode irrational pairs $(x, y) \in [0, 1]^2$ by doubly infinite sequences

$$(k) := [\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots], k_i \in Z_+$$

in such a way that the map $\widetilde{T}$ above becomes the shift of such a sequence denoted T:

$$T(k)_s = k_{s+1}. \tag{1.9}$$

(c) Continued fractions and Mixmaster chaos. Any point (p_a, p_b, p_c) in (1.6) can be obtained by choosing a unique $u \in [1, \infty]$, putting

$$p_1^{(u)} := -\frac{u}{1 + u + u^2} \in [-1/3, 0], \; p_2^{(u)} := \frac{1 + u}{1 + u + u^2} \in [0, 2/3],$$

$$p_3^{(u)} := \frac{u(1 + u)}{1 + u + u^2} \in [2/3, 1] \tag{1.10}$$

and then rearranging the exponents $p_1^{(u)} \le p_2^{(u)} \le p_3^{(u)}$ by a bijection $(1, 2, 3) \to (a, b, c)$.

As we have already explained, a "typical" solution γ of Einstein equations (vacuum, or with various energy momentum tensors) with $SO(3)$–symmetry of the Bianchi IX type, followed from an arbitrary (small) value $t_0 > 0$ in the reverse time direction $t \to +0$, oscillates close to a sequence of Kasner type solutions.

Somewhat more precisely, introduce the local logarithmic time Ω along this trajectory with inverted orientation. Its differential is $d\Omega := -\dfrac{dt}{abc}$, and the time itself is counted from an arbitrary but fixed moment. Then $\Omega \to +\infty$ approximately as $-\log t$ as $t \to +0$, and we have the following picture.

As $\Omega \cong -\log t \to +\infty$, a "typical" solution γ of the Einstein equations determines a sequence of infinitely increasing moments $\Omega_0 < \Omega_1 < \dots < \Omega_n < \dots$ and a sequence of irrational real numbers $u_n \in (1, +\infty)$, $n = 0, 1, 2, \dots$.

The time semi–interval $[\Omega_n, \Omega_{n+1})$ is called the *n–th Kasner era* for the trajectory γ (in [1], our eras are called epochs). Within the n–th era, the evolution of a, b, c is approximately described by several consecutive Kasner's formulas. Time intervals where scaling powers (p_i) are constant are called *Kasner's cycles* (in [1], our cycles are called eras).

The evolution in the n–th era starts at time Ω_n with a certain value $u = u_n > 1$ which determines the sequence of respective scaling powers during the first cycle (1.10):

$$p_1 = -\frac{u}{1+u+u^2},\ p_2 = \frac{1+u}{1+u+u^2},\ p_3 = \frac{u(1+u)}{1+u+u^2}$$

The next cycles inside the same era start with values $u = u_n - 1$, $u_n - 2, \dots$, and scaling powers (1.10) corresponding to these numbers, rearranged corresponding to a bijection $(1, 2, 3) \to (a, b, c)$ which is in turn identical to the previous one, or interchanges b and c (see [1] or [25] for a modular interpretation).

After $k_n := [u_n]$ cycles inside the current era, a jump to the next era comes, with parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}. \tag{1.11}$$

Moreover, ensuing encoding of γ's and respective sequences (u_i)'s by continued fractions (1.7) of real irrational numbers $x > 1$ is bijective on the set of full measure.

Finally, when we want to include into this picture also the sequence of logarithmic times Ω_n starting new eras, we naturally pass to the two–sided continued fractions and the transformationn T. Here are some details.

(d) Doubly infinite sequences and modular geodesics. Let $H := \{z \in C,\ rIm\, z > 0\}$ be the upper complex half–plane with its Poincaré metric

$|dz|^2/|rIm\,z|^2$. Denote also by $\overline{H} := H \cup \{Q \cup \{\infty\}\}$ this half–plane completed with cusps.

The vertical lines $rRe\,z = n, n \in Z$, and semicircles in $\overline{H}$ connecting pairs of finite cusps $(p/q, p'/q')$ with $pq' - p'q = \pm 1$, cut $\overline{H}$ into the union of geodesic ideal triangles which is called the *Farey tessellation*.

Following [8], [9], consider the set of *oriented* geodesics β's in H with ideal irrational endpoints in R. Let $\beta_{-\infty}$, resp. β_∞ be the initial, resp. the final point of β. Let B be the set of such geodesics with $\beta_{-\infty} \in (-1, 0)$, $\beta_\infty \in (1, \infty)$. Put

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \dots], \quad \beta_\infty = [k_1, k_2, k_3, \dots], \quad k_i \in Z_+, \tag{1.12}$$

and encode β by the doubly infinite continued fraction

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]. \tag{1.13}$$

The geometric meaning of this encoding can be explained as follows. Consider the intersection point $x = x(\beta)$ of β with the imaginary semiaxis in H. Moving along β from x to β_∞, one will intersect an infinite sequence of Farey triangles. Each triangle is entered through a side and left through another side, leaving the ideal intersection point (a cusp) of these sides either to the left, or to the right. Then the infinite word in the alphabet $\{L, R\}$ encoding the consecutive positions of these cusps wrt β will be $L^{k_1}R^{k_2}L^{k_3}R^{k_4}\dots$ Similarly, moving from $\beta_{-\infty}$ to x, we will get the word (infinite to the left) $\dots L^{k_{-1}}R^{k_0}$.

We can enrich the new notation $\dots L^{k_{-1}}R^{k_0}L^{k_1}R^{k_2}L^{k_3}R^{k_4}\dots$ (called *cutting sequence* of our geodesic in [9]) by inserting between the consecutive powers of L, R notations for the respective intersection points of β with the sides of Farey triangles. So $x_0 := x = x(\beta)$ will be put between R^{k_0} and L^{k_1}, and generally we can imagine the word

$$\dots L^{k_{-1}}x_{-1}R^{k_0}x_0L^{k_1}x_1R^{k_2}x_2L^{k_3}x_3R^{k_4}\dots \tag{1.14}$$

Since the Farey tessellation is acted upon by the modular group $PSL(2, Z)$ and its hyperbolic extension including orientation changing isometries of H, we may present another version of the geometric description of geodesic flow. This is an equivalent dynamical system which is the triangular hyperbolic billiard with infinitely distant corners ("pockets"): see [1], [3], [8], [9].

Here we use the term "hyperbolic" in order to indicate that sides (boards) of the billiard and trajectories of the ball ("particle") are geodesics with respect to the hyperbolic metric of constant curvature -1 of the billiard table. This is not the standard meaning of the hyperbolicity in this context, where it usually refers to non–vanishing Lyapunov exponents.

(e) Proposition. *All hyperbolic triangles of the Farey tessellation of $\overline{H}$ are isomorphic as metric spaces.*

For any two closed triangles having a common side there exists unique metric isomorphism of them identical along this side. It inverts orientation induced by H. Starting with the basic triangle Δ with vertices $\{0, 1, i\infty\}$ and consecutively using these identifications, one can unambiguously define the map $b: \overline{H} \to \Delta$.

Any oriented geodesic on H with irrational end–points in R is sent by the map b to a billiard ball trajectory on the table Δ never hitting corners.

All this is essentially well known since at least [8].

It is also worth noticing that although all three sides of Δ are of infinite length, this triangle is *equilateral* in the following sense: there exists a group S_6 of hyperbolic isometries of Δ acting on vertices by arbitrary permutations. This group has a unique fixed point $\rho := \exp(\pi i/3)$ in Δ, *the centroid* of Δ.

In fact, this group is generated by two isometries: $z \mapsto 1 - z^{-1}$ and symmetry with respect to the imaginary axis.

Three finite geodesics connecting the centre ρ with points $i, 1+i, \frac{1+i}{2}$ respectively, subdivide Δ into three geodesic quadrangles, each having one infinite (cusp) corner. We will call these points *centroids* of the respective sides of Δ, and the geodesics (ρ, i) etc. *medians* of Δ.

Each quadrangle is the fundamental domain for $PSL(2, Z)$.

(f) Billiard encoding of oriented geodesics. Consider the first stretch of the geodesic β encoded by (1.14) that starts at the point x_0 in $(0, i\infty)$. If $k_0 = 1$,the ball along β reaches the opposite side $(1, i\infty)$ and gets reflected to the third side $(0, 1)$. If $k_0 = 2$, it reaches the opposite side, then returns to the initial side $(0, i\infty)$, and only afterwards gets reflected to $(0, 1)$.

More generally, the ball always spends k_0 unobstructed stretches of its trajectory between $(0, i\infty)$ and $(1, i\infty)$, but then is reflected to $(0, 1)$ either from $(1, i\infty)$ (if k_0 is odd), or from $(0, i\infty)$ (if k_0 is even). We can encode this sequence of stretches by the formal word ∞^{k_0} showing exactly how many times the ball is reflected "in the vicinity" of the pocket $i\infty$, that is, does not cross any of the medians.

A contemplation will convince the reader that this allows one to define an alternative encoding of β by the double infinite word in *three letters* , say a, b, c, serving as names of the vertices $\{0, 1, i\infty\}$.

(g) Kasner's eras in logarithmic time and doubly infinite continued fractions. Now we will explain, how the double infinite continued fractions enter the Mixmaster formalism when we want to mark the consecutive Kasner eras upon the t–axis, or rather upon the Ω-axis, where $\Omega := -\, r\log \int dt/abc$

In the process of construction, these continued fractions will also come with their enrichments.

We start with fixing a "typical" space-time γ whose evolution with $t \to +0$ undergoes (approximately) a series of Kasner's eras described by a continued fraction $[k_0, k_1, k_2, \dots]$, where k_s is the number of Kasner's cycles within s–th era $[\Omega_s, \Omega_{s+1})$. We have enriched this encoding by introducing parameters u_s which determine the Kasner exponents within the first cycle of the era number s by (1.5). A further enrichment comes with putting these eras on the Ω–axis. According to [11], [12], [23], if one defines the sequence of numbers δ_s from the relations

$$\Omega_{s+1} = [1 + \delta_s k_s (u_s + 1/\{u_s\})]\Omega_s,$$

then complete information about these numbers can be encoded by the extension to the left of our initial continued fraction:

$$[\dots, k_{-1}, k_0, k_1, k_2, \dots] \tag{1.15}$$

in such a way that

$$\delta_s = x_s^+/(x_s^+ + x_s^-)$$

where

$$x_s^+ = [0, k_s, k_{s+1}, \dots], \quad x_s^- = [0, k_{s-1}, k_{s-2}, \dots]. \tag{1.16}$$

The following result established in [3] shows that cosmological time can be approximately measured in terms of geodesic length of path of the billiard ball.

1.3. Theorem. *Let a "typical" Bianchi IX Mixmaster Universe be encoded by the double–sided sequence (1.15). Consider also the respective geodesic in H with its enriched encoding (1.14).*

Then we have "asymptotically" as $s \to \infty$, $s \in Z_+$:

$$r\log \Omega_{2s}/\Omega_0 \simeq 2 \sum_{r=0}^{s-1} rdist\,(x_{2r}, x_{2r+1}), \tag{1.17}$$

where $rdist$ denotes the hyperbolic distance between the consecutive intersection points of the geodesic with sides of the Farey tesselation as in (1.14).

The formula (1.17) shows that the distance measured along a geodesic can be compared to (doubly) logarithmic cosmological time.

During the stretch of *time/geodesic length* which such a geodesic spends in the vicinity of a vertex of Δ, the respective space–time in a certain sense can be approximated by its degenerate version, corresponding to the vertex itself, and this will justify considering below the respective segments of geodesics as the "instanton Kasner eras".

1.4. Riemannian signature: Bianchi IX models with $SU(2)$–symmetry. Consider the $SU(2)$ Bianchi IX model with metric of the form

$$g = F\left(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2}\right). \tag{1.18}$$

Here μ is cosmological time, (σ_j) are $SU(2)$–invariant forms along space–sections with $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations of $(1, 2, 3)$, and F is a conformal factor.

By analogy with the $SO(3)$-case and metric $dt^2 - a(t)^2dx^2 - b(t)^2dy^2 - c(t)^2dz^2$, we may and will treat W_i (as well as some natural monomials in W_i and F) as $SU(2)$-*scaling factors.*

However, contrary to the $SO(3)$-case, generic solutions of Einstein equations in the $SU(2)$-case can be written explicitly in terms of elliptic modular functions, whereas their chaotic behaviour along geodesics in the complex half-plane of time is only a reflection of the chaotic behaviour of the respective billiard ball trajectories.

We will use explicit formulas given in [26], where they were deduced from the basic results of [27]. The central role in them is played by theta-functions depending on the the complex arguments $i\mu \in H$, $z \in C$, with parameters (p, q) called theta–characteristics:

$$\vartheta[p,q](z,i\mu) := \sum_{m \in Z} \exp\{-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\}. \tag{1.19}$$

It can be expressed through the theta-function with vanishing characteristics:

$$\vartheta[p,q](z,i\mu) = \exp\{-\pi p^2\mu + 2\pi ipq\} \cdot \vartheta[0,0](z + pi\mu + q, i\mu). \tag{1.20}$$

All these functions satisfy classical automorphy identities with respect to the action of $PGL(2, Z)$.

Theorem 1 ([26], [27], [28]). *Put*

$$\vartheta[p,q] := \vartheta[p,q](0, i\mu) \tag{1.21}$$

and

$$\vartheta_2 := \vartheta[1/2, 0], \ \vartheta_3 := \vartheta[0,0], \ \vartheta_4 := \vartheta[0, 1/2]. \tag{1.22}$$

(A) Consider the following scaling factors as functions of μ with parameters (p, q):

$$W_1 := \frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\delta}{\delta q}\vartheta[p, q+1/2]}{e^{\pi ip}\vartheta[p,q]}, \ W_2 := \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\delta}{\delta q}\vartheta[p+1/2, q+1/2]}{e^{\pi ip}\vartheta[p,q]},$$

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\delta}{\delta q}\vartheta[p+1/2, q]}{\vartheta[p,q]}, \tag{1.23}$$

Moreover, define the conformal factor F with non-zero cosmological constant Λ by

$$F := \frac{2}{\pi\Lambda} \frac{W_1 W_2 W_3}{(\frac{\delta}{\delta q}\log\vartheta[p,q])^2}. \tag{1.24}$$

The metric (1.18) with these scaling factors for real $\mu > 0$ is real and satisfies the Einstein equations if either

$$\Lambda < 0, \ p \in R, \ q \in \frac{1}{2} + iR, \tag{1.25}$$

or

$$\Lambda > 0,\ q \in R,\ p \in \frac{1}{2} + iR. \tag{1.26}$$

(B) Consider now a different system of scaling factors

$$W_1' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log \vartheta_2,\ W_2' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log \vartheta_3,$$

$$W_3' := \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log \vartheta_4 \tag{1.27}$$

and

$$F' := -C(\mu + q_0)^2\, W_1'W_2'W_3', \tag{1.28}$$

where $q_0, C \in R$, $C > 0$. The metric (1.18) with these scaling factors for real $\mu > 0$ is real and satisfies the Einstein equations with vanishing cosmological constant.

We will now consider values of $i\mu \in \Delta \subset \overline{H}$ in the vicinity of $i\infty$ but not necessarily lying on the imaginary axis. Since we are interested in the instanton analogs of Kasner's solutions, we will collect basic facts about asymptotics of scaling factors for $i\mu \to i\infty$.

For brevity, we will call a number $r \in R$ *general*, if $r \notin Z \cup (1/2 + Z)$.

For such r, denote by $\langle r\rangle \in (-1/2, 0) \cup (0, 1/2)$ such real number that $r + m_0 = \langle r\rangle$ for a certain (unique) $m_0 \in Z$.

Theorem 2. *The scaling factors of the Bianchi IX spaces listed in Theorem 1.5 have the following asymptotics near* $\mu = +\infty$*:*

(i) For $\Lambda = 0$*:*

$$W_1' \sim -\frac{\pi}{2}, \quad W_2' \sim W_3' \sim \frac{1}{\mu + q_0}. \tag{1.29}$$

(ii) For $\Lambda < 0$ *and general* p*:*

$$W_1 \sim -\pi\langle p\rangle \exp\{\pi i(\langle p\rangle - p)\},\ W_2 \sim \pm W_3,$$

$$W_3 \sim -2\pi i\,\langle p + 1/2\rangle \cdot \exp\{\pi i\, sgn\,\langle p\rangle q\} \cdot \exp\{\pi\mu(|\langle p\rangle| - 1/2)\}. \tag{1.30}$$

(iii) For $\Lambda > 0$*, real* q *and* $p = 1/2 + ip_0, p_0 \in R$*:*

$$W_1 \sim \pi p_0 \tan\{\pi(q - p_0\mu)\} - \frac{1}{2}, \quad W_2 \sim -W_3,$$
$$W_3 \sim 2\pi p_0 \cdot (\cos \pi(q - p_0\mu))^{-1}. \tag{1.31}$$

Theorem 1.6 (proved in [3]) shows that for general members of all solution families from [26], after eventual sign changes of some W_i's and outside of the pole singularities on the real time axis, we have asymptotically $W_2 = W_3$, $W_1 \neq W_2$.

In the next section, we will show that precisely such a condition allows one to quantize the respective geometric picture in terms of Connes–Landi [29]. This gives additional substance to our vision that chaotic Mixmaster evolution along hyperbolic geodesics reflects a certain "dequantization" of the hot quantum early Universe.

1.7. Gravitational instantons and Painlevé VI.

Hitchin's classification of gravitational instantons that led to Theorem 1.5 was based upon the reduction of the relevant Einstein equations to a Painlevé VI equation. We will briefly recall basics facts about them; see [22] for a more general context.

Equations of the type Painlevé VI form a four-parametric family. Denote parameters $(\alpha, \beta, \gamma, \delta)$, and the independent variable by t. The corresponding equation for a function $X(t)$ looks as follows:

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$
$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right].$$

Gravitational instantons correspond to the case

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}).$$

Solutions in elliptic functions of this equation describe Bianchi IX space-times with SU(2)-symmetry: see [27].

One more interesting case is $(\alpha, \beta, \gamma, \delta) = (\frac{9}{2}, 0, 0, \frac{1}{2})$. According to B. Dubrovin, a specific solution of this equation describes "the mirror of P^2" in a general context of Mirror Symmetry.

In 1907, R. Fuchs has rewritten PVI in the form

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}\right]\int_{\infty}^{(X,Y)} \frac{dx}{\sqrt{x(x-1)(x-t)}} =$$

$$= \alpha Y + \beta\frac{tY}{X^2} + \gamma\frac{(t-1)Y}{(X-1)^2} + (\delta - \frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}. \qquad (1.32)$$

Here he enhanced $X := X(t)$ to $(X, Y) := (X(t), Y(t))$ treating the latter pair as a section $P := (X(t), Y(t))$ of the generic elliptic curve $E = E(t):\ Y^2 = X(X-1)(X-t)$.

Up to a simple change of notations, the abelian integral $\int_{\infty}^{(X,Y)}$ in (1.32) can be directly identified with the abelian integral $\int_0^{R(t)}$ in (1.3) so that this integral is a version of cosmological time. The meaning of the right hand side of (1.32) was clarified in my paper [21]. After having noticed that Painlevé VI can be written on any one–dimensional family of elliptic curves (its dependent variable becoming a (multi)section of such a family), I have applied this remark to the analytic family $E_\tau := C/(Z + Z\tau) \mapsto \tau \in H$. Denoting by z a fixed coordinate on C, we can rewrite (1.32) in the form

$$\frac{d^2z}{d\tau^2} = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j \wp_z(z + \frac{T_j}{2}, \tau) \qquad (1.33)$$

Here $(\alpha_0, \dots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$, $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1+\tau)$, and

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2}\right).$$

Moreover, equation of the generic elliptic curve becomes

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2}, \tau),$$

so that $e_1 + e_2 + e_3 = 0$.

Since $PGL(2, Z)$ acts on the total space of this family, the "time variable" τ (an abelian integral along closed path on a curve) can be restricted to the fundamental domain of this group or its finite index subgroup, and this leads to the hyperbolic billiard picture.

2. Quantum Big Bang?

2.1. Canonical quantisation of the billiard system and Maass forms. The most straightforward way to produce from the Mixmaster chaotic system its quantum version consists in applying canonical quantisation to the billiard ball moving in one of the version of hyperbolic billiard table discussed above.

This immediately leads to the consideration of Maass wave functions: eigenvectors Ψ of the Laplace-Beltrami operator on the hyperbolic half-plane, invariant with respect to an appropriate subgroup of the (extended) modular group. They play now role of quantum wave-functions of early hot Universe.

We refer to [1], sec. VI and VII, for a detailed discussion of this quantisation scheme and relevant references. See also [30].

Below we will discuss a different quantisation scheme, developed in the framework of non commutative geometry (cf. [29]). We will then connect it with the complex geometry of gravitational instantons, described in subsections 1.4-1.6 above. This was done in our article [3].

2.2. Theta deformations. In Section 5 of [2] we showed that the gluing of space-times across the singularity using an algebro-geometric blowup can be made compatible with the idea of spacetime coordinates becoming noncommutative in a neighborhood of the initial singularity where quantum gravity effects begin to dominate.

This compatibility is described there in terms of Connes-Landi theta deformations [29] and Cirio-Landi-Szabo toric deformations [31]–[33] of Grassmannians.

It turns out that the Bianchi IX models with $SU(2)$-symmetry can be made compatible with the hypothesis of noncommutativity at the Planck scale, using isospectral theta deformations.

The metrics on the S^3 sections, in this case, are only left $SU(2)$-invariant. It turns out that among all the $SU(2)$ Bianchi IX spacetime, the only ones that admit isospectral theta–deformations of their spatial S^3-sections are those where the metric tensor

$$g = w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1} \, \sigma_1^2 + \frac{w_1 w_3}{w_2} \, \sigma_2^2 + \frac{w_1 w_2}{w_3} \, \sigma_3^2 \tag{2.2}$$

is of the special form satisfying $w_1 \neq w_2 = w_3$ (the two directions σ_2 and σ_3 have equal magnitude). In these metrics, the S^3 sections are Berger spheres. This class includes the general Taub-NUT family [34], [35], and the Eguchi–Hanson metrics [36], [37]. The theta–deformations are obtained, as in the case of the deformations S^3_θ of [29] of the round 3-sphere, by deforming all the tori of the Hopf fibration to noncommutative tori.

In other words, a Bianchi IX Euclidean spacetime X with $SU(2)$-symmetry admits a noncommutative theta-deformation X_θ, obtained by deforming the tori of the Hopf fibration of each spacial section S^3 to noncommutative tori, if and only if its metric has the $SU(2) \times U(1)$-symmetric form

$$g = w_1 w_3^2 \, d\mu^2 + \frac{w_3^2}{w_1} \, \sigma_1^2 + w_1 \, (\sigma_2^2 + \sigma_3^2). \tag{2.3}$$

(see [3], Proposition 4.2).

This is in stark contrast with the situation described in [38], where (Lorentzian and Euclidean) Mixmaster cosmologies of the form

$$\mp dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

were considered, with T^3-spatial sections, which always admit isospectral theta-deformations (see also [39], [40]).

We have recalled in the previous section how the general self-dual solutions (with $w_1 \neq w_2 \neq w_3$) can be written explicitly in terms of theta constants [26], and are obtained from a Darboux-Halphen type system [41]. In the case of the family of Bianchi IX models with $SU(2) \times U(1)$-symmetry, this system has algebraic solutions that give

$$w_2 = w_3 = \frac{1}{\mu - \mu_0}, \quad w_1 = \frac{\mu - \mu_*}{(\mu - \mu_0)^2}, \tag{2.4}$$

with singularities at μ_* (curvature singularity), μ_0 (Taubian infinity) and ∞ (nut). The condition $\mu_* < \mu_0$ avoids naked singularities, by hiding the curvature singularity at μ_* behind the Taubian infinity, see the discussion in Section 5.2 of [41].

Consider the operator

$$D_B = -i \begin{pmatrix} \frac{1}{\lambda} X_1 & X_2 + iX_3 \\ X_2 - iX_3 & -\frac{1}{\lambda} X_1 \end{pmatrix}, \tag{2.5}$$

where $\{X_1, X_2, X_3\}$ constitute a basis of the Lie algebra orthogonal for the bi-invariant metric. Assume moreover that the left-invariant metric on S^3 is diagonal in this basis, with eigenvalues $\{w^2/w_1, w_1, w_1\}$, with $w = w_2 = w_3$ and $\lambda = w/w_1$, and where the w_i are as in (4.4). Consider also the operator

$$D = \frac{1}{w_1^{1/2} w} \left(\gamma^0 \left(\frac{\partial}{\partial \mu} + \frac{1}{2} \left(\frac{\dot{w}}{w} + \frac{1}{2} \frac{\dot{w}_1}{w_1} \right) \right) + w_1 \; D_B|_{\lambda = \frac{w}{w_1}} \right). \tag{2.6}$$

2.3. Proposition. *The operators D of (2.6) give Dirac operators for isospectral theta deformations of the $SU(2) \times U(1)$-symmetric space-times.*

As in [38], the Dirac operator of Proposition 2.3 can be seen as involving an anisotropic Hubble parameter H. In the case of the metrics of [38] this was of the form

$$H = \frac{1}{3} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right)$$

with a, b, c the scaling factors in (2.3).

In the case of the $SU(2)$ Bianchi IX models, the anisotropic Hubble parameter is again of the form $H = \frac{1}{3}(H_1 + H_2 + H_3)$, where now the H_i correspond to the three directions of the vectors dual to the $SU(2)$-forms σ_i in (2.2). For a metric of the form (2.3), or equivalently

$$g = uw \, d\mu^2 + u^2 \lambda^2 \, \sigma_1^2 + u^2 \sigma_2^2 + u^2 \sigma_3^2,$$

we take the anisotropic Hubble parameter to be

$$H = \frac{1}{3} \left(\frac{\dot{u}\lambda + u\dot{\lambda}}{u\lambda} + 2\frac{\dot{u}}{u} \right) = \frac{1}{3} \left(3\frac{\dot{u}}{u} + \frac{\dot{\lambda}}{\lambda} \right),$$

where

$$\frac{\dot{u}}{u} = \frac{1}{2} \frac{\dot{w}_1}{w_1}, \quad \frac{\dot{\lambda}}{\lambda} = \frac{\dot{w}}{w} - \frac{\dot{w}_1}{w_1},$$

so that

$$H = \frac{1}{3}\left(\frac{\dot{w}}{w} + \frac{1}{2}\frac{\dot{w}_1}{w_1}\right),$$

so that we can write the 4–dimensional Dirac operator in the form

$$D = \gamma^0 \frac{1}{uw}\left(\frac{\partial}{\partial\mu} + \frac{3}{2}H\right) + D_B,$$

where $D_B = (w_1^{1/2}/w)\ D_B|_{\lambda=\frac{w}{w_1}}$ is the Dirac operator on the spatial sections S^3 with the left $SU(2)$-invariant metric.

Notice that in the construction above, we have considered the same modulus θ for the noncommutative deformation of all the spatial sections S^3 of the Bianchi IX spacetime, but one could also consider a more general situation where the parameter θ of the deformation is itself a function of the cosmological time μ.

This would allow the dependence of the noncommutativity parameter θ on the energy scale (or on the cosmological timeline), with $\theta = 0$ away from the singularity where classical gravity dominates and noncommutativity only appearing near the singularity. Since a non-constant, continuously varying parameter θ crosses rational and irrational values, this would give rise to a Hofstadter butterfly type picture, with both commutativity (up to Morita equivalence, as in the rational noncommutative tori) and true noncommutativity (irrational noncommutative tori), cf. also [42].

Another interesting aspect of these noncommutative deformations is the fact that, when we consider a geodesic in the upper half plane encoding Kasner eras in a mixmaster dynamics, the points along the geodesic also determine a family of complex structures on the noncommutative tori T^2_θ of the theta-deformation of the respective spatial section.

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Манин Юрий Иванұлы ЖАЛҒАН ЖӘНЕ НАҚТЫ УАҚЫТ АРАЛЫҒЫНДА: ҚАНДАЙ ГЕОМЕТРИЯЛАР ҮЛКЕН ЖАРЫЛЫС МАҢЫНДАҒЫ ҒАЛАМДЫ СИПАТТАЙДЫ?

Соңғы жүз жыл ішінде теориялық физиканың басты мәселелерінің бірі салыстырмалылықтың жалпы теориясы мен кванттық теорияның бірігуі болып отыр, ол атап айтқанда, ертедегі ғаламның құрылымы мен эволюциясын ұғыну үшін және оның бүгінгі жағдайына көшу үшін қажет. Бұл мақаланың мақсаттарының бірі - осы мәселемен айналысатын физиктер мен математиктердің ықпалдасуына үлес қосу болып саналады. Жұмыстың мазмұнды бөлігі ертедегі "таза кванттық" сатыдан Эйнштейн теңдеуімен анықталатын, қазіргі заманғы классикалық эволюцияға көшу моделін тұрғызудан тұрады.

Манин Юрий Иванович МЕЖДУ МНИМЫМ И ВЕЩЕСТВЕННЫМ ВРЕМЕНЕМ: КАКИЕ ГЕОМЕТРИИ ОПИСЫВАЮТ ВСЕЛЕННУЮ ВБЛИЗИ БОЛЬШОГО ВЗРЫВА?

В течение последних ста лет, одной из центральных задач теоретической физики остается соединение общей теории отчосительности и квантовой теории, необходимое, в частности, для понимания струкруры и эволюции ранней Вселенной и перехода к ее нынешнему состоянию. Одна из целей этой статьи - внести вклад в сотрудничество физиков и математиков, занимающихся этой проблемой. Содержательная часть работы состоит в построении модели перехода от ранней "чисто квантовой"фазы к современной классической эволюции, определяемой уравнением Эйнштейна.

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SKEW-SYMMETRIC IDENTITIES OF FINITELY GENERATED MALCEV ALGEBRAS

I. Shestakov

Dedicated to Prof. Askar Dzhumadildaev on the occasion of his 60-th anyversary

Annotation: We prove that every multilinear skew-symmetric polynomial on more then $3 + C_n^1 + C_n^2 + C_n^3$ variables vanishes in any n-generated Malcev algebra over a field of characteristic 0. Before a similar result was known only for a series of skew-symmetric polynomials g_m of special type on $2m + 1$ variables constructed by the author in [1], where $m > [\frac{n^3+5n}{12}]$

Keywords: Malcev algebra, basic rank, superalgebra, skew-symmetric identities.

1. Introduction

Let $\mathcal{V}$ Denote by $\mathcal{V}_n$ the subvariety of $\mathcal{V}$ generated by the $\mathcal{V}$-free algebra on n free generators; then we have

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots, \quad \mathcal{V} = \cup_n \mathcal{V}_n.$$

A minimal number n for which $\mathcal{V}_n = \mathcal{V}$ (if such a number exists) is called *the basic rank* of the variety $\mathcal{V}$ and is denoted as $r_b(\mathcal{V})$ (see [2]). If $\mathcal{V} \neq \mathcal{V}_n$ for any n then the basic rank of $\mathcal{V}$ is called to be infinite.

It was first shown by A.I. Malcev that the basic rank of the variety of associative algebras is equal to 2 [3, p. 331]. A.I. Shirshov proved that the same result is true for the variety of Lie algebras [4] and the variety generated by the special Jordan algebras [5]. In [6, problem 1.159], A.I. Shirshov posed a problem on the basic rank of the varieties of alternative Alt, Jordan Jord, Malcev Mal,

Ключевые слова: *Алгебра Мальцева, базисный ранг, супералгебра, кососимметричные тождества.*

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and other algebras. In [1] the author constructed a series of multilinear skew-symmetric polynomials g_m on $2m+1$ variables over a field of characteristic $\neq 2$ which vanish in any n-generated alternative or Malcev algebra for $m > [\frac{n^3+5n}{12}]$, but do not vanish in free alternative and free Malcev algebras of infinite rank. In particular, this implies that $r_b(Alt)$ and $r_b(Mal)$ are infinite.

Here we prove more general result for Malcev algebras. Namely, let $N = 3 + C_n^1 + C_n^2 + C_n^3$, then *EVERY* multilinear skew-symmetric polynomial on more then N variables vanishes in any n-generated Malcev algebra.

Observe that in case of Lie algebras any skew-symmetric multilinear polynomial on 3 or more variables vanishes in any Lie algebra. In case of Malcev algebras, the condition of finite generation cannot be omitted since the polynomials g_m mentioned above do not vanish in the free Malcev algebra of infinite rank for any m.

For the proof, we use the properties of the polynomials g_m from [1] and the base of the free Malcev superalgebra on one odd generator constructed in [7].

Throughout the paper, if otherwise is not stated, all algebras are considered over a field of characteristic 0.

2. Skew-symmetric elements and one-odd-generated superalgebras

Let us recall the definition of a superalgebra in a given variety of algebras $\mathcal{V}$. In general, a superalgebra A is just a Z_2-graded algebra: $A = A_0 \oplus A_1$, where $A_iA_j \subseteq A_{i+j\,(mod\,2)}$. Let $G = \mathrm{alg}\langle 1, e_1, \dots, e_n, \dots \,|\, e_ie_j = -e_je_i\rangle$ be the Grassmann superalgebra, where G_0 is spanned by 1 and the even products $e_{i_1}e_{i_2}\cdots e_{i_k}$, $1 < i_1 < i_2 < \cdots < i_k$, k even, and G_1 is spanned by the odd products $e_{i_1}e_{i_2}\cdots e_{i_k}$, $1 < i_1 < i_2 < \cdots < i_k$, k odd. Then a superalgebra $A = A_0 + A_1$ is called a *$\mathcal{V}$-superalgebra* if its *Grassmann envelope* $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$, considered as an algebra, belongs to $\mathcal{V}$.

When working with superalgebras, we always assume that considered elements are homogeneous, that is, *even* (belonging to A_0) or *odd* (belonging to A_1). If the defining identities of the variety $\mathcal{V}$ are known, one can easily write down the defining *super-identites* for the variety of $\mathcal{V}$-superalgebras. For example, an algebra A is called *alternative* (see [8]) if it satisfies the identities

$$(x, y, y) = 0, \quad (x, x, y) = 0,$$

where $(x,y,z)=(xy)z-x(yz)$ denotes the *associator* of the elements x,y,z. Now, a superalgebra $A=A_0\oplus A_1$ is called an *alternative superalgebra* if it satisfies the super-identities

$$(x,y,z)+(-1)^{|y||z|}(x,z,y)=0,$$
$$(x,y,z)+(-1)^{|x||y|}(y,x,z)=0,$$

where $|u|=i$ if $u\in A_i$ denotes the parity of a homogeneous element u. An algebra M is called *Malcev algebra* if it satisfies the identities

$$xx=0,$$
$$J(x,xy,z)=J(x,y,z)x,$$

where $J(x,y,z)=(xy)z+(yz)x+(zx)y$ denotes the *Jacobian* of the elements x,y,z. Clearly, every Lie algebra is Malcev algebra. If A is an alternative algebra then the *commutator algebra* $A^{(-)}$ obtained by introducing on the vector space A a new product $[x,y]=xy-yx$ is a Malcev algebra.

Now, a superalgebra $M=M_0\oplus M_1$ is called a *Malcev superalgebra* if it satisfies the super-identities

$$xy+(-1)^{|x||y|}yx=0,$$
$$J(x,ty,z)+(-1)^{|x||t|}J(t,xy,z)=$$
$$=(-1)^{|t|(|y|+|z|)}J(x,y,z)t+(-1)^{|x|(|t|+|y|+|z|)}J(t,y,z)x.$$

As for usual (non-graded) algebras, the *commutator superalgebra* $A^{(-)}$ for an alternative superalgebra A, with the product given by super-commutator $[x,y]_s=xy-(-1)^{|x||y|}yx$, is a Malcev superalgebra.

For a variety $\mathcal{V}$, denote by $\mathcal{V}[X;Y]$ the free $\mathcal{V}$-superalgebra over a field F generated by a set X of even generators and a set Y of odd generators.

Recall some results on relation between the space of skew-symmetric elements in the free $\mathcal{V}$-algebra of countable rank $\mathcal{V}[X;\emptyset]$ and the free $\mathcal{V}$-superalgebra $\mathcal{V}[\emptyset;x]$ generated by one odd element x (see [7, 9, 10]).

Let $u(x)$ be a homogeneous of degree n element from $\mathcal{V}[\emptyset;x]$. Write $u(x)$ in the form $u(x)=v(x,x,\ldots,x)$ where $v(x_1,\ldots,x_n)$ is a multilinear element. Set

$$(\mathrm{Skew}\,u)(x_1,x_2,\ldots,x_n)=\sum_{\sigma\in\Sigma_n}(-1)^{sgn\,\sigma}v(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}).$$

For example, if $u = xx$ then $(\mathrm{Skew}\, u)(x_1, x_2) = [x_1, x_2]$; if $u = (xx)x$ then $(\mathrm{Skew}\, u)(x_1, x_2, x_3) = \sum_{\sigma\in\Sigma_3}(-1)^{sgn\,\sigma}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}$.

Proposition 1. *[9, 7, 10]. Let F be a field of characteristic 0. The map $u \mapsto \mathrm{Skew}\, u$ establishes an isomorphism between the linear space $\mathcal{V}[\emptyset; x]$ and the space of skew-symmetric elements in the free $\mathcal{V}$-algebra $\mathcal{V}[X] = \mathcal{V}[X; \emptyset]$, where $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. In particular, $u(x) = 0$ in the superalgebra $\mathcal{V}[\emptyset; x]$ if and only if $\mathrm{Skew}\, u = 0$ in the free $\mathcal{V}$-algebra $\mathcal{V}[X]$.*

Let $\mathcal{M} = Mal[\emptyset; x]$ be the free Malcev superalgebra generated by an odd generator x. The following theorem summarizes some properties of the superalgebra $\mathcal{M}$ proved in [7].

Theorem 1 [7] *(i) The superalgebra $\mathcal{M}$ has a base formed by the elements*

$$ {}^{k},\ x^{4k}x^2,\ x^{4k+1}x^2,\ k > 0, \tag{1}$$

where $x^{i+1} = x^i x$.

(ii) The nonzero products of basic elements are given by

$$x^i \cdot x = x^{i+1},$$

$$x^{4k+2-i} \cdot x^i = -(-1)^{\frac{i(i-1)}{2}} x^{4k}x^2,\ i > 1,$$

$$x^{4k+3-i} \cdot x^i = -(-1)^{\frac{i(i-1)}{2}} x^{4k+1}x^2,\ i > 1,$$

In [1] the author introduced the following series of multilinear skew-symmetric polynomials g_m on $2m+1$ variables:

$$g_1(x_1, x_2, x_3) = J(x_1, x_2, x_3),$$

and if the skew-symmetric polynomial $g_{m-1}(x_1, \ldots, x_{2m-1})$ is already defined,

$$g_m(x_1, \ldots, x_{2m+1}) =$$

$$= \sum_{1\le i<j<k\le 2m+1} (-1)^{i+j+k} g_{m-1}(J(x_i, x_j, x_k), x_1, \ldots, \check{x}_i, \check{x}_j, \check{x}_k, \ldots, x_{2m+1}),$$

where $\check{x}_i$ means that this variable is absent. It is clear that the polynomial g_m is skew-symmetric on the variables $x_1, \ldots, x_{2m+1}$.

THEOREM 2 [1]. *The polynomial g_m is a nonzero element in the free Malcev algebra of countable rank but it vanishes in any n-generated Malcev algebra over a field of characteristic $\neq 2$ for $m > [\frac{n^3+5n}{12}]$.*

LEMMA 1. *For any $m > 0$ there exist α_m, α'_m, $\beta_m \in F$ such that $\alpha_m \alpha'_m \neq 0$ and*

$$g_{2m} = \alpha_m Skew(x^{4m+1}), \tag{2}$$

$$g_{2m+1} = Skew(\alpha'_m x^{4m+3} + \beta_m x^{4m+1} x^2). \tag{3}$$

PROOF. The element g_m is a skew-symmetric polynomial on $2m+1$ variables in the free Malcev algebra $Malc[X]$. By Proposition 1, there exists a homogeneous element $u \in \mathcal{M}$ of degree $2m+1$ such that $g_m = \mathrm{Skew} u$. Comparing the degrees of elements from the base (1) of $\mathcal{M}$, we see that for m even $u = \alpha x^{2m+1}$ and for m odd $u = \alpha' x^{2m+1} + \underline{\mathrm{x}}^{2m-1} x^2$, for some $\alpha, \alpha', \beta \in F$. Since $g_m \neq 0$ in $Malc[X]$, it is clear that $\alpha \neq 0$. Furthermore, it was proved in [1] that g_m does not vanish in the free metabelian Malcev algebra of countable rank. Since $\mathrm{Skew} x^{2m-1} x^2 = 0$ in any metabelian algebra, we have that $\alpha' \neq 0$. □

COROLLARY 1. *For any $m > 0$ there exist l_m, l'_m, $\nu_m \in F$ such that $l_m l'_m \neq 0$ and*

$$Skew(x^{4m+1}) = l_m g_{2m}, \tag{4}$$

$$Skew(x^{4m+2}) = \pm l_m \sum_i (-1)^i g_{2m}(x_1 \ldots, \check{x}_i, \ldots, x_{4m+1}) x_i, \tag{5}$$

$$Skew(x^{4m+3}) = l'_m g_{2m+1} +$$

$$+\nu_m \sum_{i<j} (-1)^{i+j} g_{2m}(x_1, \ldots, \check{x}_i, \check{x}_j, \ldots, x_{4m+3})[x_i, x_j], \tag{6}$$

$$Skew(x^{4m+4}) = \pm l'_m \sum_i (-1)^i g_{2m+1}(x_1 \ldots, \check{x}_i, \ldots, x_{4m+4}) x_i. \tag{7}$$

PROOF. The first equality follows from (2). For the second and the third, note first that

$$\mathrm{Skew}(x^k x) = \pm \sum_i (-1)^i \mathrm{Skew}(x^k)(x_1, \ldots, \check{x}_i, \ldots, x_{k+1}) x_i, \tag{8}$$

$$\operatorname{Skew}(x^k x^2) = \pm \sum_{i<j} (-1)^{i+j} \operatorname{Skew}(x^k)(x_1, \ldots, \check{x}_i, \check{x}_j, \ldots, x_{k+2})[x_i, x_j]. \qquad (9)$$

Now, (5) follows from (8) and (4), and (6) follows from (3), (9) and (4).

Finally, by Theorem 1 we have $(x^k x^2)x = 0$ in $\mathcal{M}$, therefore

$$\operatorname{Skew}(x^{4m+4}) = \operatorname{Skew}(x^{4m+3}x) =$$

$$= \pm \sum_i (-1)^i \operatorname{Skew}(x^{4m+3})(x_1, \ldots, \check{x}_i, \ldots, x_{4m+4})x_i$$

$$\overset{by\ (3)}{=} \pm \sum_i (-1)^i (l' g_{2m+1} + {}'_{-m}\operatorname{Skew}(x^{4m+1}x^2))(x_1, \ldots, \check{x}_i, \ldots, x_{4m+4})x_i$$

$$= \pm \sum_i (-1)^i l' g_{2m+1}(x_1, \ldots, \check{x}_i, \ldots, x_{4m+4})x_i \pm {}'_{-m}\operatorname{Skew}((x^{4m+1}x^2)x)$$

$$= \pm \sum_i (-1)^i l' g_{2m+1}(x_1, \ldots, \check{x}_i, \ldots, x_{4m+4})x_i.$$

□

Now we can prove our main result.

THEOREM 1. *Every skew-symmetric multilinear polynomial on more then $N(n) = 4 + C_n^1 + C_n^2 + C_n^3$ variables vanishes in any n-generated Malcev algebra.*

PROOF. Let $f = f(x_1, \ldots, x_k)$ be a skew-symmetric polynomial from $Mal[X]$ on $k > N(n)$ variables. By Proposition 1, there exists a homogeneous of degree k element $u(x) \in \mathcal{M}$ for which $f = \operatorname{Skew} u$. In view of base (1), have $u(x) = \alpha x^k + \beta x^{k-2}x^2$ for some $\alpha, \beta \in F$, where $\beta \neq 0$ only if $k \in \{4m+2, 4m+3\}$. Let M_n be the free Malcev algebra of rank n. By [1], $g_s = 0$ is an identity in M_n when $2s + 1 > 1 + C_n^1 + C_n^2 + C_3^n = N(n) - 3$. By Corollary 1, $\operatorname{Skew} x^k$ vanishes in M_n for $k > N(n) - 1$. Finally, for $k = 4m+2, 4m+3$ we have by (7), (4)

$$\operatorname{Skew}(x^{4m}x^2) = l'_{m-1} \sum_{i,j,l} \pm (g_{2m-1}(x_1 \ldots, \check{x}_i, \check{x}_j, \check{x}_l, \ldots, x_{4m+2})x_i)[x_j, x_l],$$

$$\operatorname{Skew}(x^{4m+1}x^2) = l_m \sum_{i,j} \pm g_{2m}(x_1 \ldots, \check{x}_i, \check{x}_j, \ldots, x_{4m+3})[x_i, x_j],$$

which proves that $\mathrm{Skew}(x^k x^2)$ vanishes in M_n for $k > N(n)$, finishing the proof. □

It is natural to ask whether a similar result could be proved for alternative algebras, namely:

Is it true that there exists a natural natural number N such that every multilinear skew-symmetric identity on more then N variables which vanishes in the free associative algebra, vanishes in any n-generated alternative algebra?

Using the results of [1] and [11], one can prove that the question above reduces to a question on a particular identity:

Is it true that there exists a natural natural number N such that any n-generated alternative algebra satisfies the identity

$$\sum_{i,j,k}(-1)^{i+j+k} St_N(x_1,\ldots,\check{x}_i,\check{x}_j,\check{x}_j,\ldots,x_N)(x_i,x_j,x_k)=0,$$

where $St_n(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}(-1)^{sgn\,\sigma}(\cdots(x_{\sigma(1)}x_{\sigma(2)})\cdots)x_{\sigma(n)}$ is the well known standart identity?

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Шестаков И.П. КОСОСИММЕТРИЧЕСКИЕ ТОЖДЕСТВА В КОНЕЧНОПОРОЖДЕННЫХ АЛГЕБРАХ МАЛЬЦЕВА

Мы доказываем, что всякий полилинейный кососимметрический неассоциативный многочлен от более чем $3 + C_n^1 + C_n^2 + C_n^3$ переменных обращается в ноль в любой n-порожденной алгебре Мальцева надполем характеристики 0. Ранее аналогичный результат был известен только для одной серии кососимметрических многочленов g_m специального вида от $2m + 1$ переменных, построенной автором в [1], где $m > [\frac{n^3+5n}{12}]$.

Шестаков И.П. АҚЫРЛЫ ТУЫНДАҒАН МАЛЬЦЕВ АЛГЕБРАЛАРЫНДАҒЫ КОСОСИММЕТРИЯЛЫ ТЕПЕ-ТЕҢДІКТЕР

Біз кез келген полисызықты кососимметриялы ассоциативті емес айнымалы саны $3 + C_n^1 + C_n^2 + C_n^3$-нен артық болатын көпмүшелік кез келген 0 сипаттамалы өрістегі n-туындаған Мальцев алгебрасында нөлге айналатынын дәлелдейміз. Бұрын осы тектес нәтиже тек $2m + 1$ айнымалылы арнайы түрдегі g_m кососимметриялы көпмүшеліктердің бір топтамасы үшін ғана белгілі болатын, бұл нәтижені автор [1] жұмысында $m > [\frac{n^3+5n}{12}]$ болғанда тұрғызды.

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K-THEORETIC DEFORMATIONS OF SCHUR FUNCTIONS

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Annotation: Stable Grothendieck polynomials are kind of symmetric functions that can be considered as K-theoretic analogs of Shur polynomials. In the paper deformations of these polynomials with two additional parameters are presented. Different properties of these functions are established.

Keywords: Symmetric functions, Schur functions, Grothendieck polynomials.

1. Introduction

I am happy to present this note in a conference dedicated to 60th birthday of my teacher Askar Serkulovich Dzhumadil'daev. I would like to express my deepest thanks to him, I am very grateful for his encouragement and continuous support of my work. I would also like to especially mention Dzhumadil'daev's regular seminar during which I (as well as many other of his students) learn a lot. Results in this note were reported in that seminar.

The ring Λ of symmetric functions in infinitely many indeterminates $x = (x_1, x_2, \ldots)$, is a polynomial ring with algebraically independent generators $e_1, e_2, \ldots$, where

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

are elementary symmetric functions.

Bases of Λ are indexed by *partitions*, that are (positive) integer sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$. One of the most important bases for Λ is given by the Schur functions $\{s_\lambda\}$, that (as a finite variable polynomials) can be defined by

Ключевые слова: *Симметрическая функция, функция Шура, полином Гротендика.*

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the double alternant formula

$$s_\lambda(x_1,\ldots,x_n)=\frac{\det\left[x_i^{\lambda_j+n-j}\right]_1^n}{\prod_{1\le i<j\le n}(x_i-x_j)}$$

or as a generating series

$$s_\lambda(x_1,x_2,\ldots)=\sum_{T\in SSYT(\lambda)}\prod_{i\ge 1}x_i^{\#i\text{'s in }T},$$

where the sum runs over *semistandard Young tableaux* (i.e. filling of boxes of the diagram of partition λ by positive integers weakly increasing in rows and strictly in columns) of shape λ.

There is a standard involutive automorphism $\omega:\Lambda\to\Lambda$ given on generators by setting $w(e_k)=h_k$, where

$$h_k=\sum_{i_1\le\cdots\le i_k}x_{i_1}\cdots x_{i_k}$$

are complete homogeneous symmetric functions. Under this involution, the Schur functions are self-dual, $\omega(s_\lambda)=s_{\lambda'}$, where λ' denotes the conjugate partition of λ. For the theory of symmetric functions we refer to [5, 6].

Schur functions play important role in many areas, including representation theory and algebraic geometry. For instance, in Schubert calculus they are related to cohomology ring of the Grassmann variety $\mathrm{Gr}(k,\mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$.

There is a K-theory analog of Schur functions, *the stable Grothendieck polynomials* G_λ. These are symmetric power series first studied by Fomin and Kirillov [2], which arise as a stable limit of Grothendieck polynomials introduced by Lascoux and Schützenberger [4].

The functions G_λ have many similarities with s_λ. For example, there is a formula

$$G_\lambda(x_1,\ldots,x_n)=\frac{\det\left[x_i^{\lambda_j+n-j}(1-x_i)^{j-1}\right]_1^n}{\prod_{1\le i<j\le n}(x_i-x_j)}$$

and a combinatorial presentation given by the series

$$G_\lambda(x_1,x_2,\ldots)=\sum_T(-1)^{|T|-|\lambda|}\prod_{i\ge 1}x_i^{\#i\text{'s in }T},$$

where the sum runs over *set-valued tableaux* of shape λ; a generalization of semistandard Young tableaux, where boxes may contain sets of integers [1].

Let $\hat{\Lambda}$ be the completion of Λ which includes infinite linear combinations of the basis elements (in some distinguished basis of Λ, e.g. Schur functions). Note that $G_\lambda \in \hat{\Lambda}$, for instance $G_{(1)} = e_1 - e_2 + e_3 - \cdots$. It is remarkable that the product of stable Grothendieck polynomials has a finite decomposition

$$G_\lambda G_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c^\nu_{\lambda\mu} G_\nu, \quad c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}, \quad |\nu| \geq |\lambda| + |\mu|.$$

This result was proved by Buch [1] and he described the coefficients $c^\nu_{\lambda\mu}$ combinatorially via set-valued tableaux, generalizing the Littlewood-Richardson rule for Schur functions. Buch studied the Grothendieck ring of the Grassmann variety $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$ as the quotient ring $\Gamma / \left\langle G_\lambda, \lambda \not\subseteq (n-k)^k \right\rangle$, where $\Gamma = \bigoplus_\lambda \mathbb{Z} \cdot G_\lambda$ is a ring with a basis of Grothendieck polynomials ($(n-k)^k$ is a partition of rectangular shape $k \times (n-k)$).

There is a dual family for G_λ via the Hall inner product (which makes orthonormal the Schur basis, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$). These dual stable Grothendieck polynomials $g_\lambda \in \Lambda$ were explicitly described by Lam and Pylyavskyy [3] via plane partitions.

In our work we study two-parameter versions of these polynomials, the dual families of symmetric functions $G^{(\alpha,\beta)}_\lambda(x_1, x_2, \ldots)$, $g^{(\alpha,\beta)}_\lambda(x_1, x_2, \ldots)$. We call them the *canonical stable Grothendieck functions* and the *dual canonical stable Grothendieck polynomials*. In special cases they correspond to:

(a) Schur functions $s_\lambda = G^{(0,0)}_\lambda = g^{(0,0)}_\lambda$; the case $\alpha + \beta = 0$ corresponds to certain deformed Schur functions;

(b) stable Grothendieck polynomials $G_\lambda = G^{(0,-1)}_\lambda$ and its dual $g_\lambda = g^{(0,1)}_\lambda$;

The functions $G^{(\alpha,\beta)}_\lambda \in \hat{\Lambda}$, $g^{(\alpha,\beta)}_\lambda \in \Lambda$ are non-homogeneous symmetric,

$$G^{(\alpha,\beta)}_\lambda = s_\lambda + \text{ higher degree terms}, \quad g^{(\alpha,\beta)}_\lambda = s_\lambda + \text{ lower degree terms}.$$

The families $\{g^{(\alpha,\beta)}_\lambda\}$, $\{G^{(-\alpha,-\beta)}_\lambda\}$ are dual via the Hall inner product and the duality map acts on them as follows:

$$\omega(G^{(\alpha,\beta)}_\lambda) = G^{(\beta,\alpha)}_{\lambda'}, \qquad \omega(g^{(\alpha,\beta)}_\lambda) = g^{(\beta,\alpha)}_{\lambda'}.$$

The structure constants conserve (with scaling), we have

$$G_\lambda^{(\alpha,\beta)} G_\mu^{(\alpha,\beta)} = \sum_\nu (\alpha+\beta)^{|\nu|-|\lambda|-|\mu|} c_{\lambda\mu}^\nu G_\nu^{(\alpha,\beta)}.$$

In some sense, the first parameter α in $G_\lambda^{(\alpha,\beta)}$ uncovers duality (under the involution ω) of the β-Grothendieck polynomials $G_\lambda^{(0,\beta)}$. Combining the 'unifying' and duality properties described above, the reason for calling these symmetric functions as *canonical* is also the following. In the specialization $(\alpha, \beta) = (q, q^{-1})$, the elements $\{g_\rho^{(\alpha,\beta)} : \rho$ is a row or column$\}$ admit a similar characterization as the Kazhdan-Lusztig canonical bases. The elements $\{g_{(k)}^{(\alpha,\beta)} : k \in \mathbb{Z}_{>0}\}$ then characterize the dual functions $g_\lambda^{(\alpha,\beta)}$ as generators for Λ.

Our results (a detailed version [7] is available as a preprint) about the functions $G_\lambda^{(\alpha,\beta)}$, $g_\lambda^{(\alpha,\beta)}$ also include e.g.: combinatorial formulas, based on several new types of tableaux called *hook-valued tableaux* (for $G_\lambda^{(\alpha,\beta)}$) and *rim border tableaux* (for $g_\lambda^{(\alpha,\beta)}$); associated noncommutative operators; Schur expansions and related combinatorics; Jacobi-Trudi type identities.

2. Dual stable Grothendieck polynomials and their deformations

We start by describing the *dual stable Grothendieck polynomials* $\{g_\lambda\}$ which form an inhomogeneous basis of the ring Λ. These polynomials can be defined combinatorially as follows:

$$g_\lambda(x_1, x_2, \ldots) = \sum_{T \in RPP(\lambda)} \prod_{i \geq 1} x_i^{\#\text{columns of } T \text{ containing } i},$$

where the sum is over *reverse plane partitions* (RPP, tableaux with entries weakly increasing in rows and columns) of shape λ [3]. In particular,

$$g_\lambda = s_\lambda + \text{lower degree terms.}$$

It is easy to see that

$$g_{(k)} = h_k, \qquad g_{(1^k)} = \sum_{i=1}^{k} \binom{k-1}{i-1} e_i = e_k(\underbrace{1, \ldots, 1}_{k-1 \text{ times}}, x_1, x_2, \ldots)$$

and the elements $\{g_{(k)}\}$ or $\{g_{(1^k)}\}$ are algebraically independent generators of Λ. There is an involutive automorphism $\tau : \Lambda \to \Lambda$ given on generators by setting $\tau(g_{(k)}) = g_{(1^k)}$ for which $\tau(g_\lambda) = g_{\lambda'}$.

We shall now consider deformations of these polynomials with two additional parameters α, β and coefficients in $\mathbb{Z}[\alpha, \beta]$. First, we add one parameter β. Let

$$g^\beta_\lambda(x_1, x_2, \ldots) = \sum_{T \in RPP(\lambda)} \beta^{|\lambda| - |T|} \prod_{i \geq 1} x_i^{\#\text{columns of } T \text{ containing } i}.$$

In particular,

$$g^\beta_{(k)} = h_k, \qquad g^\beta_{(1^k)} = \sum_{i=1}^{k} \beta^{k-i} \binom{k-1}{i-1} e_i = e_k(\underbrace{\beta, \ldots, \beta}_{k-1 \text{ times}}, x_1, x_2, \ldots).$$

(Note that $g^0_\lambda = s_\lambda$ is Schur function.)

Define $Z[\alpha > \beta] = \{\sum_{0 \leq j < i} a_{ij} \alpha^i \beta^j\}$ as the set of polynomials in α, β whose specializations $(\alpha, \beta) \to (q, q^{-1})$ are in $q\mathbb{Z}[q]$.

THEOREM 1 (Canonical generators). *There is a unique set $\{C_k : k \in \mathbb{Z}_{>0}\}$ with coefficients in $\mathbb{Z}[\alpha, \beta]$ satisfying*

$$C_k \in g^\beta_{(k)} + \sum_{i<k} \mathbb{Z}[\alpha > \beta] g^\beta_{(i)}, \qquad \omega(\overline{C_k}) \in g^\beta_{(1^k)} + \sum_{i<k} \mathbb{Z}[\alpha > \beta] g^\beta_{(1^i)},$$

where $C \to \overline{C}$ is an involution switching α and β.

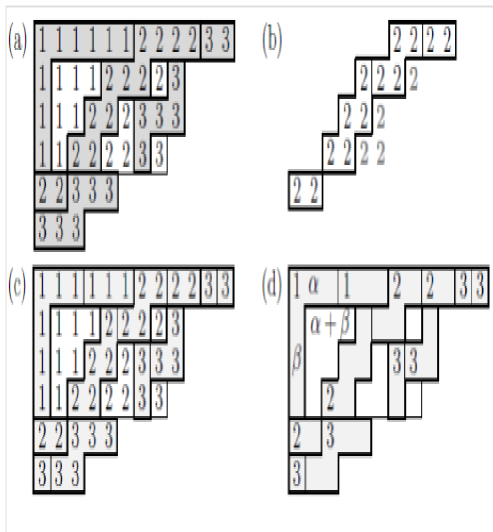
One can obtain that

$$C_k = \sum_{i=1}^{k} \alpha^{k-i} \binom{k-1}{i-1} h_i = \omega(\overline{g^\beta_{(1^k)}}).$$

Let $\varphi : \Lambda \to \Lambda$ be a homomorphism defined by $\varphi(h_k) = C_k$ and define the polynomials $g^{(\alpha,\beta)}_\lambda := \varphi(g^{(\alpha+\beta)}_\lambda)$. It is not hard to see that

$$g^{(\alpha,\beta)}_\lambda = s_\lambda + \text{lower degree terms}$$

and hence $\{g^{(\alpha,\beta)}_\lambda\}$ is an inhomogeneous basis for Λ. Note also that in the specializations $\alpha = \beta = 0$, we obtain Schur functions $s_\lambda = g^{(0,0)}_\lambda$ and for $\alpha = 0$



we have g_λ^β. Since $g_\lambda^{(\alpha,\beta)}$ is defined by applying the homomorphism φ, these polynomials inherit structural properties of the dual Grothendieck poylnomials g_λ.

Theorem 2. *The symmetric functions* $g_\lambda^{(\alpha,\beta)}$ *satisfy the following properties:*

(i) self-duality $\omega(g_\lambda^{(\alpha,\beta)}) = g_{\lambda'}^{(\beta,\alpha)}$;

(ii) $g_\lambda^{(\alpha,\beta)}$ *is Schur-positive, i.e. its expansion in the Schur basis* s_μ *has positive coefficients in* $\mathbb{Z}_{>0}[\alpha,\beta]$.

Combinatorial presentation of these polynomials is based on the following types of tableaux, which we call *rim border tableaux* (RBT, based on RPP). Consider a reverse plane partition as in the figure (a) below and let us refine the 'borders' of each of the equal entries, e.g. as it is shown in (b). These are certain *ribbons*, which we cut by some vertical lines. The remaining parts are regarded as 'white' (or inner) parts of RPP and the resulting tableaux is shown in (c).

The monomial weight x^T for such tableaux T is given by the number of ribbons containing an element, for example here $x^T = x_1^2 x_2^4 x_3^6$ since there are two ribbons with 1, four ribbons with 2 and six ribbons with 3, see (c), (d).

There is also an auxiliary (α,β)-weight $w_T(\alpha,\beta)$ which keeps track of widths, heights of the ribbons and number of boxes in inner parts. This weight easily reads from (d), which is given by $w_T(\alpha,\beta) = \alpha^{14}\beta^9(\alpha+\beta)^{11}$; here 14 is the total width of all ribbons, 9 is the total height, and 11 is the total number of boxes in white parts of the tableau.

THEOREM 3. *The following formula holds*

$$g_\lambda^{(\alpha,\beta)}(x_1, x_2, \ldots) = \sum_{T \in RBT(\lambda)} w_T(\alpha,\beta) x^T.$$

For example, from the figure we have

$$g_\lambda^{(\alpha,\beta)} = \cdots + \alpha^{14}\beta^9(\alpha+\beta)^{11} x_1^2 x_2^4 x_3^6 + \cdots$$

3. STABLE GROTHENDIECK POLYNOMIALS $G_\lambda^{(\alpha,\beta)}$

We can now define the family $\{G_\lambda^{(\alpha,\beta)}\}$ of symmetric functions, dual to $g_\lambda^{(-\alpha,-\beta)}$ via the Hall inner product. That is, via a pairing $\langle\cdot,\cdot\rangle : \Lambda \times \hat{\Lambda} \to \mathbb{Z}[\alpha,\beta]$ (given on Schur basis as $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$) for which $\langle g_\mu^{(-\alpha,-\beta)}, G_\lambda^{(\alpha,\beta)}\rangle = \delta_{\lambda\mu}$

THEOREM 4. *The symmetric functions $G_\lambda^{(\alpha,\beta)} \in \hat{\Lambda}$ satisfy the following properties:*

(i) self-duality $\omega(G_\lambda^{(\alpha,\beta)}) = G_{\lambda'}^{(\beta,\alpha)}$;

(ii) product has finite decomposition

$$G_\lambda^{(\alpha,\beta)} G_\mu^{(\alpha,\beta)} = \sum_\nu (\alpha+\beta)^{|\lambda|+|\mu|-|\nu|} c_{\lambda\mu}^\nu G_\nu^{(\alpha,\beta)}, \quad c_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}.$$

More explicitly, the functions $G_\lambda^{(\alpha,\beta)}$ can be defined as a ratio of two determinants, similarly as for Schur functions, which looks as follows

$$G_\lambda^{(\alpha,\beta)}(x_1, \ldots, x_n) = \frac{\det\left[x_i^{\lambda_j+n-j} \dfrac{(1-\beta x_i)^{j-1}}{(1-\alpha x_i)^{\lambda_j}}\right]_1^n}{\prod_{1\leq i<j\leq n}(x_i - x_j)}.$$

In the specialization $\alpha+\beta = 0$ we have $G_\lambda^{(\alpha,-\alpha)}(x) = s_\lambda(x/(1-\alpha x))$.

Combinatorial presentation for these symmetric functions is based on *hook-valued tableaux* (HVT). Boxes of these tableaux contain SSYT's of hook shapes. For example, as in the figure below, where the ordering rule is given so that when we replace each hook by any of its elements, the resulting tableau is always an SSYT.

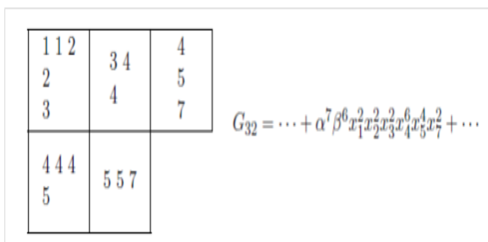


Figure 1a – Power w.r.t. two-dimensional logistic with independent standard logistic components ($n = 250$)

THEOREM 5. *We have*

$$G_\lambda^{(\alpha,\beta)}(x_1, x_2, \ldots) = \sum_{T \in HVT(\lambda)} \alpha^{arms(T)} \beta^{legs(T)} \prod_{i \geq 1} x_i^{\#i\text{'s in } T},$$

where $HVT(\lambda)$ is the set of hook-valued tableaux of shape λ and $arms(T)$ (resp. $legs(T)$) are the sums of first rows minus one (resp. first columns minus one) of all hooks in T.

4. Other properties

4.1 Pieri-type formulas

For any skew shape μ/λ define:

$r(\mu/\lambda)$ - the number of rows;

$c(\mu/\lambda)$ - the number of columns;

$b(\mu/\lambda)$ - the number of connected components;

$i(\mu/\lambda) = |\mu/\lambda| - c(\mu/\lambda) - r(\mu/\lambda) + b(\mu/\lambda)$ - the number of boxes in inner part.

Theorem 6. *The following formulas hold*

(i) For $G_\lambda^{(\alpha,\beta)}$*:*

Type 1:

$$G_{(k)}^{(\alpha,\beta)} G_\lambda^{(\alpha,\beta)} = \sum_{\mu/\lambda \text{ horisontal strip}} (\alpha+\beta)^{|\mu/\lambda|-k} \binom{r(\mu/\lambda)-1}{|\mu/\lambda|-k} G_\mu^{(\alpha,\beta)};$$

Type 2:

$$h_k G_\lambda^{(-\alpha,-\beta)} = \sum_\mu v_{\mu/\lambda}^k(\alpha,\beta) G_\mu^{(-\alpha,-\beta)},$$

where

$$v_{\mu/\lambda}^k(\alpha,\beta) = \beta^{r(\mu/\lambda)-b(\mu/\lambda)}(\alpha+\beta)^{i(\mu/\lambda)}\alpha^{c(\mu/\lambda)-k}\binom{c(\mu/\lambda)-b(\mu/\lambda)}{c(\mu/\lambda)-k};$$

(ii) For $g_\lambda^{(\alpha,\beta)}$*:*

Type 1:

$$g_{(k)}^{(\alpha,\beta)} g_\mu^{(\alpha,\beta)} = \sum_{\lambda/\mu \text{ hor. strip}} (-(\alpha+\beta))^{k-|\lambda/\mu|} \binom{r(\mu/\bar{\lambda})}{k-|\lambda/\mu|} g_\lambda^{(\alpha,\beta)};$$

Type 2:

$$h_k g_\mu^{(\alpha,\beta)} = \sum_{\lambda/\mu \text{ hor. strip}} q_{\lambda/\mu}(\alpha,\beta) g_\lambda^{(\alpha,\beta)},$$

where

$$q_{\lambda/\mu}(\alpha,\beta) = \sum_\ell (-(\alpha+\beta))^{\ell-|\lambda/\mu|}\binom{r(\mu/\bar{\lambda})}{\ell-|\lambda/\mu|}(-\alpha)^{k-\ell}\binom{k-1}{\ell-1}.$$

4.2 Jacobi-Trudi-type identities

Recall classical Jacobi-Trudi identities for Schur functions:

$$s_\lambda = \det\left[e_{\lambda'_i-i+j}\right] = \det\left[h_{\lambda_i-i+j}\right].$$

For stable Grothendieck polynomials we obtain the following generalizations.

THEOREM 7. *The following formulas hold*

(i) $G_\lambda^{(\alpha,\beta)} = \det\left[\widetilde{e}^{(i)}_{\lambda'_i - i + j}\right] \qquad \widetilde{e}^{(i)}_p = \sum_k (\alpha+\beta)^k \binom{\lambda'_i - 1 + k}{k} e_{p+k}\left(\frac{x}{1-\alpha x}\right);$

(ii) $g_\lambda^{(\alpha,\beta)} = \det\left[\widetilde{g}^{(i)}_{\lambda_i - i, j}\right],$

$$\widetilde{g}^{(i)}_{p,j} = \sum_k \widetilde{f}^{(i)}_{p,k} h_{k+j} \qquad \widetilde{f}^{(i)}_{p,q} = \sum_{k=-\infty}^{\infty} \binom{p+i-k-1}{i-1}\binom{k}{k-q}(\alpha+\beta)^{p-k}\alpha^{k-q}.$$

In particular,

$$g_\lambda^{(0,\beta)} = \det\left[\sum_k \beta^k \binom{k+i-1}{k} h_{\lambda_i - i + j - k}\right] = \det\left[\sum_k \beta^k \binom{\lambda'_i - 1}{k} e_{\lambda'_i - i + j - k}\right].$$

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Елеусизов Д. K-ТЕОРЕТИЧЕСКИЕ ДЕФОРМАЦИИ МНОГОЧЛЕНОВ ШУРА

Стабильные многочлены Гротендика это определенные симметрические функции, которые можно рассматривать как K-теоретические аналоги многочленов Шура. В работе представлены деформации этих многочленов с двумя дополнительными параметрами. Показаны различные свойства этих функций.

Елеусизов Д. ШУРДЫҢ КӨПМҮШЕЛІКТЕРІНІҢ K-ТЕОРЕТИКАЛЫҚ ДЕФОРМАЦИЯЛАРЫ

Гротендиктің көпмүшеліктері Шурдың көпмүшеліктерінің K-теоретикалық аналогтары сияқты қарастыруға болады. Жұмыста осы симметриялық функцияларының кейбір деформациялары берілген, және әртүрлі қасиеттері көрсетілген.

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LIE ALGEBRAS OF GROUPS OF EXPONENT 5

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Dedicated to Askar Dzhumadildaev on the occasion of his 60th birthday

Annotation: Let L be a Lie algebra of characteristic 5 satisfying the Engel identity $[[[[x,y],y],y],y]=0$. We prove that the multiplication algebra $R(L)$ of the algebra L is nil of bounded index.

Keywords: The Burnside problem, pro-p groups, PI-algebras, Lie algebras.

Let L be a Lie algebra over a field F. We say that L satisfies the Engel identity E_n if $[\dots[a,\underbrace{b],b],\dots,b}_{n}] = 0$ for arbitrary elements $a,b\in L$.

Let p be a prime number and let G be a group of exponent p, i.e., $g^p=1$ for an arbitrary element $g\in G$. Consider the lower central series $G=G_1>G_2>\cdots$ of the group G and the associated Lie algebra $L(G)=\bigoplus\limits_{i=1}^{\infty}G_i/G_{i+1}$ over the field $F=\mathbb{Z}/p\mathbb{Z}$ (see [1], [2]). W. Magnus [3] showed that the Lie algebra $L(G)$ satisfies the Engel identity E_{p-1}. A. I. Kostrikin ([4, 5]) proved that a Lie algebra over a field of characteristic p satisfying the identity E_{p-1} is locally nilpotent, thus settling the Restricted Burnside Conjecture for groups of prime exponent p. For further developments on the Restricted Burnside Conjecture, see [6, 7, 8, 9].

On the other hand, Yu. P. Rasmyslov [10] constructed examples of nonsolvable Lie algebras of characteristic 5 satisfying the identity E_4.

Let $R(L)$ be the multiplication algebra of L, that is, the associative subalgebra of the algebra $End_F(L)$ of all linear transformations of L generated by adjoint operators $\mathrm{ad}(a):L\to L$, $\mathrm{ad}(a):x\to[a,x]$, $a\in L$.

Ключевые слова: *The Burnside problem, pro-p groups, PI-algebras, Lie algebras.*
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We say that an associative algebra R is nil of bounded index if there exists an $n \geq 1$ such that $a^n = 0$ for an arbitrary element $a \in R$. See [11], [9] for more information about nil algebras of bounded index.

Conjecture. *Let L be a Lie Algebra over a field of characteristic $p > 0$ satisfying the identity E_{p-1}. Then the multiplication algebra $R(L)$ is nil of bounded index.*

In this paper, we will prove the Conjecture for $p = 5$.

Theorem 1. *Let L be a Lie algebra of characteristic 5 satisfying the Engel identity E_4. Then the multiplication algebra $R(L)$ is nil of bounded index.*

Notice that for Lie algebras over a field of characteristic 5 satisfying E_4, G. Higman [12] proved a stronger assertion than local nilpotence: an arbitrary element of such an algebra generates a nilpotent ideal of degree ≤ 6.

Let $f(x_1, \ldots, x_n)$ be a nonzero element of the free associative F-algebra in $X = \{x_i, i \geq 0\}$. We say that an associative algebra R satisfies the polynomial identity $f = 0$ if $f(a_1, \ldots, a_n) = 0$ for arbiratry elements $a_1, \ldots, a_n \in R$. If an algebra satisfies some polynomial identity, then we call it a PI-algebra.

Let A be a (not necessarily associative) F-algebra. Let $\mathcal{V}$ be the variety generated by the algebra A (see [13]) and let $F_{\mathcal{V}}\langle X\rangle$ be the free algebra in the variety $\mathcal{V}$ on the countable set of free generators $X = \{x_i, i \geq 0\}$. Let P_n be the linear span of all products $x_{\sigma(1)} \ldots x_{\sigma(n)}$ with arbitrary brackets, where σ runs over the symmetric group S_n. The dimensions $d_n = \dim_F P_n$, $n \geq 1$, are called codimensions of the algebra A. A. Regev [14] proved that for an arbitrary associative PI-algebra, the sequence of codimensions is exponentially bounded, i.e., there exists $a \geq 1$ such that $d_n \leq a^n$ for all n.

In [15] V. N. Latyshev gave a new proof of Regev's theorem based on a theorem of R. P. Dilworth [16].

Fix $m \geq 2$ and consider all permutations $\sigma \in S_n$ with the following property: there does not exist a sequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ of length m such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_m)$. One of the formulations of the Dilworth Theorem asserts that the number of such permutations is $\leq D(m)^n$, where $D(m)$ is a constant that depends only on m. Codimensions of an associative algebra satisfying an identity of degree $\leq n$ are bounded by $D(m)^n$.

Let L be a Lie algebra over a field F of characteristic 5 satisfying the identity E_4.

LEMMA 1. *Codimensions of the algebra L are exponentially bounded.*

ДОКАЗАТЕЛЬСТВО. Let $L\langle X\rangle$ be the free algebra in the variety generated by the algebra L on the countable set of free generators $X = \{x_i, i \geq 0\}$.

By Higman's theorem [12], every element of the algebra $L\langle X\rangle$ generates a nilpotent ideal of degree ≤ 6. The subspace P_n is spanned by left-normed commutators

$$[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = [\dots[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], \dots, x_{\sigma(n)}],$$

$\sigma \in S_n$.

Consider the lexicographical order in the set of all permutations: for two distinct permutations σ and τ, we say that $\sigma > \tau$ if there exists k, $1 \leq k \leq n$, such that $\sigma(i) = \tau(i)$, $1 \leq i \leq k-1$, but $\sigma(k) > \tau(k)$.

Suppose that there exists a sequence $1 \leq i_1 < \cdots < i_6 \leq n$ such that $\sigma(i_1) > \cdots > \sigma(i_6)$. Denote $\rho = [x_{\sigma(1)}, \dots, x_{\sigma(n)}]$. Fixing all variables except for $x_{\sigma(i_1)}, \dots, x_{\sigma(i_6)}$, we write $\rho = \rho(x_{\sigma(i_1)}, \dots, x_{\sigma(i_6)})$. By Higman's theorem, $\rho(\underbrace{x, x, \dots, x}_{6}) = 0$. Linearizing this identity (see [11]), we get $\sum\limits_{s \in S_6} \rho(y_{s(1)}, \dots, y_{s(6)}) = 0$. Hence,

$$\rho = -\sum [x_{\tau(1)}, \dots, x_{\tau(n)}],$$

where all permutations $\tau \in S_n$ on the right hand side differ from σ only at the positions $i_1, \dots, i_6$; the elements $x_{\sigma(i_1)}, \dots, x_{\sigma(i_6)}$ have been nontrivially permuted. This makes all the permutations τ lexicographically smaller than σ.

We showed that P_n is spanned by left-normed commutators $[x_{\sigma(1)}, \dots, x_{\sigma(n)}]$ such that there does not exist a sequence $1 \leq i_1 < i_2 < \cdots < i_6 \leq n$ with $\sigma(i_1) > \cdots > \sigma(i_6)$. By Dilworth's Theorem, $\dim\ P_n \leq D(6)^n$, which completes the proof of the lemma.

LEMMA 2. *The multiplication algebra $R(L)$ is a PI-algebra.*

Доказательство. First, we will establish existence of a nonzero multilinear associative polynomial $f(y_1, \ldots, y_n)$ such that for arbitrary elements $a_i, b_i, c_i \in L$, $1 \le i \le n$, we have

$$f(\mathrm{ad}(a_1)\mathrm{ad}(b_1)\mathrm{ad}(c_1), \ldots, \mathrm{ad}(a_n)\mathrm{ad}(b_n)\mathrm{ad}(c_n)) = 0.$$

Consider again the free Lie algebra $L\langle X\rangle$ in the variety generated by the algebra L, $X = \{x_i, i \ge 0\}$.

Denote $U_i = \mathrm{ad}(x_{3i-2})\mathrm{ad}(x_{3i-1})\mathrm{ad}(x_{3i})$, $i \ge 1$, and consider $n!$ elements $U_{\sigma(1)} \ldots U_{\sigma(n)} x_0$, $\sigma \in S_n$. These elements lie in P_{3n+1}. By Lemma 1, $\dim_F P_{3n+1} \le D(6)^{3n+1}$. Choose $n \ge 1$ such that $D(6)^{3n+1} < n!$ Then the elements $U_{\sigma(1)} \ldots U_{\sigma(n)} x_0$ are linearly dependent. In other words, there exist scalars α_σ, $\sigma \in S_n$, not all equal to zero, such that $\sum\limits_{\sigma} \alpha_\sigma U_{\sigma(1)} \cdots U_{\sigma(n)} x_0 = 0$. Now it remains to choose $f(y_1, \ldots, y_n) = \sum\limits_{\sigma} \alpha_\sigma y_{\sigma(1)} \cdots y_{\sigma(n)}$.

A.I. Kostrikin [4] notices that the multiplication algebra $R = R(L)$ is spanned by operators $\mathrm{ad}(a_1) \cdots \mathrm{ad}(a_k)$, $a_i \in L$, $k \le 3$. We claim that R^3 is spanned by operators $\mathrm{ad}(a_1)\mathrm{ad}(a_2)\mathrm{ad}(a_3)$, $a_i \in L$. Indeed, R^3 is spanned by products $\mathrm{ad}(a_1) \ldots \mathrm{ad}(a_s)$, $a_i \in L$, $s \ge 3$. This product can be rewritten as a linear combination of operators $\mathrm{ad}(\rho_1)\mathrm{ad}(\rho_2) \cdots \mathrm{ad}(\rho_k)$, $k \le 3$, where the ρ_i's are commutators in $a_1, \ldots, a_s$ and the sum of lengths of $\rho_1, \ldots, \rho_k$ is equal to s. If $k = 1$ or $k = 2$, then some ρ_i is a commutator of length ≥ 2. Using the Jacobi identity, we can increase k.

Since R^3 is a PI-algebra, it follows that R is a PI-algebra as well, which completes the proof.

Consider the associative commutative algebra

$$E = \langle e_i, i \ge 1 | e_i e_j = e_j e_i, e_i^2 = 0; i, j \ge 1\rangle.$$

A. R. Kemer [17] proved that for an arbitrary PI-algebra A over a field F of positive characteristic, the algebra $A \otimes_F E$ is nil of bounded index. Let the algebra $R \otimes_F E$ be nil of bounded index $\le 5^m$.

Let $w_1, \ldots, w_s$ be products of adjoint operators, $w_i = \mathrm{ad}(a_{i1}) \cdots \mathrm{ad}(a_{ir_i})$, $a_{ij} \in L$, $1 \le i \le s$. Let $\alpha_1, \ldots, \alpha_s \in F$. Then,

$$\left(\sum_{i=1}^{s} \alpha_i w_i\right)^{5^m} = \sum \beta w_{i_1} \cdots w_{i_{5^m}},$$

where $\beta \in F$ and in each product $w_{i_1} \ldots w_{i_{5m}}$, at least one operator w_j occurs more than once. Iterating, we get

$$\left(\sum_{i=1}^{s} \alpha_i w_i\right)^{5^{2m}} = \left(\sum \beta w_{i_1} \cdots w_{i_{5m}}\right)^{5^m} = \sum \gamma w_{j_1} \cdots w_{j_{5^{2m}}},$$

where $\gamma \in F$ and in each product $w_{j_1} \cdots w_{j_{5^{2m}}}$, at least one operator w_j occurs ≥ 4 times.

Finally, for any $\ell \geq 1$,

$$\left(\sum_{i=1}^{s} \alpha_i w_i\right)^{5^{\ell m}} = \sum \mu w_{k_1} \cdots w_{k_{5^{\ell m}}},$$

where $\mu \in F$ and in each product $w_{k_1} \cdots w_{k_{5^{\ell m}}}$, at least one operator w_j occurs $\geq 2^\ell$ times. Choose $\ell = 3$. Then, $2^\ell > 6$ and therefore $\left(\sum\limits_{i=1}^{s} \alpha_i w_i\right)^{5^{3m}} = 0$, which completes the proof of the theorem.

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Зельманов Е. АЛГЕБРА ЛИ ГРУППЫ ЭКСПОНЕНТЫ 5

Пусть L – алгебра Ли удовлетворяющая тождеству $[[[[x, y], y], y], y] = 0$ над полем характеристики 5. Мы докажем, что алгебра умножения $R(L)$ алгебры L является ниль-алгеброй ограниченного индекса.

Зельманов Е. 5 ЭКСПОНЕНТТІ ТОПТЫҢ ЛИ АЛГЕБРАСЫ

L алгебрасы – сиппатамасы 5 ке тең болатын өріс үстінде анықталған $[[[[x, y], y], y], y] = 0$ тепе-теңдігін қанағаттандыратын Ли алгебрасы болсын. $R(L)$ – L алгебрасының көбейтулер алгебрасы болсын. Бұл жұмыста $R(L)$-дің ақырлы индексті ниль-алгебрасы екенін дәлелдейміз.

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LIE ALGEBRAS AND AROUND: SELECTED QUESTIONS

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To my teacher and friend Askar Serkulovich Dzhumadil'daev on his 60th birthday

Annotation: Several open questions are discussed. The topics include cohomology of current and related Lie algebras, algebras represented as the sum of subalgebras, structures and phenomena peculiar to characteristic 2, and variations on themes of Ado, Whitehead, and Banach.

Keywords: Lie algebra, Zassenhaus algebra, cohomology, deformations, Koszul dual operads, sum of subalgebras, characteristic 2, Ado Theorem, Whitehead Lemma, ternary maps.

1. Introduction

I am presenting here a, perhaps, haphazard collection of questions I am interested in. Being haphazard as it is, this collection features somewhat unexpected and, hopefully, fascinating connections between different topics.

This is a modest contribution in honor of Askar Dzhumadil'daev. I first met him in 1983, when, as an undergraduate student, I started to participate in his seminar on Lie algebras. Since then and throughout many years, I enjoyed his wisdom, unfailing enthusiasm, friendship, and support. Most of what I know in mathematics I owe to him.

Ключевые слова: *Алгебра Ли, алгебра Цассенхауза, когомологии, деформации, двойственные по Кошулю операды, сумма подалгебр, характеристика 2, теорема Адо, лемма Уайтхеда, тернарные отображения.*

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2. Cohomology of Lie algebras coming from Koszul dual operads

Current Lie algebras – that is, Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is a commutative associative algebra – as well as algebras close to them, are ubiquitous in mathematics and physics (sufficient is to mention Lie algebras of smooth functions on a manifold prominent in gauge theory, Kac–Moody Lie algebras, modular semisimple Lie algebras, etc.). The Lie bracket is defined in an obvious factor-wise way:

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$

where $x, y \in L$, $a, b \in A$. A vast generalization of this construction comes from the operad theory: if A is an algebra over a binary quadratic operad $\mathcal{P}$, and B is an algebra over the operad Koszul dual to $\mathcal{P}$, then the tensor product $A \otimes B$ equipped with the bracket

$$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b, \tag{1}$$

where $a, a' \in A$, $b, b' \in B$, is a Lie algebra.

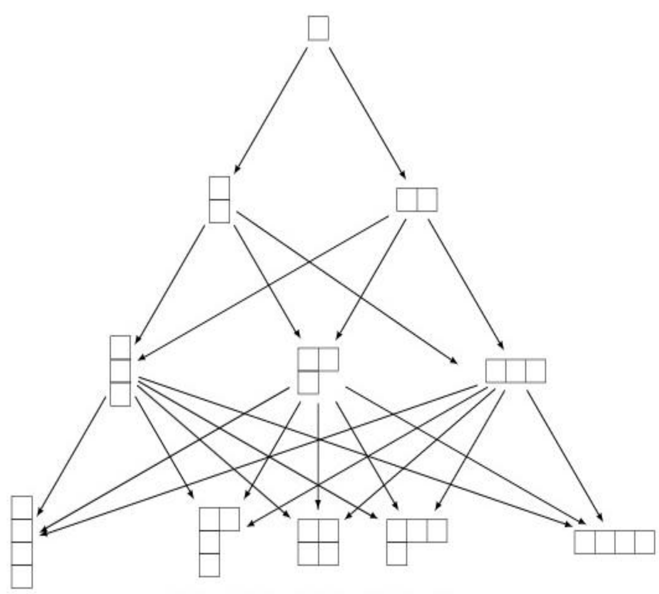
Due to a big flexibility of this construction, many interesting Lie algebras can be represented in this form, for a suitable pair of Koszul dual operads and algebras over them. Perhaps the most remarkable examples, beyond current Lie algebras, are various algebras appearing in physics (Schrödinger–Virasoro, Heisenberg–Virasoro, etc.): they are represented as tensor products of algebras over left and right Novikov operads. This remarkable observation was implicit already in the pioneering works of I. Gelfand and S.P. Novikov and their collaborators ([1] and [2]); after that, Pei and Bai ([3] and references therein) noted that Lie algebras in question admit realization as affinizations of certain left Novikov algebras; Dzhumadil'daev noted in [4] that left and right Novikov operads are Koszul dual to each other; the explicit claim was made in [5, §5] by putting all the pieces together.

Therefore, it seems to be interesting to develop a general method for computing cohomology and other invariants of Lie algebras given by bracket (1). By this, we mean to express cohomology or other invariants of such Lie algebras in terms of invariants of tensor factors A and B, similarly how it was done earlier for low-degree cohomology of current Lie algebras.

Let us try to pursue further an approach for computation of (co)homology of current Lie algebras described in [6, §4]. (For simplicity, we will consider cohomology with coefficients in the trivial module K, assume some finiteness conditions, and zero characteristic of the base field, but similar considerations may be applied in more general settings). Let us decompose the modules $\bigwedge^n(A \otimes B)$ according to the well-known Cauchy formula:

$$\bigwedge^n (A \otimes B) \simeq \bigoplus_{\lambda \vdash n} \Big(Y_\lambda(A) \otimes Y_{\lambda^\sim}(B) \Big).$$

Here the direct sum of vector spaces is taken over all Young diagrams of size n, $\lambda^\sim$ is the Young diagram transposed to λ, and Y_λ is the Young symmetrizer corresponding to λ. Assuming that at least one of the algebras A, B is finite-dimensional, passing to the decomposition of the dual vector spaces, and decomposing the differential $d : (\bigwedge^n(A \otimes B))^* \to (\bigwedge^{n+1}(A \otimes B))^*$ in the Chevalley–Eilenberg complex accordingly, we get the following maps on the Young graph:



Here each Young diagram λ denotes the vector space $Y_\lambda(A)^* \otimes Y_{\lambda^\sim}(B)^*$, and arrows denote the corresponding components of the differential d.

Intuitively it should be clear that the more arrows here vanish, the easier it would be to compute the corresponding cohomology. In the case of the pair of operads (Lie, associative commutative), i.e. for current Lie algebras $L \otimes A$, a miracle happens: approximately half of the arrows vanish (roughly, those going from "left" to "right"), what allows to define a certain spectral sequence on the Chevalley–Eilenberg complex computing the cohomology $H^*(L \otimes A, K)$. In the low cohomology degrees and/or for particular types of algebras, this spectral sequence allows to express $H^*(L \otimes A, K)$ in terms of cohomology and other invariants of the tensor factors L and A. Unfortunately, this miracle fails for the other pairs of Koszul dual operads, even such a classical one as (associative, associative).

Question 1. *What makes the pair (Lie, associative commutative) special in this regard? In which other situations (i.e., for a particular pair of Koszul dual operads, or for a particular kind of algebras over some pair of Koszul dual operads) "many "arrows in the corresponding Young graph (2) will vanish? In particular, for which types of Lie algebras expressed as the tensor products of left Novikov and right Novikov algebras, this will happen?*

3. Algebras represented as the sum of subalgebras

My mathematical debut, under the guidance of Askar Dzhumadil'daev, was the proof of the Kegel–Kostrikin conjecture about solvability of a modular finite-dimensional Lie algebra L represented as the vector space sum $L = N + M$ of two nilpotent subalgebras N, M ([7]; around the same time this was established also by Panyukov, [8]). The statement is true over fields of characteristic $p > 2$, but in characteristic 2 there is a counterexample found by Petravchuk, [9]. Take the 3-dimensional characteristic 2 analog of sl_2: the simple Lie algebra S, with the basis $\{e_{-1}, e_0, e_1\}$ subject to multiplication

$$[e_{-1}, e_0] = e_{-1}, \quad [e_{-1}, e_1] = e_0, \quad [e_0, e_1] = e_1.$$

Its 2-envelope $S^{[2]}$ is 5-dimensional and admits the decomposition $S^{[2]} = N \oplus M$, where the 2-dimensional abelian subalgebra N is linearly spanned by $e_0 + e_{-1} + e_{-1}^{[2]}$ and $e_0 + e_1 + e_1^{[2]}$, and the 3-dimensional nilpotent subalgebra M is linearly spanned by $e_{-1}^{[2]}$, e_0 and $e_1^{[2]}$ (the vector space sum in this case is direct).

Do such examples exist in higher dimensions? Of course, any (finite- or infinite-dimensional) current Lie algebra $S \otimes A$ admits such a decomposition:

$$S \otimes A = (N \otimes A) \oplus (M \otimes A),$$

so it provides such an example provided it itself is non-solvable (for example, when A contains a unit).

A slightly more interesting example can be obtained as an extension of the corresponding current Lie algebras of the form $S \otimes A + \mathcal{D}$, where $\mathcal{D}$ acts on A by derivations, what includes semisimple Lie algebras. Namely, we have the decomposition

$$S \otimes A + \mathcal{D} = (N \otimes A + \mathcal{D}) \oplus (M \otimes A). \tag{2}$$

An easy induction on n proves that for a Lie algebra $L = S \otimes A + \mathcal{D}$ with A unital, and for any positive integer n, we have

$$L^n = S^n \otimes A + \sum_{\substack{i+j=n \\ i>1, j\geq 1}} S^i \otimes A\mathcal{D}^j(A) + S \otimes \mathcal{D}^{n-1}(A) + \mathcal{D}^n.$$

This implies that if N is nilpotent, and $\mathcal{D}$ is nilpotent as an algebra of derivations of A (and hence is nilpotent as an abstract Lie algebra), then the algebra $N \otimes A + \mathcal{D}$ is nilpotent too. Therefore, (2) provides a decomposition of a nonsolvable Lie algebra into the sum of nilpotent subalgebras.

Yet it would be more interesting to generalize Petravchuk's decomposition for an arbitrary Zassenhaus algebra $W_1'(n)$ in characteristic 2. Zassenhaus algebras appear prominently in ongoing efforts of classification of simple Lie algebras in characteristic 2 (cf. [10, V. I, §7.6], [11], [12], and references therein). In characteristic $p = 2$, unlike for $p > 2$, the Zassenhaus algebra has dimension $2^n - 1$ and can be defined as the algebra with the basis $\{e_i | -1 \leq i \leq 2^n - 3\}$ subject to multiplication

$$[e_i, e_j] = \begin{cases} \binom{i+j+2}{i+1} e_{i+j} & \text{if } -1 \leq i+j \leq 2^n - 2 \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $S = W_1'(2)$ is the first algebra in the series. The 2-envelope $W_1'(n)^{[2]}$ of $W_1'(n)$ coincides with the derivation algebra of $W_1'(n)$, has dimension $2^n + n - 1$, and is spanned, in addition to elements of $W_1'(n)$, by elements $(ade_{-1})^{2^k}$, $k = 1, 2, \dots, n-1$, and $(ade_{2^{n-1}-1})^2$.

QUESTION 2. *Find a link with combinatorial interpretation of the number $2^n + n - 1$ as the shortest length of a sequence of 0 and 1 containing all subsequences of length n (see [13, A052944]).*

QUESTION 3. *Is it true that $W_1'(n)^{[2]}$ admits a decomposition into the sum of two nilpotent subalgebras?*

Virtually nothing is known about the Kegel–Kostrikin question in the infinite-dimensional case – beyond almost obvious cases when one imposes some sort of finiteness conditions on one or both of the summands; all such cases are reduced quickly to the finite-dimensional situation.

As a first step, one may wish to prove that such an algebra satisfies a nontrivial (Lie) identity. According to [14, Corollary 2.5], a Lie algebra L does not satisfy a nontrivial identity if and only if a free Lie algebra is embedded into an ultraproduct of L. As the ultraproduct construction obviously commutes with the decomposition into the sum of subalgebras, the question whether a Lie algebra $L = N + M$ does not satisfy a nontrivial identity is equivalent to the question whether its ultraproduct $L^{\mathcal{U}} = N^{\mathcal{U}} + M^{\mathcal{U}}$ does not contain a free Lie subalgebra. As being nilpotent (of a fixed nilpotency index) is the first-order property, by the Łoś theorem the Lie algebras $N^{\mathcal{U}}$ and $M^{\mathcal{U}}$ are also nilpotent. Thus the question whether the sum of two nilpotent Lie algebras satisfies a nontrivial identity, is equivalent to an apriori more special

QUESTION 4. *Is it true that an infinite-dimensional Lie algebra represented as the sum of two nilpotent subalgebras, cannot contain a free Lie algebra as a subalgebra?*

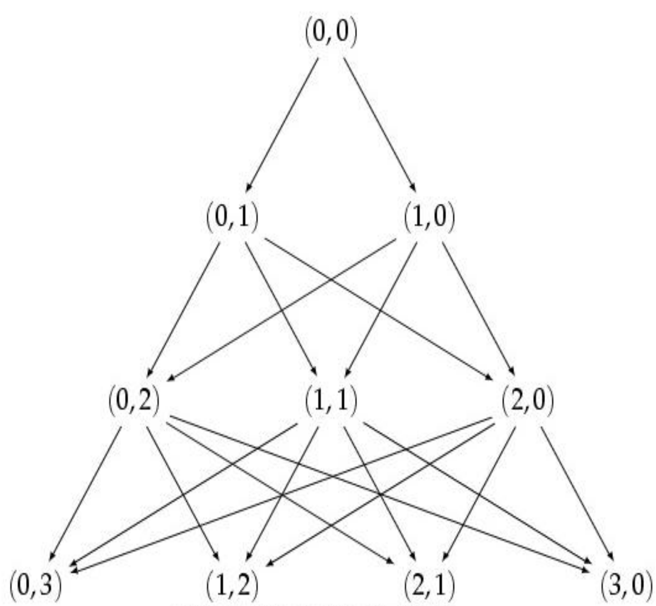
In the theory of (associative) PI algebras, there is a similar long-standing open question: whether the sum of two PI algebras is PI? (See [15] and (numerous) references therein). By the same reasonings as in the Lie case, this question is equivalent to

QUESTION 5. *Is it true that an associative algebra represented as the sum of two PI subalgebras, cannot contain a free associative algebra as a subalgebra?*

Another interesting topic is to develop a machinery to express the (co)homology (Chevalley–Eilenberg, Hochschild, etc.) of such algebras in terms of factors and their action on each other. In the case when the sum of subalgebras is direct, one may attempt to mimic the approach of §2, albeit in a more simple situation, as we have direct sums instead of tensor products. If say, a Lie algebra $L = M \oplus N$ is represented as the vector space direct sum of subalgebras M and N, then, instead of the Cauchy formula we have a more simple isomorphism of vector spaces:

$$\bigwedge^n(N \oplus M) \simeq \bigoplus_{\substack{i+j=n \\ i,j \geq 0}} \bigwedge^i(N) \otimes \bigwedge^j(M).$$

Passing to the dual vector spaces, and decomposing the differentials in the Chevalley–Eilenberg complex, as in §2, we get the following picture (now instead of the Young graph we have a more simpler triangle):



Here each pair (i, j) denotes the vector space $\bigwedge^i(N)^* \otimes \bigwedge^j(M)^*$.

QUESTION 6. *Are there any patterns (vanishing or otherwise) in the graph (5) in the general situation? In some special cases?*

Ideally, a positive answer to this question should allow to develop a cohomological machinery which would unify and generalize various situations: some particular instances of the Hochschild–Serre spectral sequence, a stuff related to "Tate Lie algebras", "Japanese cocycles" (see, e.g., [16, §2.7]), etc.

A somewhat similar machinery is contained in an interesting and seemingly entirely forgotten paper [17] (there, the author presents an alternative derivation of the Lyndon–Hochschild–Serre spectral sequence for the semidirect product of groups, but similar considerations seem to be applicable as well to the group-theoretic analog of our situation, i.e. for a group $G = AB$ decomposed into the product of its subgroups A and B; the promised sequels to [17] treating the Lie-algebraic and associative cases have never appeared).

4. DEFORMATIONS AND "COMMUTATIVE" COHOMOLOGY IN CHARACTERISTIC 2

Classification of finite-dimensional simple Lie algebras over algebraically closed fields of characteristic zero is a classical piece of mathematics, crystallized at the end of XIX–beginning of XX centuries. It took the mankind another some 100 years to achieve the same classification over fields of characteristic $p > 3$ (see [10]). The cases $p = 2$ and 3 remain widely open. In [12], an attempt was made to advance the case $p = 2$ basing on earlier results of Skryabin [11]. The general line of attack is more or less the same as in "big" characteristics: one first classifies algebras of small toral rank, and then, taking advantage of appropriate root space decompositions, glue the results together; also, many questions are reduced to computation of deformations of certain classes of algebras. In particular, in [11] simple Lie algebras having a Cartan subalgebra of toral rank 1 are characterized as certain filtered deformations of semisimple Lie algebras L such that

$$S \otimes \mathcal{O}_1(n) \subset L \subseteq Der(S) \otimes \mathcal{O}_1(n) + K\partial, \qquad (3)$$

where $\mathcal{O}_1(n)$ is the divided powers algebra, ∂ its standard derivation, and either $n = 2$ and $S \simeq W_1'(n)$, or $n = 1$ and S is isomorphic to a two-variable Hamiltonian algebra.

In [12], these deformations were computed in the simplest case $S \simeq S$, what allowed to classify simple Lie algebras of absolute toral rank 2 and having a Cartan subalgebras of toral rank 1 – a small but necessary step in the classification program. To further advance along this road, one need to compute these deformations in all the cases.

QUESTION 7. *Compute deformations of semisimple Lie algebras in characteristic* 2 *of the form (6).*

In the process of these computations, it became apparent that a new cohomology theory peculiar to characteristic 2 plays a role. This cohomology is defined via the standard formula for the coboundary map in the Chevalley–Eilenberg complex, with the alternating cochains being replaced by symmetric ones. Note that we can (profitably) consider commutative 2-cocycles of Lie algebras in arbitrary characteristic ([18] and [19]), albeit they do not lead to any cohomology; while in characteristic 2 we have a bona fide cohomology theory. Unlike the usual cohomology, this complex is apriori not restricted by the dimension of the algebra, so new interesting phenomena, similar to those appearing in cohomology of Lie superalgebras (in any characteristic), occur. Generally, this "commutative" cohomology is different from the Chevalley–Eilenberg cohomology. For example, while the second cohomology of the Zassenhaus algebra $W_1'(n)$ with coefficients in the trivial module is zero (note that this is in striking difference with the cases of "big" characteristics; if $p > 3$, the corresponding cohomology is 1-dimensional, leading to the modular analog of the famous Virasoro algebra, cf. [20]), the analogous "commutative" cohomology has dimension n and is generated by "commutative" 2-cocycles

$$e_i \vee e_j \mapsto \begin{cases} 1 & \text{if } i = j = 2^k - 2, \text{or } \{i,j\} = \{-1, 2^{k+1} - 3\} \\ 0 & \text{otherwise.} \end{cases}$$

for $k = 0, \dots, n-1$.

(This can be established by considering subalgebras of $W_1'(n)$ spanned by a "small" number of root vectors – what corresponds to the cases $n = 2$ and 3 – similarly how it was done in computation of commutative 2-cocycles on simple Lie algebras of classical type in [18]).

Besides a few isolated computations like just presented, virtually nothing is known about this kind of cohomology, so any result about it would be of interest. For example, to compute deformations in Question 4, one need to compute low-degree "commutative" cohomology with various coefficients of simple Lie algebras involved – the Zassenhaus and Hamiltonian algebras.

Question 8. *Compute the "commutative"cohomology for various Lie algebras in characteristic* 2.

Question 9. *Is it possible to represent the "commutative" cohomology as a derived functor?*

5. Variations on a theme of Ado

The Ado Theorem, one of the cornerstones of the modern theory of Lie algebras, says that each finite-dimensional Lie algebra has a finite-dimensional faithful representation. Somewhat surprisingly, its proof is not that straightforward as one may expect for such a basic result: it involves universal enveloping algebras – infinite dimensional objects, and is strikingly different for the cases of zero and positive characteristics. There exists a substantial body of literature with variants of the proof of the Ado theorem, but all of them follow, more or less, the same pattern. In [21] a different proof was given, not appealing to the notion of universal enveloping algebra and intrinsic to the category of finite-dimensional Lie algebras. Unfortunately, the proof is valid for nilpotent Lie algebras and in characteristic zero only.

Question 10. *Give a characteristic-free, "short" and "natural" (i.e., not utilizing the notion of universal enveloping algebra or any other infinite-dimensional objects) proof of the full Ado Theorem.*

In the standard proofs of the Ado Theorem, the case of nilpotent Lie algebras is the most laborious part. Then, the general case is derived from the nilpotent one via the passage to the algebraic envelope, and employing a certain structure of a faithful module built, again, with the help of universal enveloping algebra (and the PBW theorem).

To get a partial answer to Question 10, we may try to move along the same route, but employing ideas of [21]. As any finite-dimensional Lie algebra is embedded into its algebraic envelope (first constructed by Malcev [22], and independently by Goto [23] and Matsushima [24]), it is enough to prove the Theorem for algebraic Lie algebras. In characteristic zero, the Levi–Malcev decomposition of any algebraic Lie algebra is of the form $L = S + T + N$

(direct sum of vector spaces), where S is the semisimple part, T is a torus acting on the nilradical N, and $[S,T] = 0$. As in the proof of the nilpotent case of the Theorem in [21], it is enough to establish the existence of a faithful representation of L not vanishing on any nonzero central element of L, and the latters lie in N. If, say, N is $\mathbb{N}$-graded, then arguing like in [21, Lemma 2.5], we may construct a representation of N in $L \otimes tK[t]/(t^n)$, for a suitable n, with required properties. Since S and T act on N by derivations, we may extend this representation to the whole L. In this way we get a proof of the Theorem for Lie algebras whose algebraic envelope has an $\mathbb{N}$-graded nilpotent radical.

As for characteristic-free requirement, it is easy to see that all the reasonings of [21] remain valid over a field of characteristic p, if the index of nilpotency of the Lie algebra in question is $< p$. But to give a full-blown characteristic-free proof will require, apparently, new ideas.

6. Variations on a theme of Whitehead

The Second Whitehead Lemma is another classical result saying that the second cohomology of a finite-dimensional semisimple Lie algebra of characteristic zero, with coefficients in arbitrary finite-dimensional module, vanishes.

In [25] a curious "almost converse" was proved: a finite-dimensional Lie algebra of characteristic zero such that the second cohomology in any its finite-dimensional module vanishes, is either semisimple, or one-dimensional, or is the direct sum of a semisimple and one-dimensional algebra. According to [26], over an algebraically closed field this is exactly the list of finite-dimensional Lie algebras having the tame representation type.

Question 11. *What is the reason that these two classes of Lie algebras coincide?*

Note that in the modular case the situation is entirely different, due to the result of Dzhumadil'daev [20]: for any finite-dimensional Lie algebra over a field of positive characteristic, and any degree not exceeding the dimension of the algebra, there is a finite-dimensional module with non-vanishing cohomology in that degree. In particular, the one-dimensional Lie algebra is the only finite-dimensional Lie algebra with vanishing second cohomology in any finite-dimensional module.

QUESTION 12. *Is it true that over an (algebraically closed) field of positive characteristic, the only finite-dimensional Lie algebra of tame representation type is one-dimensional?*

If this question has an affirmative answer, then one may ask for a characteristic-free variant of Question 6; a satisfactory answer should establish a bijection between these two classes of Lie algebras, without addressing peculiarities related to characteristic of the base field.

It is known that analogs of the Second Whitehead Lemma hold for other classes of algebraic structures: Jordan algebras, alternative algebras, Lie triple systems, etc.

QUESTION 13. *Do some sort of converses hold for all these analogs of the Second Whitehead Lemma?*

7. VARIATIONS ON A THEME OF BANACH

In [27] an attempt was made to trace the possible origins of a (vaguely formulated) question by Stefan Banach about ternary maps which are superpositions of binary maps.

This is possibly the most isolated topic in the present collection, though a nice relation with Lie theory exists: an answer to possible interpretations of the Banach question may be obtained using an idea borrowed from a pioneering paper by Jacobson about Lie triple systems.

In the process the following question arose. Let us count the number of ternary maps $f : X \times X \times X \to X$ on a finite set X of n elements, which can be represented as a superposition of binary maps $* : X \times X \to X$. We get the following table:

n	$T_L(n)$	$T_{LR}(n)$	$T_{comm}(n)$
1	1	1	1
2	14	21	5
3	19292	38472	48

Here $T_L(n)$ denotes the number of ternary maps represented in the form

$$f(x, y, z) = (x * y) * z \tag{4}$$

for some binary map $*$, $T_{LR}(n)$ denotes the number of ternary maps represented either in the form (4), or in the form

$$f(x, y, z) = x * (y * z), \tag{5}$$

and $T_{comm}(n)$ denotes the number of ternary symmetric maps (i.e., invariant under any permutation in S_3) represented in the form (4). In the given range, the latter number coincides with the number of ternary symmetric maps represented in the form (4) for some *commutative* $*$.

As of time of this writing, the 3-term sequences for $T_L(n)$ and $T_{LR}(n)$ were absent in The Online Encyclopedia of Integer Sequences, and among a dozen or so sequences containing the 3-term sequence for $T_{comm}(n)$, nothing seems to be relevant.

QUESTION 14. *Continue this table. Give formulas (closed form, or recurrent) for the numbers $T_L(n)$, $T_{LR}(n)$, $T_{comm}(n)$.*

QUESTION 15. *Is it true that for every n, any symmetric ternary map represented in the form (4) for some $*$, can be represented in the form (4) for some commutative $*$?*

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Зусманович Паша АЛГЕБРЫ ЛИ И ВОКРУГ НИХ: ИЗБРАННЫЕ ВОПРОСЫ

Обсуждается несколько открытых вопросов. Затронутые сюжеты включают в себя когомологии алгебр Ли токов и связанных с ними алгебр, алгебры, представимые в виде суммы подалгебр, структуры и явления, специфичные для характеристики 2, и вариации на темы Адо, Уайтхеда и Банаха.

Зусманович П. ЛИ АЛГЕБРАЛАРЫ ЖӘНЕ ОЛАРДЫҢ АЙНАЛАСЫНДА: ТАҢДАМАЛЫ МӘСЕЛЕЛЕР

Бірнеше ашық мәселелер талқыланады. Қозғалатын тақырыптар өзіне токтардың Ли алгебралары мен олармен байланысқан алгебралар когомологияларын, ішкіалгебралар қосындысы түрінде кейіптелетін алгебраларды және 2 сипаттамасы үшін арнайы құбылыстарды, әрі Адо, Уайтхед және Банах тақырыптарына вариацияларды қамтиды

Правила "Математического журнала" для авторов статей

Общие положения

В "Математическом журнале" публикуются оригинальные статьи по основным разделам современной математики: теория функций, функциональный анализ, обыкновенные дифференциальные уравнения, уравнения с частными производными, алгебра, математическая логика, теория чисел, геометрия, топология, теория вероятностей и математическая статистика, вычислительная математика, математическая физика, математическое моделирование. Журнал выпускается ежеквартально, четыре номера составляют том.

Статья должна быть написана на высоком научном уровне, содержать новые, четко сформулированные математические результаты и их доказательства. Во введении необходимо привести имеющиеся результаты по теме представленной работы, дать краткое содержание статьи и отразить актуальность, новизну полученных автором результатов.

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В "Математическом журнале" публикуются статьи объемом до 25 журнальных страниц. Статьи объемом более 25 страниц публикуются по специальному решению редколлегии журнала. Принимаются статьи, написанные на казахском, русском и английском языках. Статьи рецензируются.

Требования к оформлению статей

1. Рукопись статьи должна быть подготовлена в издательской системе LaTeX-2e и представлена в виде двух твердых копий, а также в виде tex и pdf - файлов на любом электронном носителе или прислана по электронной почте zhurnal@math.kz, mat-zhurnal@mail.ru. Статья должна быть подписана всеми авторами. Правила оформления рукописи и стилевые файлы можно найти на сайте Института математики и математического моделирования МОН РК http://www.math.kz в разделе "Математический журнал".
2. В левом верхнем углу необходимо указать индекс УДК, далее инициалы

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Литература

1 Мынбаев К.Т., Отелбаев М.О. Весовые функциональные пространства и спектр дифференциальных операторов. — М.: Наука, 1988. — 288 с. (для монографий)

2 Женсыкбаев А.А. Моносплайны минимальной нормы и наилучшие квадратурные формулы // Успехи матем. наук. — 1981. — Т. 36, вып. (или №) 4. — С. 107–159.

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