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The collection contains papers about actual problems of the theory of models, associated with the study of Jonsson, Generalized Jonsson, and Horn theories, elementary pairs, homogeneous models of stable theories, properties of polygons, unars theories. We consider issues like: behavior of the theories, while we expand the signature, expressive power of some software logics, also related to the model-theoretic algebra. Most papers are written by the results of work, which was done by dint of the direct supervision of scientific direction or T.G. Mustafin (R.I.P.). This collection is dedicated to him.

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August 24, 1994 at the age of 52 years died a prominent scientist-mathematician, Doctor of Physical and Mathematical Sciences, Professor, Director of the Institute of Applied Mathematics of the National Academy and the Ministry of Education of the Republic of Kazakhstan, Tulendy Garifovich Mustafin.

The pupil of the famous Siberian scientific school, he graduated from Novosibirsk State University in 1965 and defended his PhD (1971) and doctorate (1990) thesis in Akademgorodok

After NSU he began as an assistant of Tselinograd Pedagogical Institute. In 1972, when he was head of the Department of Algebra and Geometry, he has been invited to the new Karaganda State University, where he worked until the end of his life (22 years). In Karaganda State University he was head of the department, the dean, vice-rector for academic affairs, last year of his career life he was head of the Institute of applied mathematics of double subordination. Tulendy Garifovich was a bright person, a real scientist and teacher. From the first years of work in the University, while he was creating the scientific school, he has oriented education of that school to the international level. His aim was to get the recognition of foreign mathematicians. In 1966 (at age 23), he gave a presentation at the International Congress of Mathematics in Moscow - the highest forum of mathematicians around the world. At the same forum in 1994 in Zurich (Switzerland), he was the only representative from Karaganda. In 1987 TG Mustafin, together with colleagues from Almaty and Siberia organizes International scientific cooperation of Karaganda, Almaty, Novosibirsk and France universities (II-A-117 project).

In 1990, by this cooperation (the first time in the USSR), on the territory of Karaganda State University, there was successful Soviet-French symposium on model theory, which was attended by 17 leading scientists from France, the USA, Germany and Switzerland. All activities in the organization of this conference was directed by T.G. Mustafin. In the same year (1990) he went to Helsinki, to Logic Colloquium as a guest speaker, which means a high recognition of his scientific authority.

In 1992 Tulendy Garifovich has stayed in Paris for two months, where he has given lectures and done scientific work on a special grant of the Ministry of Science and Technology of France. Then, two more times, he leaved to France, because of the invitation of French colleagues.

For a high level of scientific papers Tulendy Garifovich was awarded by personal grant from the Soros Foundation, and his laboratory of Pure and Applied Logic was awarded by The international grant INTAS. TG Mustafin has published over 75 scientific papers, mostly in domestic and foreign famous journals and collections. He is the author of three books, founder

of some new directions in mathematics, which are picked up and now are developing by experts from France, Russia, Kazakhstan and other countries. His works have a high citation index, which mean the great scientific achievements in international practice. His theorems, methods and concepts are timeless.

Tulendy Garifovich founded a scientific model theory school in Karaganda, which has received worldwide recognition. His students, most of whom are PhDs, are continuing his work - the development of algebra and mathematical logic in Kazakhstan.

Wherever Tulendy Garifovich was working and whatever he was doing, he always was an example of high responsibility, deep vision and broad scope of the issues involved. Although he was head of some departments and he has been honored, he was always open and available to everyone. Surroundings have been always amazed by his optimism and life loving..

Tulendy Garifovich Mustafin was not only a great scientist of world renown, but also a great teacher and organizer of science, a man who was characterized by the noble features of a patriot and a citizen.

EXPANDING OF O-MINIMAL MODEL BY CONVEX UNARY PREDICATES*

In work [1] there was formulated the following problem: let M be o-minimal, M^* - the expansion of M , obtained by adding unary predicate, emitting M in a convex subset, so is M^* weakly o-minimal?

In this paper we prove a theorem, which is a solution of that problem.

Definition 1 [2,3]. *Linearly ordered structure D is o-minimal if every definable (definable with parameters) subset of D is a finite union of points and open intervals $I = (a, b)$, $a \in DU[-\infty]$, $b \in DU[\infty]$.*

Definition 2 [3]. *The theory T is o-minimal if any model of theory T is o-minimal.*

Definition 3 . *A subset $D \subset U$ of linearly ordered structure D is called convex if for any $a, b \in U$ is true $[a < b \rightarrow \forall t(a < t < b \rightarrow t \in U)]$.*

Definition 4 [4]. *Linearly ordered model D is called weakly o-minimal if every definable (with constants definable) subset of D is a finite union of convex subsets.*

Definition 5 [4]. *A theory T is called weakly o-minimal if any model of theory T is weakly o-minimal.*

Theorem 6 [5, 6]. Any theory of the o-minimal model is o-minimal.

Note that the theory of weakly o-minimal model is not necessarily weakly o-minimal. [1].

O-minimal theory has nice model-theoretic properties.

Theorem 7 [3]. For any set of o-minimal model D there exists a simple atomic model.

Theorem 8 [3]. Let D be o-minimal structure, $a \in \text{acl}(A \cup [b] \text{acl}(A))$, $A \cup [a, b] \subset D$. Then $b \in \text{acl}(A \cup [a]) \text{acl}(A)$

Definition 9 [2]. 1 - type o-minimal model D is called single, if for any α , realizing type in model $D(\alpha)$, true $\forall \beta \in D(\alpha)$ if β realizes $\beta = \alpha$.

Definition 10 . The partition of o-minimal structure on two convex subsets is called a section. If the section has the largest element in the lower subset, the section is called a rational [1]. Otherwise the section is called irrational.

Note that the smallest and largest elements can be $-\infty$ or ∞ .

We will identify the section with the appropriate quantifier-free type in the signature j

Theorem 11 [3]. Linearly ordered model D is o-minimal if and only if any one type of D is defined by section.

We note that B. Kulpeshov has given a description of the weakly o-minimal model in terms of the types and sections.

Designation . Let D be elementary submodel of o-minimal model D_1 . By $D(A)$ we will define a simple atomic model of the $D \cup A$ in the D_1 model. If $A = \alpha_1, \alpha_2, \dots, \alpha_n$ then $D(A) = D(\alpha_1, \alpha_2, \dots, \alpha_n)$

It is obvious that to single type there corresponds irrational section.

Definition 12 . The type determined by rational section, we will call rational. Not single, irrational 1 type we will call social.

Theorem 13 [7, lemma 7]. Irrational type o - minimal model is social if and only if there exists an unary function $h(x, d), d \in D$ that $\forall x [h(x, d) > x]$ and is right for any γ $[\gamma \text{ realizes} \leftrightarrow h(x, d) \text{ realizes}]$.

Agreement. Instead of a formula $x < z < y$ we will write $z \in I$, where x, y variable symbols and $I = (x, z)$.

Often instead of $\exists x \exists y (x < y)$ we will write $\exists I$. If $a < z < b$ and a, b elements from D , we write $z \in J, J = (a, b)$. The record $J \subset D$ means that

$a, b \in D$. For example, if $J \subset D$, $\alpha \in D_1$, then record $\alpha \in J$ means that α lies in D - a definable interval of J . Thus, I always uncertain interval, J always an interval with the trailer elements lying in a certain model.

Definition 14 (a). Let $p(x), q(y) \in S_1(D)$ - types o-minimal model D . We will tell that orthogonal to q if for any α , realizing type p , in model $D(\alpha)$ the type q isn't realized.

Definition 15 (b). The family of types $[p_i | i \in K] \subset S_1(D)$ is called orthogonal if for any $i \in K$ for any α_j , ($j \in K, \text{ by } j \neq i, \alpha_1$ realizes p_1) in model $D([\alpha_j | j \in K \setminus i])$ don't realize the type p_1 .

Definition 16 [5, 9]. Let M o-minimal structure. (a) If $X = [a]$, where $a \in M$, the X cage and $\dim X = 0$. If X is the interval (a, b) ($a, b \in M \cup -\infty, \infty$), the X cage and $\dim X = 1$.

(b) We will assume $X \subset \mathbb{R}^n$ is a cage with $\dim X = k$. We will define two types of cages in M^{n+1} associated with X :

(1) Let $h : X \rightarrow \mathbb{R}$ formular and continuous function. Then $X_1 = \text{graph } h$
(h) $= \{x, h(x) : x \in X\}$ is the cage in M^{n+1} , and we assume that $\dim X_1 = k$.

Theorem 17 [5, 9]. If $X \subset \mathbb{R}^n$ is the M -definable formular set, we can break X on final number of cages. If X is defined by means of constants only from A , then each cage is defined only by means of constants from A .

Further for lemmas of model D , M always o-minimal.

Lemma 18 . Let $p, q \in S_1(D)$.

(a) $[p \text{ is orthogonal to } q \leftrightarrow q \text{ is not orthogonal to } p]$; (b) If p is orthogonal to q , then $(x) \cup q(y)$ full 2 type from $S_2(D)$.

Proof. (b) follows from (a) and definition of orthogonality.

(a) Let p is not orthogonal to q . Then in model $D(a)$ by definition 16(a) there exists an element $\beta \in D(a)$, realizing type q . Then for β there is a full formula $\Theta(y, \alpha, \bar{d})$, $\bar{d} \in D$, not divisible by formulas over $D \cup \{\alpha\}$ that $D(a) \models \Theta(\beta, \alpha, \bar{d})$. For any $\gamma \in \Theta(D(a), \alpha, \bar{d})$ $tp(\gamma/D) = tp(\beta/D) = q$. The formula $\gamma(y, \alpha, \bar{d})$ is equivalent to an open interval or realizes one element. If it is true that $\exists! \gamma(y, \alpha, \bar{d})$, then $\beta \in dcl(\alpha, \bar{d})$ and by theorem 8 $\alpha \in dcl(\beta, \bar{d})$, that means existence of a formula $\psi(\cdot, \beta, \bar{d})$ such that $D(a) \models \exists! \psi(x, \beta, \bar{d})$ and $\psi(\alpha, \beta, \bar{d})$. Then formula $\exists! \psi(x, y, \bar{d}) \in q$. And then the formula $\mu(\cdot, \bar{d}) = \psi(x, y, \bar{d})$ and $\Theta(y, x, \bar{d})$ defines bijection. Let $g(y, \bar{d}) = \leftrightarrow \mu(\cdot, \bar{d})$. Then $g(\beta, \bar{d}) = \alpha$. The formula $\exists \mu(\cdot, \bar{d})$ breaks up to finite number of intervals and points. We will consider opened definable interval $\bar{d}$ - of J containing β . On this interval $g(y, \bar{d})$ is monotonous [5, 9]. $J_2 = \exists(\mu(\cdot, \bar{d}))$. and $\alpha \in J_2$ Then function $g(y, \bar{d})$ monotonously maps J_1 to J_2 . We will consider that g increasing as $\alpha, \bar{d} \in D$.

Statement. For any $J \subset D$, $\alpha \in J \subset J_2$, $J \subset D$, $\beta \in J \subset J_2$, $g(J, \bar{d}) = J$.

As J' it is possible to take $g^{-1}(J, \bar{d})$.

For any β , realizing type q , in model $D(\beta)$ the type p is realized. Really, the element $g(\beta', \bar{d})$ from $D(\beta')$ has to realize $(\cdot)$. If it doesn't realize type (x) , owing to the theorem 10 there exists $J \subset D$, $x \in J \in p(x)$, $g(\beta', \bar{d}) \in J$. But then $g^{-1}(g(\beta, \bar{d})) = \beta' \in J$. But according to the statement the formula $y \in J$ belongs to type q that contradicts a choice β' . We will notice that if $\Theta(D(\alpha), \alpha, \bar{d}) = J_0 = (\delta_1, \delta_2)$, then $tp(\delta_1/D) = q$ or $tp(\delta_2/D) = q$. If $\delta_1 \in D_1$ that as in an interval (δ_1, β) there is no elements from D , then according to the theorem 10 $tp(\delta_1/D) = tp(\beta/D)$. From this we have that $\delta_1 \in dcl(D \cup \alpha)$.

If $\delta_1, \delta_2 \in D$, in the range of J_0 there exists an element from D that contradicts a choice $\Theta(y, \alpha, \bar{d})$. Then for $\delta_1 \in D$ it is possible to carry out a similar reasoning, as well as for β .

Lemma 19 . Let $p, q \in S_1(D)$. p is not orthogonal to q . Then

- a) $\text{rational} \leftrightarrow q \text{ rational}$;
- b) $\text{- social} \leftrightarrow q \text{ - social}$;
- c) $\text{- single} \leftrightarrow q \text{ - single}$.

Proof. Existence of function $g(y, \bar{d}, d \in D)$ follows from the proof of a lemma 1 (a), $d \in D$ transferring any realization of type q to realization of type p and for any realization a of type p exists in $D(\alpha)$ an element β , realizing type q that $f(\beta, \bar{d}) = a$. We will notice that in model $D(\alpha)$ the set of realization of type p is a convex set of R_p . And as between two convex sets R_p and R_q formed by types p and q , exists bijective continuous map, a type of these convex sets same. Convex set R_p single-elemented if and only if R_q single-elemented. R_p has the smallest upper limit in D if and only if R_q has the smallest upper limit from D or the greatest lower limit from D .

Lemma 20 . (a) Let the D elementary submodel of $\mathcal{O}$ -minimal model M . If the M can be received from the D by consecutive realization of social and single types, for any $a, b_1, b_2 (b_1 < a < b_2, a \in D, b_1, b_2 \in M \setminus D)$ there exist c_1, c_2 from D , where $b_1 < c_1 < a < c_2 < b_2$ holds.

(b) Let $Q = [q_i | i \in S]$ is orthogonal family of irrational 1-st type $\mathcal{O}$ -minimal model M , $q_i = 1 | i \in S$ β_1 realizes q_1 . Then for any β_1 , for any γ_1, γ_2 from $(\beta_1) (\beta_1 \in (\gamma_1, \gamma_2))$ there exists $c_1, c_2 \in M$, that

$$\gamma_1 < 1 < \beta_1 < 2 < \gamma_2.$$

Proof, (b) follows from definition 16(b).

(a) We will copy a lemma condition.

Let the D elementary submodel of $\mathcal{O}$ -minimal model M . Let $\bigcup D_1$, where k - some ordinal, such that $I < k$.

1) $D = D_0$;

2) $D_1 = D_2$, if $i < j < k$,

3) $D_1 + 1 = D_1(\alpha_1 + 1)$, $\alpha_1 + 1$ realizes social or single type over D ;

4) $\bigcup D_1$, if j - a limit ordinal.

. Then for any $j < k$, for any b_1, b_2 ($b_1 < a < b_2$, a from D , $b_1, b_2 \in D_1$) exist $c_1, c_2 \in D$, that $b_1 < c_1 < c_2 < b_2$. We will prove by method of transfinite induction.

Let $j = i + 1$. For any $\gamma_1, \gamma_2 \in D_1 + 1$ ($\beta_1 < a < \beta_2$) exist $\gamma_1, \gamma_2 \in D_1$, that $\beta_1 < \gamma_1 < a < \gamma_2 < \beta_2$. Because $tp((\beta_1/D_1) = tp(\beta_2/D_1))$, then by induction assumption in intervals $J_1 = (\gamma_1, a)$, $J_2 = (a, \gamma_2)$ exist elements from D .

The next lemma is specification of the theorem 14.

Lemma 21 . Let $(x) \in S(M)$ - social. Then for any $\gamma \in (\alpha)[tp(\gamma/D)] = ()$ and $\alpha < \gamma \rightarrow$ is true, there is an unary function $h(x, \bar{d})$, that $\gamma \leq h(\alpha, \bar{d})$, $tp(h(\alpha, \bar{d})/D) = ()$.

Proof. Let $\gamma \in D(\alpha)$, $tp(\alpha/D) = ()$. If the formula $\Theta(x, \alpha, \bar{d})$ full over $D \cup \alpha$, it defines an open interval or a point (owing to $\mathcal{O}$ -minimality of D).

If $\Theta(x, \alpha, \bar{d})$ defines a point $\Theta(\gamma, \alpha, \bar{d})$, it is possible to define unary function as in the proof of a lemma 1(a).

If $\Theta(x, \alpha, \bar{d})(D(\alpha)) = (\gamma_1, \gamma_2)$, then $\exists \gamma \in (\delta_1, \delta_2) tp(\gamma/D) = ()$.

If $tp(\delta_2/D) = ()$, then $\delta_2 \in D$ and therefore type is rational. Therefore, $tp(\delta_2/D) = ()$. And as $\delta_2 \in dcl(\bar{d}, \alpha)$, it is possible to define the function $h(x, \bar{d})$ which isn't removing for' limits of a convex set of realization of type (x) as in a lemma 1 (a).

Lemma 22 . (a) Let $p, q \in S_1(D)$, - irrational type, orthogonal to single type q , α - realization of type p . Then the only type $q \in S_1(D(\alpha))$, the expanding q , is single.

(b) Let $p, q_1, q_2 \in S(D)$, p - social, single type q_1 orthogonal to single type q_2 , α realizes type p . Then the single type q_1 is orthogonal to the single type $q_2, q_1 \cup q_2$ from $S_1(D(\alpha))$, $q_2 \cup q_2 \in S_1(D(\alpha))$.

Proof. (a) Uniqueness of type q' follows from a lemma 1(b). Let β realizes type q . We will consider model $D(\alpha)(\beta)$. In model $D(\alpha)(\beta)$ there exists at least one element β_1 from $D(\alpha)(\beta)$ $D(\alpha)$, that $tp(\beta_1/D(\alpha)) = tp(\beta/D(\alpha))$.

Case 1. $\beta < \beta_1$ Let $D(\beta)$ is a simple model over $D \cup \{\beta\}$ in model $D(\alpha)(\beta)$, $D(\beta) = D(\alpha)(\beta)$.

From uniqueness of a type q it follows that $\beta_1 \in D(\beta)$. We will a (y) determines type $r() \in S_1(D(\beta))$. The type $r(y)$ is rational and realizes in model $D(\beta)(\alpha)$. Therefore, the type $r(y)$ is not orthogonal to type $(x) \in S_1(D(\beta))$, $\subset$. From lemma 2(a) it follows that $r()$ is rational.

Case 2 $\beta > \beta_1$

Considered similar to the case 1. Instead of section $R_\beta(y)$ is considered section $L_{\hat{\beta}}(y) = [d < y < \beta | d < \beta, d \in D(\beta)]$.

Consequently, the convex set of realizations of the type $p(x)$ model $D(\beta)$ should be limited from the top or bottom element from $D(\beta)$. This may not be an element from $D(\beta)$ D , in otherwise types $p(x)$ and $q(y)$ are not orthogonal. Therefore, this element should be made of D , which means that $p(x)$ is rational.

(b) Let α implements type p , β_1 realize type q_1 . At the model $D(\beta_1)$ are not implemented types of p [Lemma 2 (b)] and q_2 (by the condition of the lemma). In model $D(\beta_1)(\alpha)$ not implemented q_2 such that q_2' is single (lemma 5 (a)), and p' - social [theorem 14] (here $q_2 \subset q_2' \in S_1(D(\beta_1)), p \subset p' \in S_1(D(\alpha))$). Then in model $D(\alpha)(\beta_1) \supset D(\beta_1)(\alpha)$ does not occur type q_2 . Adoption from lemma 5(b) that of the arbitrary choice of α, β_1 .

Lemma 6. Let M o - minimal structure, $Q = [q_1, \dots, q_s | q_i \in S_1(M), 1 \leq i \leq s]$ - orthogonal family types of the M , $B = (\beta_i | 1 \leq i \leq s, \beta_1$ realizes type $q_1)$. Then for any formula $\psi(y_1, \dots, y_m, d), d \in M$ if $M(\beta_1, \dots, \beta_m) \models \psi(\beta_1, \dots, \beta_m, d)$, then there are exists intervals $J_1, \dots, J_m \subset M$ such that, following done

1) $\beta_i \in J_i, 1 \leq i \leq s$;

2) $M = \forall y_1, \dots, \forall y_s ((y_1 \in J_1) \rightarrow \dots \rightarrow \psi(y_1, \dots, y_s, d))$.

Proof. Consider the formula $\psi(y, \beta_1, \dots, \beta_s, d)$ Implementation of this formula must be infinite, otherwise case Theorem $\beta_1 \in M(\beta_1, \dots, \beta_s, d)$, which contradicts to orthogonality Q family types. Because of the

o-minimal-existence should be $(\beta_2, \dots, \beta_s, d)$ - determinable interval $J_1 \subset M(\beta_2, \dots, \beta_s)$, that $\beta_1 \in J_1$ and right

$$M(\beta_2, \dots, \beta_s) \models \forall y_1 [y_1 \in J_1 \rightarrow \psi(y_1, \beta_2, \dots, \beta_s, d)] \quad (1)$$

Obviously, for any $J_1 \in J_1$ formula (1) will be performed from lemma 3(b) the existence of a $J_1 \subset M$, that $J_1 \subset J_1$ and $\beta_1 \in J_1$.

Let $\psi_1(\beta_2, \dots, \beta_s, d) := \forall y_1 [y_1 \in J_1 \rightarrow \psi(y_1, \beta_2, \dots, \beta_s, d)]$. Repeating (s-1) times the argument, we obtain $J_2, \dots, J_s \subset M$.

Lemma 7. Let D o-minimal model. $f(\bar{u}, \bar{b})$ and $g(\bar{u}, \bar{b})$ two functions ($\bar{b} \in D$) such that $D \models \forall \bar{u} [f(\bar{u}, \bar{b}) < g(\bar{u}, \bar{b})]$. The formula $\theta(\bar{u}, \bar{d})$ such that $D \models \forall \bar{u}_1 \forall \bar{u}_2 [\theta(\bar{u}_1, \bar{d}) \ \& \ \theta(\bar{u}_2, \bar{d}) \rightarrow f(\bar{u}_1, \bar{b}) < g(\bar{u}_2, \bar{b})]$. Then right

$$D \models \exists x [\forall \bar{u} [\theta(\bar{u}, \bar{d}) \rightarrow [f(\bar{u}, \bar{b}) \leq x < g(\bar{u}, \bar{b})]]] \text{ or}$$

$$D \models \exists x [\forall \bar{u} [\theta(\bar{u}, \bar{d}) \rightarrow [f(\bar{u}, \bar{b}) < x \leq g(\bar{u}, \bar{b})]]]$$

Proof. Let $H(x, b, d) := \forall \bar{u} [\theta(\bar{u}, \bar{d}) \rightarrow f(\bar{u}, \bar{b}) < x]$. Since D o-a minimal model, then $H(D, b, d)$ splits into a finite number of intervals $\bar{b} \& \bar{d}$ - definable and $H(D, \bar{b}, \bar{d}) \neq \emptyset$ and, since any $g(\bar{u}, \bar{b})$ with condition $\theta(\bar{u}, \bar{d})$ satisfies $H(x, \bar{b}, \bar{d})$. Consequently, there is a lower boundary point. Suppose a lower bound $H(D, \bar{b}, \bar{d}), c \in D$.

Case 1. $c \in H(D, \bar{b}, \bar{d})$

In this case, for any $\bar{u}$, if $D \models \theta(\bar{u}, \bar{d})$. $\rightarrow f(\bar{u}, \bar{b}) \leq c$. Hence $g(\bar{u}, \bar{b}) > c$.

Case 2. $c \in H(D, \bar{b}, \bar{d})$

In this case for any $\bar{u}$, if $D \models \theta(\bar{u}, \bar{d}) \rightarrow g(\bar{u}, \bar{b}) \geq c$. Consequently, $f(\bar{u}, \bar{b}) < c$.

Hence the conclusion of Lemma 2.

Lemma 8. Let Y b-definable cell in D , $\bar{b} \in D$. $K(\bar{b}, D^l) = Y \subset D^1$. Then, if some extension D_1 models D exist $a_1, \dots, a_i \in D_1$ D , which is carried out

$$1) < a_1, \dots, a_1 > \in K(b, D_1^1) = Y(D_1^1);$$

$$2) [a_1, \dots, a_1] \text{ - algebraically independent over } D, \text{ then } \dim Y = l.$$

Proof. By induction 1.

$1 = 1$. Obviously, $K(b, z) \equiv Z \in I \subset D$, where I b-definable open interval.

$1 + 1$ $D_1 = K(b, a_1, \dots, a_i, a_{i+1})$. By definition, the cells $< a_1, \dots, a_i >$ lies in the $X(D_1^1)$, where X is a cell dimension 1. $a_{i+1} \in \text{acl}(D, a_1, \dots, a_i)$.

Therefore, $K(b, z_1, \dots, z_i, z_{i+1})$ determines a cell of dimension $1 + 1$.

Lemma 9. For theory T signatures Σ , comprising linear order axioms, had only weakly o-minimal expansion, it is necessary and sufficient that for any formula $\phi(x, y)$ there was a positive integer N_ϕ that

$$\neg \exists \bar{y} \exists x_1. \quad \exists x_{N_\phi} (\& \& \neg \varepsilon_\phi(x_i, x_j, \bar{y})) \in T$$

$$< \varepsilon_\phi(x, z, \bar{y}) := (\phi(x, y) \& \phi(z, \bar{y}) \& (z \neq X - - > ((z < x - - > \forall t (z < t < x - - > \phi(t, y))) \& (x < z - - > \forall t (x < t < z - - > \phi(t, \bar{y})))))).$$

The proof follows easily from the definition of weak o-minimal theory. Let D model of the minimum-theory Q - orthogonal family of single types of $S_1(D)$, R family social types of $S_1(D)$. Fix an enumeration R . We assume that D is elementary submodel some $|P \cup Q \cup D|^+$ - saturated model and all and the elements in question are in this model fixed $|P \cup Q \cup D|^+$ -saturated model.

We define by induction on $\langle n, m \rangle$ ($n < k = |P|, m < \omega$) The following model elements and types:

$$A_o = \emptyset. \quad D_o = D;$$

$$A_n = [\alpha_{i,j} | i \leq n; j < \omega; \alpha_{i,j} < \alpha_{i,j+1} \text{ implement one type. } p \in P];$$

$$D_n = D(A_n);$$

$$A_{n,m} = \bigcup_{i < n} A_i \cup [\alpha_{n,j} | j \leq m, \alpha_{n,i} \text{ implement type } P.$$

with minimum ordinal in $D(\bigcup_{i < n} A_i)]$.

Note 1. $A_{n,m}$ - algebraically independent set on D .

Note 2. $\text{tp}(\alpha_{n_1}^-, \alpha_{n_2}^-, \dots, \alpha_{n_k}^- / D) = \text{tp}(\alpha_{n_1}^+, \alpha_{n_2}^+, \dots, \alpha_{n_k}^+ / D)$ for all $\alpha_{n_1}^-, \alpha_{n_i}^- \in A_{n_i} \setminus \bigcup_{j < n_i} A_j$ equally ordered sequences of the same length.

Let $p = \text{tp}(\alpha_{n,1} / D)$. But the construction of p - social. Therefore, the section defined by p , consists of two non-empty convex subsets of D . We denote by C_n top convex set. Consider the following set besnivantornyh formulas of the signature of the $\Sigma_0 = \{<\}$.

$$R_{n,m+1}(x) = (a < x < c | c \in C; a \in D_{n,m} \text{ such that } \forall c \in C_n, a < c').$$

Proposition 3. $R_{n,m+1}(X)$ defines an irrational section in $D_{n,m}$, and therefore the only way expands to the full over the $D_{n,m}$.

Proof. Compatibility of $R_{n,m}(x)$ should be of a kind convex set C_n defined social type p . Suppose, $R_{n,m+1}(x)$ defines an irrational section. Consequently, there is $a \in D_{n,m} \setminus D$, that is, with any $e \in D_{n,m} \setminus D$ is true $[a < e \rightarrow c \in C(a < c < e)]$ and the type $\text{tp}(a/D) = \text{tp}(\alpha_{n,1}/D) = p$.

Theorem 14 implies the existence of a function $h(x,d), d \in D$, that $h(a,d) \models a$ and $\text{tp}(h(a,d)/D) = p$. This contradicts and selecting a , as $h(a,d) \in D_{n,m}$. Consequently, $R_{n,m+1}(x)$ defines the irrational section that only way completed to type over $S_1(D_{n,m})$ and is social over the $D_{n,m}$. Take an arbitrary realization of the type $R_{n,m}(x)$. Let derive it through $\alpha_{n,m+1}$

1) $A_{n,m+1} = A_{n,m} \cup [\alpha_{n,m+1}]$ is algebraically independent on D . By the construction $\alpha_{n,m} \in D(A_{n,m})$.

2) By the induction hypothesis, $\text{tp}(\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n / D) = (\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n / D)$.

Consider the types of $\text{tp}(\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n, \alpha_{n,m+1} / D)$;
(2)

$$\text{tp}(\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n, \alpha_{n,j}^+ / D) \quad (3)$$

In models $D(\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n)$ and $D(\alpha_{n_1}^-, \dots, \alpha_{n_k}^-, \bar{\alpha}_n)$ the elements $\alpha_{n,m+1}$ and $\alpha_{n,j}$ respectively, implement a single social type, defined by a convex set C_n . Uniqueness is follow from lemma 4. Therefore, the types (2) and (3) match. Let $B = [\beta_1 \dots \beta_s]$ implementation of the type $q_s \in Q$. Let $D := D(\cup_{n < k} A_n \cup B)$.

Proposition 4. a) $\cup_{n < k} A_n \cup B$ - algebraically independent set on D ;

b) $\text{tp}(\beta_s / D(A \cup (B \setminus [\beta_s])))$ - and the single type determined by the section D ;

c) $\forall \bar{\alpha}, \bar{\alpha}' \in A, \forall \bar{\beta} \in B$.
 $[\text{tp}(\bar{\alpha}/D) = \text{tp}(\bar{\alpha}'/D) \rightarrow \text{tp}(\bar{\alpha}, \bar{\beta}/D) = \text{tp}(\bar{\alpha}', \bar{\beta}/D)]$.

Proof. a) By construction and by Lemma 5(b);

b) follows from a) and Lemma 3(b);

c) follows from Proposition 4(a), Lemma 1(b), Lemma 5(b)

We now turn to the proof of the fundamental theorem.

Let D o-minimal model. $O = \{\cup_\lambda | \lambda \in K\}$ - family of convex subsets of model D . Each convex subset defines two full one type. The set of all 1-defined types O family is divided into three sets: rational, single and social types. For each type there is a constant sound of the D , defining this type. In the set of single types choose maximum orthogonal subfamily, which denote through Q . The rest of the single types of this set determined using D - definable functions and realizations types in Q (Lemma 5(b)). Many social types denoted with P and fix these types ordinals. Consider the model D , built in the previous paragraph for the P and Q .

$$P_o = \{p \in P \mid \exists \alpha_{n,1} \in A, \alpha_{n,1} \text{ realized type } p\}$$

By construction, for any $p \in P \setminus P_o$ exist D -definable function $f(\bar{u}, \bar{d}), \exists \bar{\alpha} \alpha \in A$, that $tp(f(\bar{\alpha}, \bar{d})/D) = p$.

Consider the model of $D^+ = \langle D, \cup_\lambda, \lambda \in K \rangle$, which unary predicates $\cup'_\lambda$ defined convex sets of the same name as from O .

Proposal 1(tight). For any formula $\mu(\bar{x})$ signature model D^+ , there is exists a formula $\psi_\mu(\bar{x}, \bar{\alpha}, \bar{\beta}, \bar{d})$ model D^+ , ($\bar{\alpha} \in A, \bar{\beta} \in B, \bar{d} \in D$), such that for any $b \in D$ is true

$$[D^+ = \mu(\bar{b}) \iff D \models \psi_\mu(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})]$$

The proof is by induction on the steps of building in formulas [1].

For a binary predicate $U_n(x)$ corresponding to the formula is a conjunction of two formulas suitable to the following:

$$x \leq d; \quad x < d, \quad d < x, \quad d \leq x, x < f(\bar{\alpha}, \bar{d}), \quad g(\bar{\alpha}, \bar{d}) < x, \quad g(\bar{\beta}, \bar{d}) < x, \quad x < g(\bar{\beta}, \bar{d}), \quad x < \alpha_{n,1}, \quad \alpha_{n,1} < x, \quad \beta_n < x, \quad x < \beta_n. \text{ Here } \bar{\alpha}, \quad \alpha_{n,1} \in A, \bar{\beta}, \beta_n \in B, \quad d, \bar{d} \in D.$$

Let $\mu(\bar{y}) \neq \exists x \phi(x, \bar{y})$ the formula D^+ model signature. Then the formula for $\phi(x, \bar{y})$ by the induction hypothesis There is exists a formula $\psi_\phi(x, \bar{y}, \bar{\alpha}_n, \bar{\alpha}_m, \dots, \bar{\beta}_m, \dots, \bar{\beta}_r, d), \bar{d} \in D$, that $\forall a, \bar{b} \in D [D^+ \models \phi(a, \bar{b}) \neq D^+ \models \mu_\phi(a, \bar{b}, \bar{\alpha}_n, \bar{\alpha}_m, \dots, \bar{\beta}_m, \dots, \bar{\beta}_r, d)]$. Here $\bar{\alpha}_n = \langle \alpha_{n,i_1}, \alpha_{n,i_2}, \dots, \alpha_{n,i_k} \rangle, \quad i_1 < i_2 < \dots < i_k; \quad \bar{\alpha}_m = \langle \alpha_{m,j_1}, \alpha_{m,j_2}, \dots, \alpha_{m,j_k} \rangle, \quad j_1 < j_2 < \dots < j_k$.

By condition 2, and proposition 4(c) we can change constant as follows:

$$\alpha_{n,i_1} \dashv\dashv > \alpha_{n,3} \quad : \quad \alpha_{m,j_1} \dashv\dashv > \alpha_{m,3} \vdots$$

$$\alpha_{n,i_2} \dashv\dashv > \alpha_{n,6} \quad : \quad \alpha_{m,j_2} \dashv\dashv > \alpha_{m,6} \vdots$$

$$\alpha_{n,i_k} \dashv\dashv > \alpha_{n,3k} \quad : \quad \alpha_{m,j_i} \dashv\dashv > \alpha_{m,3i} \vdots$$

Therefore, we can assume that $\bar{\alpha}_n = \langle \alpha_{n,3}, \dots, \alpha_{n,3k} \rangle, \dots; \quad \bar{\alpha}_m = \langle \alpha_{m,3}, \dots, \alpha_{m,3i} \rangle, \dots$

By Theorem formula 18 $\psi_\phi(x, \bar{y}, \bar{z}, \bar{t}, \bar{d})$ falls into a finite number of formulas (Lemma 8), defining cells. Therefore, we can assume that the formula $\psi_\phi(x, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$ is equivalent to disjunction of the form formula

$$x = h(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d});$$

$f(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d}) < x < g(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$, where $h(\bar{y}, \bar{z}, \bar{t}, \bar{d}), f(\bar{y}, \bar{z}, \bar{t}, \bar{d}), g(\bar{y}, \bar{z}, \bar{t}, \bar{d})$ - $\bar{d}$ - definable function.

We claim that $\psi_\mu(\bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$ is a disjunction formulas of the form

$$\begin{aligned} & \exists I_m \dots \exists I_r \quad \exists x [\forall \bar{z}_n \dots \forall \bar{z}_m \forall t_s \dots \forall t_r] \\ & [((\& \& (\alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1}) \& (t_s \in I_m \& t_r \in I_r)) \leftrightarrow x = h(\bar{y}, \bar{z}_n, \dots, \bar{z}_m, \\ & t_s, \dots, t_r, d)) \& x = h(\bar{y}, \bar{\alpha}_n, \dots, \bar{\alpha}_m, \beta_s, \dots, \beta_r, d) \& \beta_s \in I_s \& \dots \& \beta_r \in I_r \& (x = \\ & h(\bar{y}, \bar{\alpha}_n, \dots, \bar{\alpha}_m, t_s, \dots, t_r, d))]; \quad (6) \end{aligned}$$

$$\begin{aligned} & \exists I_s, \dots \exists I_r \quad \exists x [\forall \bar{z}_n, \dots, \forall \bar{z}_m \forall t_s, \dots \forall t_r] \\ & [((\& \& (\alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1}) \& (t_s \in I_m \& t_r \in I_r)) \leftrightarrow f(\bar{y}, \bar{z}_n, \dots, \bar{z}_m, \\ & t_s, \dots, t_r, d)) < x < g(\bar{y}, \bar{z}_n, \dots, \bar{z}_m, t_s, \dots, t_r, d) \& \beta_s \in I_s \& \dots \& \beta_r \in I_r \& f(\bar{y}, \bar{\alpha}_n, \dots, \bar{\alpha}_m, \\ & \beta_s, \dots, \beta_r,) < x < g(\bar{y}, \bar{\alpha}_n, \dots, \bar{\alpha}_m, t_s, \dots, t_r, d))] \quad (7) \end{aligned}$$

Here $\bar{\alpha}_n := \langle \alpha_{n,3}, \dots, \alpha_{n,3k} \rangle, \quad \bar{\alpha}_m := \langle \alpha_{m,3}, \dots, \alpha_{m,3l} \rangle$

$$\bar{z}_n := \langle z_{n,1}, \dots, z_{n,k} \rangle, \quad \bar{z}_m := \langle z_{m,1}, \dots, z_{m,l} \rangle$$

$$\bar{\alpha}_n' := \langle \alpha_{n,j_1}, \alpha_{n,j_2}, \dots, \alpha_{n,j_k} \rangle, \quad j_1 < j_2 < \dots < j_k$$

$$\bar{\alpha}_m' := \langle \alpha_{m,i_1}, \alpha_{m,i_2}, \dots, \alpha_{m,i_l} \rangle, \quad i_1 < i_2 < \dots < i_l$$

$$\bar{\alpha}_n'' := \langle \alpha_{n,2}, \alpha_{n,3}, \dots, \alpha_{n,3i-l}, \alpha_{n,3i}, \alpha_{n,3i+1} \rangle$$

$$\bar{\alpha}_m'' := \langle \alpha_{m,2}, \alpha_{m,3}, \dots, \alpha_{m,3i-l}, \alpha_{m,3i}, \alpha_{m,3i+1} \rangle$$

$$\bar{\alpha} := \langle \bar{\alpha}_n, \dots, \bar{\alpha}_m \rangle, \quad \bar{\alpha}' := \langle \bar{\alpha}_n', \dots, \bar{\alpha}_m' \rangle.$$

Proposition 5. For any $b \in D$ is true

$$[D' \models \psi_\mu(\bar{b}, \bar{\alpha}', \bar{\beta}, \bar{d}) \leftrightarrow D^+ \models \mu(\bar{b})].$$

Evidence. Since in model $D(\bar{\alpha}, \bar{\beta})$ is satisfied Formula (6) or (7), there are exists intervals $J'_s, \dots, J'_r \subset D(\bar{\alpha}', \bar{\beta})$, that replacing these intervals vague intervals $I_s, \dots, I_r$, formula (6) or (7) eliminates quantifiers $\exists I_s, \dots, \exists I_r, \beta_s \in J_s, \dots, \beta_r \in J_r, J'_s \subset D(\bar{\alpha}', \bar{\beta})$ Since $\beta_s \notin D(\bar{\alpha}', \bar{\beta} \setminus \beta_s)$ and $tp(\beta_s/D(\bar{\alpha}', \bar{\beta} \setminus \beta_s))$ - single that is determined by the D section (statement 4(b)), there is exists $J_s \subset J'_s$ that $J_s \subset D$ and $\beta_s \in J_s \subset J'_s$ Consequently, there exists intervals $J_s, \dots, J_r \subset D, (\beta_s \in J_s, \dots, \beta_r \in J_r)$, for which satisfied, formula (6) or (7). Thus, we got

$$D(\bar{\alpha}' \models \exists x \forall \bar{z}_n \dots \forall \bar{z}_m \forall t_s \dots \forall t_r) [(\& \& (\alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1}) \& t_s \in J_s \& \dots \& t_r \in J_r) \leftrightarrow x = h(b, \bar{z}_n, \dots, \bar{z}_m, t_s, \dots, t_r, d)];$$

$$D(\bar{\alpha}' \models \exists x \forall \bar{z}_n \dots \forall \bar{z}_m \forall t_s \dots \forall t_r) [(\& \& (\alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1}) \& t_s \in J_s \& \dots \& t_r \in J_r) \leftrightarrow f(b, \bar{z}_n, \dots, \bar{z}_m, t_s, \dots, t_r, d) < x < g(b, \bar{z}_n, \dots, \bar{z}_m, \bar{t}, \bar{d})].$$

Terms of the form $\alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1}$ allow arguments to pass to the types, since any tuples elements from $D(\bar{\alpha}')$, sending satisfying the formulas (8),(9), we must implement one type of D, which as well (Theorem 7, Theorem 10, Lemma 4). Therefore, by the truth of the formulas (6), (7), the truth statements (10),(11). follows.

$$\text{For any } \bar{\gamma}_n, \dots, \bar{\gamma}_m, \delta_s, \dots, \delta_r, \bar{\gamma}_n', \dots, \bar{\gamma}_m', \delta_s', \dots, \delta_r' \in D \text{ true} \\ \delta_r, \delta_r' \in J_r, \text{ then } h(b, \bar{\gamma}_n, \dots, \bar{\gamma}_m, \delta) = h(b, \bar{\gamma}_n', \dots, \bar{\gamma}_m', \delta'); \quad (10)$$

$$[f(b, \bar{\gamma}_n, \dots, \bar{\gamma}_m, \delta_s, \dots, \delta_r) < g(\bar{b}, \bar{\gamma}_n', \dots, \bar{\gamma}_m', \delta_s', \dots, \delta_r')] \quad (11)$$

By the compactness theorem, there exists formula $\theta(\bar{z}, \bar{d}_1) \in tp(\alpha/D)$, that's true

$$\forall \bar{z} \quad \forall \bar{t} \quad \forall \bar{z}' \quad \forall \bar{t}' \quad [\theta(\bar{z}, \bar{d}_1) \& \theta(\bar{z}', \bar{d}_1) \& \bar{t} \in J_s \mathbf{x} \dots \mathbf{x} J_r \& \bar{t}' \in J_s \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow h(\bar{b}, \bar{z}, \bar{t}, \bar{d}) = h(\bar{b}, \bar{z}', \bar{t}', \bar{d})] \quad (12)$$

$$\text{or } [F(\bar{b}, \bar{z}, \bar{t}, \bar{d}) < g(\bar{b}, \bar{z}', \bar{t}', \bar{d})] \quad (13)$$

Formula (12) or (13) a true D' model, and hence in D, since they contain only parameter D.

Case 1. (12) the truth in D.

Take arbitrary $\bar{a}_n, \dots, \bar{a}_m, c_s, \dots, c_r \in D$, such that $D \models \theta(\bar{a}_n, \dots, \bar{a}_m, \bar{d}_1) \& c_s \in J_s \& c_r \in J_r$.

Then (12) $h(\bar{b}, \bar{a}, \bar{c}, \bar{d}) = h(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$. From here it follows that $h(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \in D$ and consequently there exists $a \in D$; $h(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) = a$. Hence the truth of the formula $\psi_\phi(a, \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$.

Case 2. (13) holds in D.

From Lemma 7 implies the existence of a $c \in D$, which

$$\begin{aligned} D \models \forall \bar{z} \forall \bar{t} [\theta(\bar{z}, \bar{d}_1) \& \quad \bar{t} \in J_s] \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow f(\bar{b}, \bar{z}, \bar{t}, \bar{d}) \leq c < g(\bar{b}, \bar{z}, \bar{t}, \bar{d}) \text{ or} \\ D \models \forall \bar{z} \forall \bar{t} [\theta(\bar{z}, \bar{d}_1) \& \quad \bar{t} \in J_s] \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow f(\bar{b}, \bar{z}, \bar{t}, \bar{d}) < c \leq g(\bar{b}, \bar{z}, \bar{t}, \bar{d})]. \end{aligned}$$

For definiteness we assume that the first formula implemented. From the fact that $\theta(\bar{z}, \bar{d}_1) \in tp(\bar{\alpha}/D)$, $\bar{\beta} \in J_s \mathbf{x} \dots \mathbf{x} J_r$, then $f(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \leq c < g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$.

Look an interval $(c, g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}))$. By lemma 3(a) there exists $a \in D$, that $c; a; g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$.

$$\text{We have } f(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}); a; g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}). \text{ Therefore, } D' \models \psi_\phi(a, \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{t}, \bar{d}) \quad (14)$$

(14) is performed in cases (1) and (2). By the induction the assumption for the formula ϕ , true $D^+ \models \phi(a, \bar{b})$. From $D^+ \models \exists x \phi(x, \bar{b})$ Therefore, $D^+ \models \mu(\bar{b})$.

Proposition 6. For any $b \in D$ is true

$$[D^+ \models \mu(\bar{b}) \leftrightarrow \psi_\mu(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})].$$

Proof. $D^+ \models \mu(\bar{b}) \leftrightarrow D^+ \models \psi(a, \bar{b})$, $a \in D$. By the induction hypothesis, $D' \models \psi_\phi(a, \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$. And the formula ψ_ϕ is equivalent to a disjunction of formulas of the form (4),(5), that is true

$$D' \models a = h(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \quad (15)$$

or

$$\begin{aligned} D' \models f(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) < a < g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \quad (16) \\ M := D(\bar{\alpha}). \text{ Then } M(\beta_1, \dots, \beta_n) = a = h(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \quad (17) \end{aligned}$$

$$\text{or} \\ M(\beta_1, \dots, \beta_n) \models f(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}); a; g(\bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) \quad (18)$$

By Lemma 6 there exists intervals $J'_1, \dots, J'_n \subset M$ such that

$$1) \beta_s \in J'_s, \dots, \beta_r \in J'_r;$$

$$2) M \models \forall t_r (t_s \in J'_s \& \dots \& t_r \in J'_r \leftrightarrow a = h(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d})) \quad (19)$$

or

$$M \models \forall t_s, \dots, \forall t_r (t_s \in J'_s \& \dots \& t_r \in J'_r \leftrightarrow f(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d}) < a < g(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d})) \quad (20)$$

As with $\beta_s \in J'_s, \dots, J'_r \subset D(\alpha)$, in $D(\alpha)$ is not realized $tp(\beta_s/D)$ and it is a single, then by Lemma 3(b), there exists $J_s \subset J'_s, J_s \subset D, \beta_s \in J_s$. Thus, $D(\bar{\alpha}) \models \forall t_s, \dots, t_r ((t_s \in J_s \& \dots \& t_r \in J_r) \leftrightarrow a = h(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d})) \quad (21)$

$$D\bar{\alpha} \models \forall t_s, \dots, \forall t_r ((t_s \in J_s \& \dots \& t_r \in J_r \leftrightarrow f(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d}) < a < g(\bar{b}, \bar{\alpha}, \bar{t}, \bar{d})) \quad (22)$$

From this for any $\bar{\gamma} \in D'$, if $\text{tp}(\bar{\gamma}/D) = \text{tp}(\bar{\alpha}/D)$, then $D' \models \forall \bar{t} (\bar{t} \in J_s \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow a = h(\bar{b}, \bar{\gamma}, \bar{t}, \bar{d})) \quad (23)$

$$D' \models \forall \bar{t} (\bar{t} \in J_n \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow (f(\bar{b}, \bar{\gamma}, \bar{t}, \bar{d}) < a < g(\bar{b}, \bar{\gamma}, \bar{t}, \bar{d})) \quad (24)$$

Determine $\bar{\gamma} \in D(\alpha')$, $\bar{\gamma} = \langle \gamma_n, \dots, \gamma_m \rangle$, $\gamma_n = \langle \gamma_{n,1}, \dots, \gamma_{n,k} \rangle$ - - - - -
 $\gamma_m = \langle \gamma_{m,1}, \dots, \gamma_{m,k} \rangle$
 $\alpha_{n,2} < \gamma_{n,1} < \alpha_{n,4} \alpha_{n,5} < \gamma_{n,2} < \alpha_{n,7}, \dots$
 $\alpha_{n,3i-1} < \gamma_{n,j} < \alpha_{n,3j+1}$
 - - - - -
 $\alpha_{m,3i-1} < \gamma_{n,j} < \alpha_{n,3j+1}$.

From Theorem 10 and Lemma 4 that $\text{tp}(\bar{\gamma}/D) = \text{tp}(\bar{\alpha}/D)$. From this $D(\bar{\alpha}) \models \forall \bar{z} \forall \bar{t} ((\& \& \alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1} \& \& \bar{t} \in J_s \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow a = h(\bar{b}, \bar{z}, \bar{t}, \bar{d})) \quad (25)$

$$D(\bar{\alpha}) \models \forall \bar{z} \forall \bar{t} ((\& \& \alpha_{n,3i-1} < z_{n,1} < \alpha_{n,3i+1} \& \& \bar{t} \in J_s \mathbf{x} \dots \mathbf{x} J_r \leftrightarrow f(\bar{b}, \bar{z}, \bar{t}, \bar{d}) < a < g(\bar{b}, \bar{z}, \bar{t}, \bar{d})) \quad (26)$$

Hence, replacing a $\exists \mathbf{x}$ and J_s to $\exists I_s$ we obtain that $\bar{b}$ satisfies (6) or (7).

So from proposition 2, 3 follow proposal 1.

Theorem 19. Let D o-minimal model, $(U_\lambda | \lambda \in K)$ family of convex subsets of D . Then theory $\text{Th}(\langle D, U_\lambda \rangle_{\lambda \in K})$ weakly o-minimal.

Proof. For the D , there are three possibilities:

- a) D - tight o-minimal model
- b) D - discrete on-minimal model;
- c) D - a mixed-minimal model.

a) Let $\mu(z, \bar{b})$ arbitrary formula with one free variable with parameters from D^+ . According to Proposition 1, there exists formula $\psi_\mu(z, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$ ($\bar{\alpha}, \bar{\beta} \in D', \bar{d} \in D$), such that for any $a \in D$

$$[D^+ \models \mu(a, \bar{b}) \iff D' = \psi_\mu(a, \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})]$$

Consider the formula $\psi_\mu(z, \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d})$. According to o-minimal of D $\psi_\mu(D', \bar{b}, \bar{\alpha}, \bar{\beta}, \bar{d}) = \cup I_1 \cup [\cup [d_1]]$ -I have a finite union intervals and singletons. Let $a_1, a_2' \in I_1 \cap D$. Then $[a_1, a_2) \in I_1$ and hence for any

$d \in [a_1, a_2] \cap D$ And $D' \models \psi_\mu(d, \bar{\alpha}, \bar{\beta}, \bar{d})$ and by Proposition 1, $D^+ \models \mu(d, \bar{b})$. Therefore, the implementation of the formula $\mu(z, \bar{b})$ falls into a finite number of convex sets as the number of such convex sets, dependent on $\bar{b}$, and from the formula $\psi_\mu(z, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$ o-minimal model D . By Theorem 6, for formula $\psi_\mu(z, \bar{y}, \bar{\alpha}, \bar{\beta}, \bar{d})$ the number of convex sets must be limited and this number does not depend on $\bar{b}$. Consequently, the number of convex sets for formulas $\mu(z, \bar{y})$ is limited. Consequently, by Lemma 9 D^+ model theory of weakly-minimal.

b) Note that, in a discrete o-minimal models do not there is a single type of [Theorem 14]. Analogue Proposition 1 for a discrete o-minimal model follows Theorem Pillay - Steinhorn and its proof (analogue) repeats the proof of Proposition 1 in a simpler form. The role of model D played by the following model $D'' = D(A)$, where $A = \delta \text{---} \text{tp}(\delta / D) \in P$. For each $p \in P$ in A there are exactly two elements that satisfy p . Here P - maximal orthogonal subset of social (Non-principal) types of $S_1(D)$ from a variety of social types, determine by means of a family of convex sets. First induction step uses only the following formula: $x \leq d, \quad d \leq x, \quad x < f(\delta), \quad f(\delta) < x, \quad x < \delta, \quad \delta < x | d \in D, \quad \delta < A, f$ - unary strictly monotone e-definable function

c) Mixed case stems from a), b) and Theorem Pillay-Steinhorn [6], in which it is proved that any Formula for minimum-theory is a Boolean combination of formulas discrete and dense parts of the model.

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