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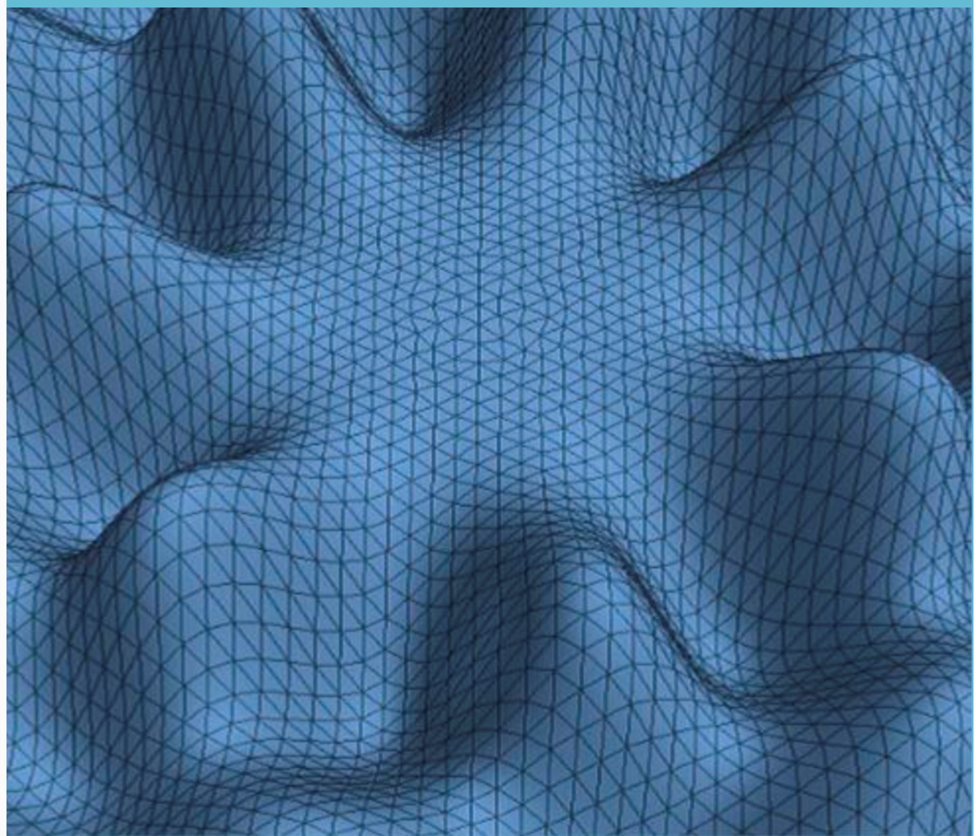

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Mixed problem in a multidimensional domain for the Lavrent'ev-Bitsadze equation

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Abstract. In this paper, unique solvability is shown and an explicit representation of the classical solution of a mixed problem in a multidimensional domain for the Lavrent'ev-Bitsadze equation is obtained. Multidimensional hyperbolic-parabolic equations describe important physical, astronomical, and geometric processes. It is known that the oscillations of elastic membranes in space according to the Hamilton principle can be modeled by a multi-dimensional wave equation. Assuming that in the position of the membrane is in equilibrium, from the Hamilton principle we also obtain the Laplace equation. Consequently, the oscillation of elastic membranes in space can be modeled as a multidimensional Lavrent'ev-Bitsadze equation. When studying these applications, it is necessary to obtain an explicit representation of the solution of the boundary value problems under study. In the paper the unique solvability and the explicit representation of the classical solution of a mixed problem in a multidimensional domain for the Lavrent'ev-Bitsadze equation are obtained.

Keywords. Multidimensional domain, mixed problem, unique solvability, spherical functions, orthogonality.

1 Introduction

It is known that vibrations of elastic membranes in space are modeled by partial differential equations. If the membrane deflection is considered as function $u(x,t)$, $x = (x_1, ..., x_m)$, $m \geq 2$, then by the Hamilton principle we arrive at a multidimensional wave equation.

Assuming that the membrane is in equilibrium in the bending position, Hamilton's principle also yields the multi-dimensional Laplace equation.

Consequently, oscillations of elastic membranes in space can be modeled as a multidimensional Lavrent'ev-Bitsadze equation. A mixed problem in the cylindrical domain for

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multidimensional hyperbolic equations in the space of generalized functions has been well studied [1], [2]. In [3], the correctness of this problem was proved and an explicit form of the classical solution was obtained. In this paper, we show the unique solvability and obtain an explicit representation of the classical solution of the mixed problem in the multidimensional domain for the Lavrent'ev-Bitsadze equation.

2 Statement of the problem and results

Let Ω_α be the finite domain of the Euclidean space E_{m+1} of points $(x_1, ..., x_m, t)$, bounded by $t > 0$ by spherical surface $\sigma : r^2 + t^2 = 1$, by cylinder $t < 0$ $\Gamma_\alpha = \{(x, t) : |x| = 1\}$ and the plane $t = \alpha < 0$, where $|x|$ is the length of the vector $x = (x_1, ..., x_m)$. Denote by Ω^+ and Ω^-_α parts of the domain Ω_α, lying in the half-spaces $t > 0$ and $t < 0$, respectively; by σ_α denote the lower base of the domain Ω^-_α.

Further, let S be the common part of the boundaries of the domains Ω^+ and Ω^-_α, representing the set $\{t = 0, 0 < |x| < 1\}$ in E_m.

In the domain Ω_α consider the multidimensional Lavrent'ev-Bitsadze equation

$$(sgnt)\Delta_x u + u_{tt} = 0, \tag{1}$$

where Δ_x is Laplace operator over the variables $x_1, ..., x_m$, $m \geq 2$.

In the future, it is convenient for us to move from Cartesian coordinates $x_1, ..., x_m, t$ to spherical $r, \theta_1, ..., \theta_{m-1}, t$, $r \geq 0$, $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi$, $i = 2, 3, ..., m-1, \theta = (\theta_1, ..., \theta_{m-1})$.

Problem 1. Find a solution to the equation (1) in the domain Ω_α at $t \neq 0$ from the class $C(\bar{\Omega_\alpha}) \cap C^1(\Omega_\alpha) \cap C^2(\Omega^+ \cup \Omega^-_\alpha)$, satisfying boundary conditions

$$u\big|_\sigma = \varphi(r, \theta), \tag{2}$$

$$u\big|_{\Gamma_\alpha} = \psi(r, \theta), \tag{3}$$

wherein $\varphi(1, \theta) = \psi(0, \theta)$.

Let $\left\{Y^k_{n,m}(\theta)\right\}$ be the system of linearly independent spherical n-th order functions, $1 \leq k \leq k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2)$, let $W^l_2(S)$ be Sobolev space, $l = 0, 1, ...$

The following lemma holds ([4, p. 142-144])

Lemma 1. *Let $f(r, \theta) \in W^l_2(S)$. If $l \geq m-1$, then the series*

$$f(r, \theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} f^k_n(r) Y^k_{n,m}(\theta), \tag{4}$$

as well as the series derived from it by the differentiation of order $p \leq l - m + 1$, converge absolutely and uniformly.

Lemma 2. *In order to $f(r,\theta)\in W_2^l(S)$, it is necessary and sufficient that the coefficients of the series (4) satisfy inequalities*

$$|f_0^1(r)|\le c_1,\ \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} n^{2l}|f_n^k(r)|^2\le c_2,\ \ c_1,\, c_2=const.$$

By $\bar\varphi_n^k(r),\psi_n^k(t),\tau_n^k(r)$ we denote the coefficients of the expansion series (4), of functions $\varphi(r,\theta),\psi(t,\theta),\tau(r,\theta)$, respectively.

Let $\varphi(r,\theta)\in W_2^l(S)$, $\psi(r,\theta)\in W_2^l(\Gamma_\alpha)$, $l>\frac{3m}{2}$.

Then the following theorem is true.

Theorem. *Problem 1 is uniquely solvable.*

Proof of the theorem. In spherical coordinates the equation (1) in the domain Ω^+ has the form

$$u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u+u_{tt}=0, \tag{5}$$

$$\delta\equiv-\sum_{j=1}^{m-1}\frac{1}{g_j\sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right),\ g_1=1,\ g_j=(\sin\theta_1...\sin\theta_{j-1})^2,\ j>1.$$

It is known ([5, p. 239]) that the spectrum of the operator δ consists of eigenvalues $\lambda_n=n(n+m-2)$, $n=0,1,...$, each of which corresponds to k_n orthonormal eigenfunctions $Y_{n,m}^k(\theta)$.

Since the desired solution of the Problem 1 in the domain Ω^+ belongs to the class $C(\bar\Omega^+)\cap C^2(\Omega^+)$, then it can be represented as

$$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\bar u_n^k(r,t)Y_{n,m}^k(\theta), \tag{6}$$

where $\bar u_n^k(r,t)$ are functions to be defined.

Substituting (6) into (5), using the orthogonality of spherical functions $Y_{n,m}^k(\theta)$ ([5, p. 241]), we will have

$$\bar u_{nrr}^k+\frac{m-1}{r}\bar u_{nr}^k+\bar u_{ntt}^k-\frac{\lambda_n}{r^2}\bar u_n^k=0, k=\overline{1,k_n},\quad n=0,1,\dots, \tag{7}$$

at the same time, the boundary condition (2), taking into account Lemma 1, can be represented in the form

$$\bar u_n^k(r,\sqrt{1-r^2})=\bar\varphi_n^k(r),\ k=\overline{1,k_n},\ n=0,1,\dots,0\le r\le 1. \tag{8}$$

In (7), (8) replacing $\bar{u}_n^k(r,t) = r^{\frac{1-m}{2}} u_n^k(r,t)$, and then using $r = \rho\cos\varphi$, $t = \rho\sin\varphi$, $\rho \geq 0$, $0 \leq \varphi \leq \pi$, we will get

$$\upsilon_{n\rho\rho}^k + \frac{1}{\rho}\upsilon_{n\rho}^k + \frac{1}{\rho^2}\upsilon_{n\varphi\varphi}^k + \frac{\bar{\lambda}_n}{\rho^2\cos^2\varphi}\upsilon_n^k = 0, \tag{9}$$

$$\upsilon_n^k(1,\varphi) = g_n^k(\varphi), \tag{10}$$

where

$$\upsilon_n^k(\rho,\,\varphi) = u_n^k(\rho\cos\varphi,\,\rho\sin\varphi),\ \bar{\lambda}_n = \frac{[(m-1)(3-m) - 4\lambda_n]}{4},$$

$$g_n^k(\varphi) = (\cos\varphi)^{\frac{(m-1)}{2}}\bar{\varphi}_n^k(\cos\varphi).$$

The solution to the problem (9), (10) will be sought in the form

$$\upsilon_n^k(\rho,\,\varphi) = R(\rho)\phi(\varphi). \tag{11}$$

Substituting (11) into (9), we will have

$$\rho^2 R_{\rho\rho} + \rho R_\rho - \mu R = 0, \tag{12}$$

$$\phi_{\varphi\varphi} + (\mu + \frac{\bar{\lambda}_n}{\cos^2\varphi})\phi = 0,\ \mu = const. \tag{13}$$

If the solution of the Euler equation (12) is sought in the form $R(\rho) = \rho^s,\ 0 \leq s = const$, then we get $s^2 = \mu$.

Next, we write equation (13) as follows

$$\phi_{\varphi\varphi} = \left[\frac{l(l-1)}{\cos^2\varphi} - s^2\right]\phi,\ l = -n - \frac{(m-3)}{2}. \tag{14}$$

In equations (14), making the replacement $\xi = \sin^2\varphi$, we come to the equation

$$\xi(\xi-1)g_{\xi\xi} + \left[(\beta+\gamma+1)\xi - \frac{1}{2}\right]g_\xi + \beta\gamma g = 0, \tag{15}$$

$$g(\xi) = \frac{\phi(\varphi)}{\cos^l\varphi},\ \beta = \frac{(l+s)}{2},\ \gamma = \frac{(l-s)}{2}.$$

The general solution of the equation (15) is represented by the formula ([6, p. 423]):

$$g_s(\xi) = c_{1s}F\left(\beta,\,\gamma,\,\frac{1}{2};\xi\right) + c_{2s}\sqrt{\xi}F\left(\beta+\frac{1}{2},\,\gamma+\frac{1}{2},\,\frac{3}{2};\xi\right), \tag{16}$$

which is periodic by φ, if $s = 0, 1, ...,$ where c_{1s}, c_{2s} are arbitrary independent constants, and $F(\beta, \gamma, \alpha; \xi)$ is Gaussian hypergeometric function. Thus, from (11), (16) it follows that the general solution of the equation (9) is written as

$$\upsilon_n^k(\rho, \varphi) = \sum_{s=0}^{\infty} \rho^s \cos^l \varphi \left[c_{1s} F\left(\beta,\, \gamma,\, \frac{1}{2}; \sin^2\varphi\right) + c_{2s} \sin\varphi F\left(\beta + \frac{1}{2}, \gamma + \frac{1}{2}, \frac{3}{2}; \sin^2\varphi\right)\right]. \quad (17)$$

Since $\left|\upsilon_n^k\left(\rho,\, \frac{\pi}{2}\right)\right| < \infty$, then from (17) we will have

$$c_{1s} F\left(\beta,\, \gamma,\, \frac{1}{2}; 1\right) + c_{2s} F\left(\beta + \frac{1}{2}, \gamma + \frac{1}{2}, \frac{3}{2}; 1\right) = 0,$$

or

$$c_{2s} = -\frac{2\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma\left(\frac{1}{2} - \beta\right)\Gamma\left(\frac{1}{2} - \gamma\right)} c_{1s}, \quad (18)$$

where $\Gamma(z)$ is the gamma function.

Substituting (18) into (17) we get

$$\upsilon_n^k(\rho,\, \varphi) = \sum_{s=0}^{\infty} c_{1s} \rho^s \cos^l \varphi \left[F\left(\beta,\, \gamma,\, \frac{1}{2}; \sin^2\varphi\right)\right.$$

$$\left. - \frac{2\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma\left(\frac{1}{2} - \beta\right)\Gamma\left(\frac{1}{2} - \gamma\right)} \sin\varphi F\left(\beta + \frac{1}{2},\, \gamma + \frac{1}{2},\, \frac{3}{2}; \sin^2\varphi\right)\right]. \quad (19)$$

It is known ([7, p. 393]), that the system of functions $\{\frac{1}{2},\, \cos 2s\varphi,\, \sin 2s\varphi,\, s = 1, 2, ...\}$ is full, orthogonal to $C([0,\, \pi])$, hence is closed.

It follows that the function $g_n^k(\varphi) \in C([0,\, \pi])$ can be represented in the form of the series

$$g_n^k(\varphi) = a_{0,n}^k + \sum_{s=1}^{\infty} (a_{s,n}^k \cos 2s\varphi + b_{s,n}^k \sin 2s\varphi), \quad (20)$$

where

$$a_{0,n}^k = \frac{1}{2\pi} \int_0^{\pi} g_n^k(\varphi) d\varphi,\ a_{s,n}^k = \frac{1}{\pi} \int_0^{\pi} g_n^k(\varphi) \cos 2s\varphi d\varphi,$$

$$b_{s,n}^k = \frac{1}{\pi} \int_0^{\pi} g_n^k(\varphi) \sin 2s\varphi d\varphi,\ s = 1, 2, \dots . \quad (21)$$

Further, assuming that the function (19) satisfies (10), taking into account the expansion (20) and assuming $\varphi = 0$, we will get

$$c_{1s} = a^k_{s,n},\ s = 0, 1, \dots . \tag{22}$$

Thus, from (6), (19), (22) it follows that the solution of the problem (5), (8) in the domain Ω^+ is the function

$$\begin{aligned} u(r,\theta,t) = &\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\sum_{s=p}^{\infty} a^k_{s,n} r^{2-m-n}(r^2+t^2)^{\frac{s}{2}+\frac{n}{2}+\frac{(m-3)}{4}} \\ &\times \left[F(-\frac{n}{2}+\frac{3-m}{4}+\frac{s}{2},\ -\frac{n}{2}+\frac{3-m}{2}-\frac{s}{2},\ \frac{1}{2};\ \frac{t^2}{r^2+t^2}\right) \\ &-\frac{2\Gamma\left(1+\frac{n}{2}+\frac{m-3}{4}-\frac{s}{2}\right)\Gamma\left(1+\frac{n}{2}+\frac{m-3}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n}{2}+\frac{m-3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}+\frac{m-3}{4}+\frac{s}{2}\right)} t(r^2+t^2)^{-\frac{1}{2}} \\ &\times F\left(-\frac{n}{2}+\frac{5-m}{4}+\frac{s}{2},\ -\frac{n}{2}+\frac{5-m}{4}-\frac{s}{2},\ \frac{3}{2};\ \frac{t^2}{r^2+t^2}\right)\Bigg] Y^k_{n,m}(\theta). \end{aligned} \tag{23}$$

From (23) at $t \longrightarrow +0$ we will have

$$u(r,\theta,0) = \tau(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\sum_{s=p}^{\infty} a^k_{s,n} r^{s+\frac{(1-m)}{2}} Y^k_{n,m}(\theta), \tag{24}$$

$$\begin{aligned} u_t(r,\theta,0) = \nu(r,\theta) = -2&\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\sum_{s=p}^{\infty} a^k_{s,n} r^{s-\frac{(1+m)}{2}} \\ \times &\frac{\Gamma\left(1+\frac{n}{2}+\frac{m-3}{4}-\frac{s}{2}\right)\Gamma\left(1+\frac{n}{2}+\frac{m-3}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n}{2}+\frac{m-3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{n}{2}+\frac{m-3}{4}+\frac{s}{2}\right)} Y^k_{n,m}(\theta), \end{aligned} \tag{25}$$

where $p \geq \frac{m+1}{2}$, $a^k_{s,n}$ are determined from (21).

It is known that if $g^k_n(\varphi) \in C^q((0,\pi))$, then the estimate ([8, p. 457]) $\left|a^k_{s,n}\right| \leq \frac{c_1}{s^{q+2}}$, $q = 0, 1, \dots$, as well as the formulas ([9, p. 71, p. 62])

$$\frac{d^q}{dz^q} F(a,b,c;z) = \frac{(a)_q (b)_q}{(c)_q} F(a+q,\ b+q,\ c+q;\ z),$$

$$(a)_q = \frac{\Gamma(a+q)}{\Gamma(a)},\ \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+\frac{1}{2z}(\alpha-\beta)(\alpha-\beta-1)+O(z^{-2})\right],$$

and the estimates ([4, p. 147])

$$|k_n| \leq c_1 n^{m-2}, \ \left|\frac{\partial^q}{\partial\theta_j^q} Y_{n,m}^k(\theta)\right| \leq c_2 n^{\frac{m}{2}-1+q}, \ j=\overline{1, m-1} \tag{26}$$

are valid.

From the embedding theorem ([5, p. 50]) it follows that $W_2^l(S) \subset C^q(S) \cap C(\bar{S})$, if $l > q + \frac{m}{2}$.

From the above analysis and taking into account Lemma 2, as well as the boundary condition $\varphi(r,\theta) \in W_2^l(S)$, $l > \dfrac{3m}{2}$, we get that the solution (23) belongs to the class $u(r,\theta,t) \in C(\bar{\Omega}^+) \cap C^1(\Omega^+ \cup S) \cup C^2(\Omega^+)$, and also from (24)–(26) it follows that $\tau(r,\theta),\ \nu(r,\theta) \in W_2^l(S),\ l > \frac{3m}{2}$.

Thus, taking into account the boundary conditions (3) and (24), (25) in Ω_α^- we arrive to the mixed problem for multidimensional wave equation

$$u_{rr} + \frac{m-1}{r} u_r - \frac{1}{r^2}\delta u - u_{tt} = 0 \tag{27}$$

with the conditions

$$u\big|_S = \tau(r,\theta),\, u_t\big|_S = \nu(r,\theta),\, u\big|_{\Gamma_\alpha} = \psi(t,\theta), \tag{28}$$

which has the unique solution (3).

The theorem is proved.

Since in [3] the solution to the problem (27), (28) is derived explicitly, it is possible to write the explicit representation of the solution for the Problem 1.

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Алдашев С.А. КӨП ӨЛШЕМДІ ОБЛЫСТА ЛАВРЕНТЬЕВ-БИЦАДЗЕ ТЕҢДЕУІ ҮШІН АРАЛАС ЕСЕП

Маңызды физикалық, астрономиялық және геометриялық құбылыстар көп өлшемді гиперболалық-параболалық теңдеулермен сипатталады. Гамильтон қағидаты бойынша, қалың мембрананың кеңістіктегі тербелістері көп өлшемді толқын теңдеуімен сипатталады. Егер де мембрана қозғалмайтын тыныш күйде деп есептесек, онда тағы да Гамильтон қағидаты бойынша көп өлшемді Лаплас теңдеуіне келеміз. Сонымен, тығыз мембрананың кеңістіктегі тербелу процесі көп өлшемді Лаврентьев-Бицадзе теңдеуімен сипатталған. Жоғарыда айтылған құбылыстарды зерттегенде, қарастыратын те?деу үшін шеттік есептердің айқын шешімдері керек болады. Осы жұмыста көп өлшемді облыста Лаврентьев-Бицадзе теңдеуі үшін аралас есептің бірмәнді шешілімділігі дәлелденген және оның классикалық шешімінің айқын түрі келтірілген.

Кілттік сөздер. Көп өлшемді облыс, аралас есеп, бірмәнді шешілімділік, сфералық функциялар, ортогоналдылық.

Алдашев С.А. СМЕШАННАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ЛАВРЕНТЬЕВА-БИЦАДЗЕ В МНОГОМЕРНОЙ ОБЛАСТИ

Важные физические, астрономические и геометрические явления описаны многомерными гиперболо-параболическими уравнениями. По принципу Гамильтона колебания плотной мембраны в пространстве описываются многомерным волновым уравнением. Если мембрана находится в состоянии покоя, опять же по принципу Гамильтона приходим к многомерному уравнению Лапласа. Таким образом, процесс колебания плотной мембраны в пространстве охарактеризован многомерным уравнением Лаврентьева-Бицадзе. При исследовании вышеуказанных явлений необходимы явные решения краевых задач этого уравнения. В этой статье доказана однозначная разрешимость смешанной задачи для уравнения Лаврентьева-Бицадзе в многомерной области, а также получен явный вид ее классического решения.

Ключевые слова. Многомерная область, смешанная задача, однозначная разрешимость, сферические функции, ортогональность.

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Solvability of a periodic problem for the fourth order system of partial differential equations with time delay

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Abstract. A periodic problem for the system of fourth order partial differential equations with finite time delay is investigated. By method of introduction of additional functions the problem reduces to equivalent problem, consisting of a family of periodic problems for a system of ordinary differential equations with finite delay and integral relations. The algorithm for finding approximate solutions of the equivalent problem is constructed and its convergence is proved. The sufficient conditions of an unique solvability to the periodic problem for the fourth order system of partial differential equations with finite time delay are obtained.

Keywords. Periodic problem, system of fourth order partial differential equations with time delay, family of periodic problems for the system of differential equations with finite delay, algorithm, unique solvability.

1 Introduction

In this paper, we study the following periodic problem for the system of fourth order partial differential equations with time delay in the domain $\Omega_\tau = [-\tau, T] \times [0, \omega]$:

$$\frac{\partial^4 u(t,x)}{\partial t \partial x^3} = A(t,x)\frac{\partial^3 u(t,x)}{\partial x^3} + A_0(t,x)\frac{\partial^3 u(t-\tau,x)}{\partial x^3}$$

$$+B(t,x)\frac{\partial^3 u(t,x)}{\partial t \partial x^2} + C(t,x)\frac{\partial^2 u(t,x)}{\partial x^2} + D(t,x)u(t,x) + f(t,x), \quad (t,x) \in [0,T] \times [0,\omega], \quad (1)$$

$$\frac{\partial^3 u(z,x)}{\partial x^3} = diag\Big[\frac{\partial^3 u(0,x)}{\partial x^3}\Big] \cdot \varphi(z), \ z \in [-\tau, 0], \ x \in [0,\omega], \quad (2)$$

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$$\frac{\partial^3 u(0,x)}{\partial x^3} = \frac{\partial^3 u(T,x)}{\partial x^3},\ x \in [0,\omega], \tag{3}$$

$$u(t,0) = \psi_0(t), \quad \frac{\partial u(t,x)}{\partial x}\Big|_{x=0} = \psi_1(t), \quad \frac{\partial^2 u(t,x)}{\partial x^2}\Big|_{x=0} = \psi_2(t), \quad t \in [-\tau, T], \tag{4}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is unknown function, $(n\times n)$-matrices $A(t,x)$, $A_0(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$ and n-vector function $f(t,x)$ are continuous in $\Omega = [0,T] \times [0,\omega]$, n-vector function $\varphi(t)$ is continuously differentiable and given in the initial set $[-\tau, 0]$ such that $\varphi_i(0) = 1$, $i = \overline{1,n}$, $\tau > 0$ is a constant delay, n-vector functions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$ are continuously differentiable in $[-\tau, T]$. The compatibility conditions are valid: $\psi_i(0) = \psi_i(T)$, $i = 0, 1, 2$.

Let

$C(\Omega_\tau, R^n)$ be the space of continuous in Ω_τ vector functions $u(t,x)$ with the norm

$$||u||_0 = \max_{(t,x)\in\Omega_\tau} ||u(t,x)||,\ ||u(t,x)|| = \max_{i=\overline{1,n}} |u_i(t,x)|;$$

$C([0,\omega], R^n)$ be the space of continuous in $[0,\omega]$ vector functions $\varphi(x)$ with the norm

$$||\varphi||_{0,1} = \max_{x\in[0,\omega]} ||\varphi(x)||;$$

$C^1([-\tau, T], R^n)$ be the space of continuously differentiable in $[-\tau, T]$ vector functions $\psi(t)$ with the norm

$$||\psi||_{1,0} = \max\Big(\max_{t\in[-\tau,T]} ||\psi(t)||, \max_{t\in[-\tau,T]} ||\dot{\psi}(t)||\Big);$$

$\Omega_0 = \{(t,x) : t = 0, 0 \le x \le \omega\}$.

The function $u(t,x) \in C(\Omega_\tau, R^n)$, that has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\Omega_\tau, R^n)$, $\frac{\partial^2 u(t,x)}{\partial x^2} \in C(\Omega_\tau, R^n)$, $\frac{\partial^3 u(t,x)}{\partial x^3} \in C(\Omega_\tau, R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\Omega_\tau \backslash \Omega_0, R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega_\tau \backslash \Omega_0, R^n)$, $\frac{\partial^3 u(t,x)}{\partial t \partial x^2} \in C(\Omega_\tau \backslash \Omega_0, R^n)$, $\frac{\partial^4 u(t,x)}{\partial t \partial x^3} \in C(\Omega_\tau \backslash \Omega_0, R^n)$ is called a *classical solution* to the periodic problem (1)–(4) if it satisfies the system (1) for all $(t,x) \in \Omega$, the condition (2) in the initial set $[-\tau, 0]$ and the boundary conditions (3), (4).

As well-known, various problems of population dynamics, mathematical biology, ecology, management of technical systems, the problem of physics, variational problems related to the regulatory process, the optimal control problem with delay systems, etc. leads to the boundary value problems for differential equations with time delay [1]–[18]. Periodic and nonlocal problems for the partial differential equations with time delay arise of mathematical modeling of numerous processes in biology, physics, chemistry, mechanics in [2], [3], [5], [6], [12], [13], [17], [19]. To investigate the questions of solvability of these classes of problems there have been applied the methods of the qualitative theory of differential equations and the theory of oscillations, the Riemann's method, the numerical-analytical method, the method of monotone iteration, asymptotic methods, the method of upper and lower solutions and others. Nevertheless, the problem of finding effective features of a unique solvability of periodic problems for the higher order system of partial differential equations with time delay still holds actual today.

The goal of this paper is to establish conditions for the existence and the uniqueness of the classical solution to the system of fourth order partial differential equations with finite time delay.

2 Reduction to equivalent family of periodic problems for the system of ordinary differential equations with finite delay and integral relations

In this section, we reduce the original problem (1)–(4) to the family of periodic problems for the system of ordinary differential equations with finite delay and integral relations by method of introduction of additional functions [19].

So, we introduce new unknown functions $v(t,x)=\dfrac{\partial^3 u(t,x)}{\partial x^3}$ and $w(t,x)=\dfrac{\partial^2 u(t,x)}{\partial x^2}$, and reduce the periodic problem for the system of fourth order partial differential equations with time delay (1)–(4) to equivalent problem:

$$\frac{\partial v(t,x)}{\partial t}=A(t,x)v(t,x)+A_0(t,x)v(t-\tau,x)+f(t,x)$$

$$+B(t,x)\frac{\partial w(t,x)}{\partial t}+C(t,x)w(t,x)+D(t,x)u(t,x),\quad (t,x)\in\Omega, \tag{5}$$

$$v(z,x)=diag[v(0,x)]\cdot\varphi(z),\quad z\in[-\tau,0],\quad x\in[0,\omega], \tag{6}$$

$$v(0,x)=v(T,x),\quad x\in[0,\omega], \tag{7}$$

$$w(t,x)=\psi_2(t)+\int_0^x v(t,\xi)d\xi,\quad \frac{\partial w(t,x)}{\partial t}=\dot{\psi}_2(t)+\int_0^x\frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{8}$$

$$u(t,x)=\psi_0(t)+\psi_1(t)x+\psi_2(t)\frac{x^2}{2!}+\int_0^x\frac{(x-\xi)^2}{2!}v(t,\xi)d\xi. \tag{9}$$

Conditions (4) are included in integral relations (8) and (9). The function $u(t,x)$ also satisfies the following integral relation

$$u(t,x)=\psi_0(t)+\psi_1(t)x+\int_0^x(x-\xi)w(t,\xi)d\xi. \tag{10}$$

Integral relations (9) and (10) are equivalent for $v(t,x)=\dfrac{\partial^3 u(t,x)}{\partial x^3}$ and $w(t,x)=\dfrac{\partial^2 u(t,x)}{\partial x^2}$.

A triple $\{v(t,x),w(t,x),u(t,x)\}$ of functions is called a *solution* to the problem (5)–(9) if the function $v(t,x)$ belonging to $C(\Omega_\tau,R^n)$ has a continuous derivative with respect to t in $\Omega_\tau\backslash\Omega_0$ and satisfies the one-parameter family of periodic problems for ordinary differential equations with finite delay (5)–(7), where the functions $w(t,x)$, $\dfrac{\partial w(t,x)}{\partial t}$ and $u(t,x)$ are connected with $v(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t}$ by the integral relations (8), (9).

Let $u^*(t,x)$ be a classical solution of the periodic problem (1)–(4). Then the triple $\{v^*(t,x), w^*(t,x), u^*(t,x)\}$, where $v^*(t,x) = \dfrac{\partial^3 u^*(t,x)}{\partial x^3}$, $w^*(t,x) = \dfrac{\partial^2 u^*(t,x)}{\partial x^2}$, $\frac{\partial w^*(t,x)}{\partial t} = \dfrac{\partial^3 u^*(t,x)}{\partial t \partial x^2}$, is the solution to problem (4)–(9). Conversely, if a triple $\{\widetilde{v}(t,x), \widetilde{w}(t,x), \widetilde{u}(t,x)\}$ is a solution to the problem (4)–(9), then $\widetilde{u}(t,x)$ determining by equality

$$\widetilde{u}(t,x) = \psi_0(t) + \psi_1(t)x + \int_0^x (x-\xi)\widetilde{w}(t,\xi)d\xi = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}\widetilde{v}(t,\xi)d\xi$$

is the classical solution to the periodic problem (1)–(4).

For fixed $w(t,x)$, $\dfrac{\partial w(t,x)}{\partial t}$ and $u(t,x)$ in the problem (4)–(7) it is necessary to find a solution to a one-parameter family of periodic problems for the system of ordinary differential equations with finite delay.

3 Algorithm and unique solvability of the periodic problem (1)–(4)

Hereby, the problem (1)–(4) reduces to equivalent problem, consisting of a family of periodic problems for the system of differential equations with finite delay and integral relations.

If we know $v(t,x)$ and its derivative $\dfrac{\partial v(t,x)}{\partial t}$, then from (8), (9) we find $w(t,x)$, $\dfrac{\partial w(t,x)}{\partial t}$ and $u(t,x)$. Conversely, if we know $w(t,x)$, $\dfrac{\partial w(t,x)}{\partial t}$, $u(t,x)$, then from (5)–(7) we can find $v(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t}$. Since $v(t,x)$ and $w(t,x)$, $u(t,x)$ are unknown, to find a solution to the problem (5)–(9) we use the iterative method. A triple $\{v^*(t,x), w^*(t,x), u^*(t,x)\}$ we determine as a limit sequence of triples $\{v^{(k)}(t,x), w^{(k)}(t,x), w^{(k)}(t,x)\}$, and $k = 0, 1, 2, ...$, by the following algorithm:

Step 0. a) Solving the family of periodic problems (5)–(7) for $w(t,x) = \psi_2(t)$, $\dfrac{\partial w(t,x)}{\partial t} = \dot{\psi}_2(t)$ and $u(t,x) = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!}$, we find $v^{(0)}(t,x)$ and its derivative $\dfrac{\partial v^{(0)}(t,x)}{\partial t}$.
b) From integral relations (8) and (9) for $v(t,x) = v^{(0)}(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial v^{(0)}(t,x)}{\partial t}$, we define $w^{(0)}(t,x)$, $\dfrac{\partial w^{(0)}(t,x)}{\partial t}$ and $u^{(0)}(t,x)$.

Step 1. a) Solving the family of periodic problems (5)–(7) for $w(t,x) = w^{(0)}(t,x)$, $\dfrac{\partial w(t,x)}{\partial t} = \dfrac{\partial w^{(0)}(t,x)}{\partial t}$ and $u(t,x) = u^{(0)}(t,x)$, we find $v^{(1)}(t,x)$ and its derivative $\dfrac{\partial v^{(1)}(t,x)}{\partial t}$.

b) From integral relations (8) and (9) for $v(t,x) = v^{(1)}(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial v^{(1)}(t,x)}{\partial t}$, we define $w^{(1)}(t,x)$, $\dfrac{\partial w^{(1)}(t,x)}{\partial t}$ and $u^{(1)}(t,x)$.

And so on.

Step k**.** a) Solving the family of periodic problems (5)–(7) for $w(t,x) = w^{(k-1)}(t,x)$, $\dfrac{\partial w(t,x)}{\partial t} = \dfrac{\partial w^{(k-1)}(t,x)}{\partial t}$ and $u(t,x) = u^{(k-1)}(t,x)$, we find $v^{(k)}(t,x)$ and its derivative $\dfrac{\partial v^{(k)}(t,x)}{\partial t}$.

b) From integral relations (8) and (9) for $v(t,x) = v^{(k)}(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial v^{(k)}(t,x)}{\partial t}$, we define $w^{(k)}(t,x)$, $\dfrac{\partial w^{(k)}(t,x)}{\partial t}$ and $u^{(1)}(t,x)$, $k = 1, 2, \dots$.

The method of introduction of additional functions divides the process of finding unknown functions into two parts: 1) From the family of periodic problems for ordinary differential equations with time delay (5), (6), (7) we find the unknown function $v(t,x)$ $\Big($and its derivative $\dfrac{\partial v(t,x)}{\partial t}\Big)$. 2) From the integral relations (8) and (9) we find the functions $w(t,x)$, $\dfrac{\partial w(t,x)}{\partial t}$ and $u(t,x)$.

Consider the following family of periodic problems for the system of ordinary differential equations with finite delay:

$$\frac{\partial v(t,x)}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(t-\tau,x) + g(t,x), \quad (t,x) \in \Omega, \quad v \in R^n, \tag{11}$$

$$v(z,x) = diag[v(0,x)] \cdot \varphi(z), \ z \in [-\tau, 0], \ x \in [0,\omega], \tag{12}$$

$$v(0,x) = v(T,x), \ x \in [0,\omega], \tag{13}$$

where n-vector function $g(t,x)$ is continuous in Ω_τ.

Continuous function $v : \Omega_\tau \to R^n$ that has a continuous derivative with respect to t in $\Omega_\tau \backslash \Omega_0$ is called a solution to the family of periodic problems with finite delay (11)–(13) if it satisfies the system (10) for all $(t,x) \in \Omega$ and has the values $v(0,x)$, $v(T,x)$ on the lines $t = 0$, $t = T$ and the equalities (12), (13) are valid for all $x \in [0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ the problem (11)–(13) is a linear periodic problem for the system of ordinary differential equations with finite delay [1], [4], [6]–[8], [10], [11], [14]–[16]. Suppose that a variable x is changed in $[0,\omega]$; then we obtain the family of periodic problems for ordinary differential equations with finite delay.

Now we state the main theorem of realization and convergence of the proposed algorithm. This assertion also provides sufficient conditions for the unique solvability of the periodic problem (1)–(4).

Theorem 1. *Let*

i) $(n \times n)$-matrices $A(t,x)$, $A_0(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$ and n-vector function $f(t,x)$ be continuous in Ω;

ii) n-vector function $\varphi(t)$ be continuously differentiable and given in the initial set $[-\tau, 0]$ such that $\varphi_i(0) = 1$, $i = \overline{1,n}$, $\tau > 0$ is a constant delay;

iii) n-vector functions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$ be continuously differentiable in $[-\tau, T]$. The compatibility conditions are valid: $\psi_i(0) = \psi_i(T)$, $i = 0, 1, 2$;

iv) the family of periodic problems for the system of ordinary differential equations with time delay (11)–(13) be uniquely solvable for any $g(t,x) \in \Omega$.

Then periodic problem for the system of fourth order partial differential equations with time delay has the unique classical solution.

The proof of the theorem is carried out according to the scheme of algorithm proposed above.

The main condition for the unique solvability of the problem (1)–(4) is the unique solvability of the auxiliary family of periodic problems (11)–(13). In [20] for investigation of the family of periodic problems for the system of ordinary differential equations with finite delay (11)–(13) we applied the parametrization method [21]. The sufficient and necessary conditions of the unique solvability of the family of periodic problems for the system of ordinary differential equations with finite delay (11)–(13) are established in the terms of initial data. In [19] the periodic problem for the system of hyperbolic equations of the second order with finite time delay is investigated. The considered problem is also reduced to the equivalent problem, consisting of the family of periodic problems for the system of ordinary differential equations with finite delay and integral equations, by method of introduction of new functions. Relationship between of the periodic problem for the system of hyperbolic equations with finite time delay and the family of periodic problems for the system of ordinary differential equations with finite delay is established. Algorithms for finding approximate solutions of the equivalent problem are constructed and their convergence is proved. Based on the results in [20], criteria of well-posedness of periodic problem for the system of hyperbolic equations with finite time delay are obtained.

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Асанова А.Т., Искакова Н.Б., Орумбаева Н.Т. УАҚЫТ БОЙЫНША КЕШІГУЛІ ТӨРТІНШІ РЕТТІ ДЕРБЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН ПЕРИОДТЫ ЕСЕПТІҢ ШЕШІЛІМДІЛІГІ

Уақыт бойынша ақырлы кешігуі бар төртінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін периодты есеп зерттеледі. Қарастырылып отырған есеп ақырлы кешігуі бар жәй дифференциалдық теңдеулер жүйесі үшін периодты есептер әулетінен және интегралдық қатынастардан тұратын пара-пар есепке қосымша функциялар енгізу әдісі арқылы келтіріледі. Пара-пар есептің жуық шешімін табу алгоритмі тұрғызылады және оның жинақтылығы дәлелденеді. Уақыт бойынша ақырлы кешігуі бар төртінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін периодты есептің бірмәнді шешілімділігінің жеткілікті шарттары алынған.

Кілттік сөздер. Периодты есеп, уақыт бойынша ақырлы кешігуі бар төртінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі, ақырлы кешігуі бар жәй дифференциалдық теңдеулер жүйесі үшін периодты есептер әулеті, алгоритм, бірмәнді шешілімділік.

Асанова А.Т., Искакова Н.Б., Орумбаева Н.Т. РАЗРЕШИМОСТЬ ПЕРИОДИЧЕСКОЙ ЗАДАЧИ ДЛЯ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ ЧЕТВЕРТОГО ПОРЯДКА С ЗАПАЗДЫВАНИЕМ ПО ВРЕМЕНИ

Исследуется периодическая задача для системы дифференциальных уравнений в частных производных четвертого порядка с запаздыванием по времени. Рассматриваемая задача методом введения дополнительных функций сведена к эквивалентной задаче, состоящей из семейства периодических задач для системы обыкновенных дифференциальных уравнений с конечным запаздыванием и интегральных соотношений. Построен алгоритм нахождения приближенных решений эквивалентной задачи и доказана его сходимость. Получены достаточные условия однозначной разрешимости периодической задачи для системы дифференциальных уравнений в частных производных четвертого порядка с запаздыванием по времени.

Ключевые слова. Периодическая задача, система дифференциальных уравнений в частных производных четвертого порядка с запаздыванием по времени, семейство периодических задач для системы обыкновенных дифференциальных уравнений с конечным запаздыванием, алгоритм, однозначная разрешимость.

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On a criterion for omissibility of a countable set of types in an incomplete theory

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Abstract. Let $\langle p_i : i < \omega\rangle$ and $\langle q_i : i < \omega\rangle$ be two sequences of complete non-isolated types in a small theory, such that for every natural number n there is a model which realizes the first n p_i's and omits the first n q_i's. In the article, we investigate the question of possibility of simultaneuos realizing the family $\langle p_i : i < \omega\rangle$ and omitting the family $\langle q_i : i < \omega\rangle$. We give a criterion characterizing omission of a countable set of types in an incomplete theory.

Keywords. Small theory, countable model, omitting types, prime model over a finite tuple, non-complete theory.

1 Introduction

A complete countable theory T is *small* if for every natural number $n \in \omega$ the set of all n-types over $\emptyset$ is countable, that is $|\bigcup S_n(T)| = \omega$. We say that a complete countable theory T has a *few number of countable models*, if the number of countable non-isomorphic models $I(T, \omega)$ is less than $2^{\aleph_0}$. It is well known that a theory with a few number of countable models is small.

Notice that for every countable model $\mathfrak{M} := \langle M, \Sigma\rangle$ of a small theory T, for every finite set $A \subset M$, the set of all 1-types over A is at most countable ($|S_1(A)| \leq \omega$) and there is a countable saturated model $\mathfrak{N} = \langle N, \Sigma\rangle$ ($\mathfrak{M} \preceq \mathfrak{N}$). Throughout the paper $\mathfrak{N}$ stands for a countable saturated model of a small theory.

Let $P := \langle p_n(x_n) : n \in \omega\rangle$ and $Q := \langle q_n(y_n) : n \in \omega\rangle$ be two sequences of non-isolated complete types over an empty set in a small theory T of a signature Σ such that for every natural number $n \in \omega$ there is a model $\mathfrak{M}_n$ of T such that for each $i \leq n$, $\mathfrak{M}_n$ realizes p_i and omits q_i. We can assume that the models $\mathfrak{M}_n$ are pairwise non-isomorphic. Otherwise there is no subject for the next question.

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Does there exist a countable model $\mathfrak{M}$ of T, such that it realizes each type from P and omits each type from Q?

In this article we present the necessary and sufficient condition for the existence of such a model, but this is not a complete answer to the question.

2 Criterion characterizing the omission of a countable set of types in an incomplete theory

Each type $q_i(y_i)$ can be represented as a strictly decreasing sequences of formulas $\{H_{i,m} : m \in \omega\}$ such that $T \vdash \forall y_i(H_{i,m+1}(y_i) \to H_{i,m}(y_i))$ and for each $\phi(y_i) \in q_i(y_i)$ there exists m such that $T \vdash \forall y_i(H_{i,m}(y_i) \to \phi(y_i))$. This is possible because for every $i \in \omega$ a type q_i is non-isolated and countable.

Let T_0 be a logical closure of $T \cup \bigcup\limits_{n\in\omega} p_n(\bar{c}_n)$ in the signature $\Sigma(C) := \Sigma \cup \{\bar{c}_n : n \in \omega\}$. Denote by $T_{0,n}$ a logical closure of $T_0 \cup \bigcup\limits_{j\leq n} p(\bar{c}_j)$ in the signature $\Sigma(C_n) := \Sigma \cup \{\bar{c}_j : j \leq n\}$. Then from the Deduction Theorem and the axiom of $\forall$-introduction for Calculus of Predicates the next lemma follows.

Lemma 1. *For any $i, n, m \in \omega$ and for any formula $\phi(y_i, c_1, c_2, ..., c_n)$ of signature $\Sigma(C_n)$ if $T_0 \vdash \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i))$, then $T_{0,n} \vdash \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i))$.*

Proof. It follows from finitary character of logical deducibility (provability) that for some $\Sigma(c_1, c_2, ..., c_n, c_{n+1}, ..., c_{n+k})$-formula $\Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}) \in T_0$ the following holds:

$$T_{0,n} \cup \{\Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k})\} \vdash \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)).$$

Then by the Deduction Theorem [1]

$$T_{0,n} \vdash \Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}) \to \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)),$$

and

$$T_{0,n} \vdash \neg\Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}) \vee \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)).$$

By $\forall$-introduction [1] we have

$$T_{0,n} \vdash \forall z_1, ..., z_k[\neg\Theta(c_1, ..., c_n, z_1, ..., z_k) \vee \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i))],$$

then we apply the law $\forall x(P(x) \vee A) \Leftrightarrow (\forall x P(x)) \vee A$ provided that x is not free in A

$$T_{0,n} \vdash \forall z_1, ..., z_k[\neg\Theta(c_1, ..., c_n, z_1, ..., z_k)] \vee \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)).$$

Then by the standard laws of equivalence of First-Order Logic we have

$$T_{0,n} \vdash \exists z_1, ..., z_k\Theta(c_1, ..., c_n, z_1, ..., z_k) \to \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)). \tag{1}$$

Again by the Deduction Theorem

$$T_{0,n} \cup \{\exists z_1, ..., z_k \Theta(c_1, ..., c_n, z_1, ..., z_k)\} \vdash \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)). \tag{2}$$

Since $\Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}) \in T_0$ we conclude

$$\exists z_1, ..., z_k \Theta(c_1, ..., c_n, z_1, ..., z_k) \in T_{0,n}. \tag{3}$$

Indeed, $T_0 \vdash \Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k})$. It means

$$T \cup \bigcup_{j \le n+k} p(c_j) \vdash \Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}).$$

From finitary character of provability it follows that $T \cup \bigcup_{j \le n} p(c_j) \cup \{\bigwedge_{n+1 \le j \le n+k} K_j(c_j)\} \vdash \Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k})$, where $K_j(c_j) \in p(c_j)$. So, by the Deduction Theorem we obtain

$$T \cup \bigcup_{j \le n} p(c_j) \vdash \bigwedge_{n+1 \le j \le n+k} K_j(c_j) \to \Theta(c_1, ..., c_n, c_{n+1}, ..., c_{n+k}).$$

Then by $\forall$-introduction we have

$$T \cup \bigcup_{j \le n} p(c_j) \vdash \forall z_1, ..., \forall z_k [\bigwedge_{1 \le r \le k} K_r(z_r) \to \Theta(c_1, ..., c_n, z_1, ..., z_k)]. \tag{4}$$

Since for any r $(1 \le r \le k)$ $K_r(z_r) \in p_{n+r}(z_r)$ the formula

$$\exists z_1, ..., z_k \bigwedge_{1 \le r \le k} K_r(z_r) \ \text{ belongs to the theory } \ T. \tag{5}$$

And by (4) and (5) we have (3). Thus by (2) and (3) we have

$$T_{0,n} \vdash \forall y_i(\phi(y_i, c_1, c_2, ..., c_n) \to H_{i,m}(y_i)).$$

□

Let $\phi(y_i, \bar{c}_n)$ be a formula such that for every $H_{i,m}(y_i) \in q_i$ the following holds: $T_0 \vdash \forall y_i(\phi(y_i, \bar{c}_n) \to H_{i,m}(y_i))$, that is the type q_i is deduced from the formula $\phi(y_i, \bar{c}_n)$. Then by Lemma 1 we have that $T_{0,n}, \phi(y_i, \bar{c}_n) \vdash q_i(y_i)$. Since $T_{0,n} \subset T_{0,n+k}$, then $T_{0,n}$ has infinitely many models of T omitting q_i, $T_{0,n} \cup \{\neg\exists y_i \phi(y_i, \bar{c}_n)\}$ is consistent and consequently, $T_0 \cup \{\neg\exists y_i \phi(y_i, \bar{c}_n)\}$ is consistent. Moreover, for any $k \in \omega$ such that $i \le n+k$, we have $\mathfrak{M}_{n+k} \models \neg\exists y_i \phi(y_i, \bar{c}_n)$.

Let T_1 be a logical closure of the following set:

$$T_0 \cup \{\neg\exists y_i \phi(y_i, \bar{c}_n) : \phi(y_i, \bar{c}_n) \text{ is a formula of } \Sigma(C) \text{ such that } \exists i \in \omega, T_0 \vdash \phi(y_i, \bar{c}_n) \to q_i(y_i)\}.$$

We notice that T_1 is consistent, because any finite subset of T_1 has an infinite number of models of $\Sigma(C_n)$ for suitable $n \in \omega$. Suppose T_1 is not complete. Denote by $T_{1,n}$

$$T_{0,n} \cup \{\neg\exists y_i \phi(y_i, \bar{c}_n) \text{ formula of } \Sigma(C_n) : \exists i \in \omega, T_0 \vdash \phi(y_i, \bar{c}_n) \to q_i(y_i)\}.$$

For every $n, i \in \omega$ we consider the following set of one-$\Sigma(C_n)$-formulas

$$\Gamma_{n,i} := \{\phi(y_i, \bar{c}_n) : \exists y_i(\phi(y_i, \bar{c}_n)) \in T_{1,n} \text{ and } T_1 \cup \{\forall y_i(\phi(y_i, \bar{c}_n) \to H_{i,m}(y_i))\} \text{ is consistent for each } m \in \omega\}.$$

We enumerate formulas in $\Gamma_{n,i}$ in some way. For each $n, i, l, m \in \omega$ we denote for l-th formula from $\Gamma_{n,i}$ the next $\Sigma(C_n)$-sentence:

$$S_{n,i,l,m}(\bar{c}_n) := \forall y_i(\phi_l(y_i, \bar{c}_n) \to H_{i,m}(y_i)).$$

It follows from the definition of $H_{i,m}$ that for any $n, i, l \in \omega$, if $\Gamma_{n,i} \neq \emptyset$ and $l < |\Gamma_{n,i}|$ the following is true, $T \vdash \forall \bar{z}(S_{n,i,l,m+1}(\bar{z}) \to S_{n,i,l,m}(\bar{z}))$ and consequently, $T \vdash \forall \bar{z}(\neg S_{n,i,l,m}(\bar{z}) \to \neg S_{n,i,l,m+1}(\bar{z}))$.

Lemma 2. *Let T'_n be a complete consistent extension of $T_{1,n}$ of the same signature and let $i \in \omega$. Then every model of T'_n realizes q_i if and only if $\Gamma_{n,i} \neq \emptyset$ and for some $l < |\Gamma_{n,i}|$, $T_{1,n} \cup \{S_{n,i,l,m}(\bar{c}_n) : m \in \omega\} \subseteq T'_n$.*

Proof. Since T is a complete small theory, every complete extension T'_n containing $T_{1,n}$ in $\Sigma(C_n)$ is a small theory, too. Let $\mathfrak{M}(\mathfrak{c})$ be a prime model of T'_n over $\bar{c}$. Then because q_i is realized in $\mathfrak{M}(\mathfrak{c})$ there is $\bar{a} \in M$ such that $\bar{a} \models q_i$ and $tp(\bar{a}/\bar{c})$ is isolated. Consequently, there is a one-$\Sigma(C_n)$-formula $\phi(y_i, \bar{c})$ such that $T'_n \vdash \forall y_i(\phi(y_i, \bar{c}_n) \to H_{i,m}(y_i))$ for any $m \in \omega$. This means that $\Gamma_{n,i} \neq \emptyset$ and $\phi(y_i, \bar{c}) = \phi_l(y_i, \bar{c}_n)$ for some $l < |\Gamma_{n,i}|$. Thus, $T'_n \vdash S_{n,i,l,m}(\bar{c}_n)$ and by completeness of T'_n we have that $S_{n,i,l,m}(\bar{c}_n) \in T'_n$ for any $m \in \omega$.

Sufficiency follows from the definition that any model of T'_n contains a realization of the type q_i. $\square$

Let $p(\bar{x})$, $q(\bar{y}) \in S(A)$ be two types over a subset A of some model of T. We say that the type $p(\bar{x})$ is *not almost orthogonal* to the type $q(\bar{y})$, $p(\bar{x}) \not\perp^a q(\bar{y})$, if there is a formula $\varphi(\bar{x}, \bar{y}, \bar{a})$, $\bar{a} \in A$, such that for some (equivalently, for any) model $\mathfrak{M} \models T$ realizing $p(\bar{x})$ with $A \subseteq M$, for some (equivalently, for any) realization $\bar{\alpha} \in p(M)$, $\emptyset \neq \varphi(\bar{\alpha}, M, \bar{a}) \subset q(M)$. Otherwise, the types are called to be almost orthogonal, $p(\bar{x}) \perp^a q(\bar{y})$.

Theorem 1. *(Tarski-Vaught criterion) [2]. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two $\mathfrak{L}$-structures, with $M \subseteq N$. Then the following conditions are equivalent:*

i) the structure $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$;

ii) for any formula $\psi(x, \bar{y})$ of the language $\mathfrak{L}$ and any $\bar{a} \in M$, if $\mathfrak{N} \models \exists x\ \psi(x, \bar{a})$, then $\mathfrak{N} \models \psi(d, \bar{a})$ for some $d \in M$.

Fact 1. *Let $p(\bar{x})$, $q(\bar{y}) \in S(A)$ be two types over a subset $A \subseteq N$ of a countable saturated model $\mathfrak{N} \models T$, such that $p(\bar{x})$ is isolated and $q(\bar{y})$ is non-isolated. Then $p(\bar{x})$ is almost orthogonal to $q(\bar{y})$: $p(\bar{x}) \perp^a q(\bar{y})$.*

Proof. Let us suppose that $p \not\perp^a q$, then by definition for some realization $\bar{\alpha} \in p(N)$ there is a formula $\varphi(\bar{x}, \bar{y})$ having the following property $\emptyset = \varphi(M, \bar{\alpha}) \subset q(M)$. Since $p(\bar{x})$ is an isolated type, there is an isolating formula $\theta(\bar{x})$ for which $p(M) = \theta(M)$. Now let us consider the following A-formula $H(\bar{x}) := \exists y(\theta(\bar{x}) \wedge \varphi(\bar{x}, \bar{y}))$. So, we have $H(M) \subset q(M)$, that contradicts q being non-isolated. $\square$

The following fact is well known:

Fact 2. *Let $\mathfrak{M} = \langle M, \Sigma \rangle$ be a countable model of a small theory T. Then for each formula $\psi(\bar{x}, \bar{b})$, $\bar{b} \in M$, there is a subformula $\psi_0(\bar{x}, \bar{b})$ such that $\psi_0(\bar{x}, \bar{b})$ determines an isolated type over $\bar{b}$.*

Proof. Note that if the formula $\psi(\bar{x}, \bar{b})$ has no subformulas defining an isolated type, then every its subformula has the same property. Consider an arbitrary subformula $\psi_1(\bar{x}, \bar{b}) \subset \psi(\bar{x}, \bar{b})$. Then the formula $\psi_0(\bar{x}, \bar{b}) := \psi(\bar{x}, \bar{b}) \wedge \neg\psi_1(\bar{x}, \bar{b})$ is a proper subformula of $\psi(\bar{x}, \bar{b})$. Therefore for every n and every finite sequence $\langle \tau_1, \tau_2, ..., \tau_n \rangle$ of 0's and 1's we can choose the following sequence of $\bar{b}$-definable formulas: $\psi_{\tau_1,\tau_2,...,\tau_n}(\bar{x}, \bar{b}) \subset ... \subset \psi_{\tau_1,\tau_2}(\bar{x}, \bar{b}) \subset \psi_{\tau_1}(\bar{x}, \bar{b})$. The last means existence of an infinite 2-branching tree of $\bar{b}$-formulas, that implies that T is not small, for a contradiction. $\square$

Fact 3. *Let $p(\bar{x})$, $q(\bar{y}) \in S(A)$ be two types over a subset $A \subseteq M$ of a countable model $\mathfrak{M} = \langle M, \Sigma \rangle$ of a small theory T and let $q(\bar{y})$ be non-isolated. Then $p(\bar{x}) \perp^a q(\bar{y})$ if and only if for every realization $\bar{\gamma}$ of the type $p(\bar{x})$ every extension $q'(\bar{y}, \bar{\gamma})$ of $q(\bar{y})$ to a complete type over $(A \cup \{\bar{\gamma}\})$ is non-isolated.*

Proof. Towards a contradiction, assume that the type $q'(\bar{y}, \bar{\gamma})$ is isolated. Then there exists a formula $\psi(\bar{y}, \bar{a}, \bar{\gamma}) \in q'(\bar{y}, \bar{\gamma})$ with $\bar{a} \in A$, which isolates the type $q'(\bar{y}, \bar{\gamma})$. Since $q'(\bar{y}, \bar{\gamma}) \supset q(\bar{y})$, we have $q(M) \supset q'(M, \bar{\gamma}) \supset \psi(M, \bar{a}, \bar{\gamma}) \neq \emptyset$. Therefore, by the definition, $p(\bar{x}) \not\perp^a q(\bar{y})$. This is a contradiction.

Let every complete extension of the type q over any realization of the type p be non-isolated. We will obtain a contradiction by supposing that $p(\bar{x}) \not\perp^a q(\bar{y})$. From the last it follows that there exists $\bar{\gamma}_0 \models p$ and an $A\bar{\gamma}_0$-formula $\psi(\bar{x}, \bar{\gamma}_0)$ such that $\psi(M, \bar{\gamma}_0) \subset q(M)$. Then by Fact 2 it follows that there exists an isolating subformula $\psi_0(M, \bar{\gamma}_0) \subset \psi(M, \bar{\gamma}_0) \subset q(M)$. Then for some isolated type q' we have the following:

$$\psi_0(M, \bar{\gamma}_0) = q'(M, \bar{\gamma}_0) \subset q(M),$$

which is a contradiction, since any extension of the type q over any realization of p is a non-isolated type. $\square$

Fact 4. *Let T be a small theory. Let $q(\bar{x}, \bar{b})$ be non-isolated, let $tp(\bar{c}\bar{d}/\bar{b}) \not\perp^a q(\bar{x}, \bar{b})$ and $tp(\bar{d}/\bar{b}\bar{c})$ be isolated, then $tp(\bar{c}/\bar{b}) \not\perp^a q(\bar{x}, \bar{b})$. Equivalently, if $q(x, \bar{b})$ is non-isolated, $tp(\bar{c}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$ and $tp(\bar{d}/\bar{b}\bar{c})$ is isolated, then $tp(\bar{c}\bar{d}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$.*

Proof. If $tp(\bar{c}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$, then every type $q(\bar{x}, \bar{b}) \subset q'(\bar{x}, \bar{b}, \bar{c})$ is non-isolated. By Fact 1, $tp(\bar{d}/\bar{b}\bar{c}) \perp^a q'(\bar{x}, \bar{b}, \bar{c})$ for every $q'(\bar{x}, \bar{b}, \bar{c})$ extending $q(\bar{x}, \bar{b})$. The last means that every

$q''(\bar{x}, \bar{b}, \bar{c}, \bar{d})$ extending $q'(\bar{x}, \bar{b}, \bar{c})$ will be non-isolated. We prove that $tp(\bar{c}\bar{d}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$. Let us suppose that $tp(\bar{c}\bar{d}/\bar{b}) \not\perp^a q(\bar{x}, \bar{b})$. Then there is a formula $\theta(M, \bar{b}, \bar{c}, \bar{d}) \subset q(M, \bar{b})$, and by the Fact 2, $\theta(\bar{x}, \bar{b}, \bar{c}, \bar{d})$ can be considered to be isolating for some isolated type $q''(\bar{x}, \bar{b}, \bar{c}, \bar{d})$. That is, $q''(\bar{x}, \bar{b}, \bar{c}, \bar{d})$ is an isolated type, which contradicts the obtained condition that all such types q'' are non-isolated. □

Fact 5. *Let $\bar{b} \in M$, let $\bar{c} \in N \backslash M$ and let $\psi_0(\bar{x}, \bar{b}, \bar{c})$ defines an isolated type over $\bar{b}\bar{c}$, $q(\bar{x}, \bar{y}) \in \mathfrak{D}(\mathfrak{M}) := \{p : p \in S(T),\ p \text{ is realized in } \mathfrak{M}\}$. Then the following holds: if $q(N, \bar{b}) \cap M = \emptyset$, then $\psi_0(N, \bar{b}, \bar{c}) \subset q(N, \bar{b})$ or $\psi_0(N, \bar{b}, \bar{c}) \cap q(N, \bar{b}) = \emptyset$.*

While formulating the fact, we used the condition of almost orthogonality in the following sense: $tp(\bar{c}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$ implies $\psi_0(N, \bar{b}, \bar{c}) \cap q(N, \bar{b}) = \emptyset$, provided that $\psi_0(\bar{x}, \bar{b}, \bar{c})$ defines an isolated type over $\bar{b}\bar{c}$.

Theorem 2. *Let theory T' of the signature $\Sigma(C)$ be a complete consistent extension of T_1. Then there is a model of T' omitting all types from Q if and only if for every $n, i \in \omega$, $\Gamma_{n,i} = \emptyset$ or for every $l \leq |\Gamma_{n,i}|$ there exists $m \leq \omega$ such that $\neg S_{n,i,l,m}(\bar{c}_n) \in T'$.*

Proof. Necessity follows from Lemma 2. Construction of a model is based on S.V. Sudoplatov's construction in his theorem which states that every countable model of a small theory can be represented as an increasing chain of prime models over some finite tuples [3].

Take a realization $c_1 \in p_1(N)$. By Fact 3 any completion $r_2(x_1, x_2)$ of $p_1(x_1) \cup p_2(x_2)$ is non-isolated. Let c_2 be a realization of $r_2(c_1, x_2)$. Repeating this construction we obtain a set $C := \{c_1, c_2, \ldots,\ c_k, c_{k+1} \ldots\}_{k<\omega}$ consisting of exactly one realization of each type in P, such that $tp(c_{k+1}|c_1, ..., c_k) = r_{k+1}(c_1, ..., c_k, x_{k+1})$ is non-isolated. Also, for each $n < \omega$ let $\bar{c}_n$ denote $\langle c_1, c_2, \ldots, c_n \rangle$, $c_i \in C$ for any $i < \omega$.

Since T' is a complete theory in the signature $\Sigma(C)$, then $T' = T \cup \bigcup\limits_{n\in\omega} r_n(\bar{c}_n)$. Here, for any $n \in \omega$, $\bigcup\limits_{0\leq j\leq n} p_j(\bar{c}_j) \subset r_n(\bar{c}_n)$. Let $T'_n := T \cup r_n(\bar{c}_n)$. Notice that T' is not necessary small, and at the same time for any n, the complete theory T'_n is small. It is possible to construct a countable model of theory T' as a submodel of an $\aleph_0$-saturated model of T. We use Tarski-Vaught criterion in order to show that the constructed set $M(\bar{c}_n)$ is a universe of an elementary substructure of $\mathfrak{N}$. On each step of the construction we fix a set of parameters and promising to realize each satisfiable 1-formula over it. We come back to some set of parameters and deal with another formula. So, the different sets of parameters are attacked in parallel.

We construct an elementary chain $\mathfrak{C}$ of prime models $\mathfrak{M}(\bar{\mathfrak{c}}_i)$ over tuples $\bar{c}_i,\ i \in \omega$, such that $\mathfrak{M}(\mathfrak{C}) = \bigcup\limits_{i\in\omega} \mathfrak{M}(\bar{\mathfrak{c}}_i)$. We construct $\mathfrak{C}$ inductively and at the step k, a finite sequence of tuples $\bar{c}_0, ..., \bar{c}_n$ is defined, and each such tuple connected to a finite set $X_i^k,\ 0 \leq i \leq n$, such that the unions of these sets for all k with respect to a fixed i define universes of models

$\mathfrak{M}(\bar{\mathfrak{c}}_i)$. If the tuple $\bar{c}_i$ is not defined before the step k, then the sets X_i^l are supposed to be empty for all $l < k$.

Note that for every $q_i \in Q$, $r_n(\bar{z}_n) \perp^a q_i(y_i)$, since for every $n, i \in \omega$, either $\Gamma_{n,i} = \emptyset$ or for every $l \leq |\Gamma_{n,i}|$ there exists $m \leq \omega$ such that $\neg S_{n,i,l,m}(\bar{c}_n) \in T'$.

Construction of $\mathfrak{M}(\mathfrak{C})$. We use the method described in [4] and [5].

Step 1. Let us consider the tuple c_1. Denote by Ψ_1 the set of all c_1-definable 1-formulas of the theory T_1', $\Psi_1 := \{\psi_i^1(x, c_1) : i < \omega\}$. Choose a formula $\psi_i^1(x, c_1) \in \Psi_1$ with the smallest index i satisfying $\mathfrak{N} \models \exists x \psi_i^1(x, c_1)$. To satisfy the Tarski-Vaught property, we must find a witness for $\psi_i^1(x, c_1)$. Since T_1' is small, there exists an isolated over c_1 subformula $\psi_{i,1}^1(x, c_1) \subseteq \psi_i^1(x, c_1)$, which, in its turn, has an isolated subformula over $\{\bar{c}_2\}$. Repeating this procedure, we obtain a locally consistent infinite decreasing chain of isolated over parameters formulas $\psi_{i,j}^1(x, \bar{c}_j)$: $... \subseteq \psi_{i,n+1}^1(N, \bar{c}_{n+1}) \subseteq \psi_{i,n}^1(N, \bar{c}_n) \subseteq ... \subseteq \psi_{i,1}^1(N, \bar{c}_2) \subseteq \psi_i^1(N, c_1)$. Take a type $q_{\psi_i^1}$ defined by this chain. And since the model $\mathfrak{N}$ is $\aleph_0$-saturated, this type is realized in $\mathfrak{N}$ by some element, denote it by d_1. Since for any $q_i \in Q$, $r_n(\bar{z}_n) \perp^a q_i(y_i)$. Thus, for any $n < \omega$, d_1 is isolated over $\bar{c}_n$. Let $X_1^1 = \{c_1, d_1\}$.

Step 2. Choose a formula $\psi_i^1(x, c_1) \in \Psi_1$ which has not been considered before and which has the smallest index i satisfying $\mathfrak{N} \models \exists x \psi_i^1(x, c_1)$. We find a realization d_2 by analogy with d_1. Since $tp(d_1/\bar{c}_n) \perp^a q_i(y_i, \bar{c}_n)$, then the element d_2 can be chosen to be isolated over the set $\bar{c}_n d_1$, for any $n < \omega$.

Now we take $\bar{c}_2$ and consider the set of all $(\{d_1\} \cup \{\bar{c}_2\})$-definable 1-formulas of the theory T_2', $\Psi_2 := \{\psi_i^2(x, d_1, \bar{c}_2) : i < \omega\}$. We choose the formula $\psi_i^2(x, d_1, \bar{c}_2) \in \Psi_2$ which has not been considered previously and has the smallest index satisfying $\mathfrak{N} \models \exists x \psi_i^2(x, d_1, \bar{c}_2)$, and find a realization d_3 (existing since $\mathfrak{N}$ is $\aleph_0$-saturated) of the following infinite decreasing chain of isolated formulas $\psi_{i,j}^2(x, d_1, \bar{c}_j)$: $... \subseteq \psi_{i,n+1}^2(x, d_1, \bar{c}_{n+1}) \subseteq \psi_{i,n}^2(x, d_1, \bar{c}_n) \subseteq ... \subseteq \psi_i^2(x, d_1, \bar{c}_2)$. Note that, if $tp(d_2/\bar{c}_n d_1)$ and $tp(d_1/\bar{c}_n)$ are isolated types, then $tp(d_2/\bar{c}_n)$ is isolated. Also, if $tp(d_2/\bar{c}_n d_1)$ and $tp(d_1/\bar{c}_n)$ are isolated types, then $tp(d_1 d_2/\bar{c}_n)$ is isolated for any $n \geq 2$. So, d_3 can be chosen to be isolated over the set $\bar{c}_n d_1 d_2$, for any $n < \omega$. Let $X_1^2 = X_1^1 \cup \{d_2\}$ and $X_2^1 = X_1^1 \cup \{c_2, d_3\}$.

By the end of the step k we will have the following sets:

– for all m, $1 \leq m \leq k$, the sets $D_1 = \{d_1\}$, $D_2 = \{d_1, d_2, d_3\}$, $D_3 = \{d_1, d_2, ..., d_6\}$, $D_m := \{d_1, d_2, ..., d_{\frac{(m+1)m}{2}}\}$ (it is possible that $d_i = d_j$ for some i and j such that $1 \leq i < j \leq \frac{(m+1)m}{2}$), the set of all c_1-definable 1-formulas Ψ_1, and for all m, $2 \leq m \leq k$, the sets of all $(D_{m-1} \cup \bar{c}_m)$-definable 1-formulas, Ψ_m;

– for all m, $1 \leq m \leq k$, the sets $X_1^m, X_2^{m-1}, ..., X_m^1$.

Step $k+1$. For each m, $1 \leq m \leq k$, we find previously unused formula $\psi_{i_m}^m \in \Psi_m$ of a minimal index, the set of realizations of which in the model $\mathfrak{N}$ is nonempty. Then we find realizations $d_{\frac{(k+1)k}{2}+m}$ of the corresponding infinite decreasing chains of isolated subformulas of the formulas $\psi_{i_m}^m$.

Denote by Ψ_{k+1} the set of all $(D_k \cup \bar{c}_{k+1})$-definable 1-formulas of the theory T'_{k+1}. And find $d_{\frac{(k+1)k}{2}+k+1}$ by the analogy with the construction above. Let D_{k+1} stand for the set $\{d_1, d_2, ..., d_{\frac{(k+1)k}{2}+k+1}\}$. We can arrange that each new d_i is isolated over $\bar{c}_n$ and the d_j's for $j < i \leq n$.

By the construction, the sets $X_i \rightleftharpoons \underset{k\in\omega}{\cup} X_i^k$ are the universes of prime models $\mathfrak{M}(\bar{\mathfrak{c}}_{\mathfrak{i}})$ over tuples $\bar{c}_i$. Moreover, $\mathfrak{M}(\bar{\mathfrak{c}}_{\mathfrak{i}}) \prec \mathfrak{M}(\bar{\mathfrak{c}}_{\mathfrak{i+1}})$ and $\mathfrak{M}(\mathfrak{C}) = \bigcup\limits_i \mathfrak{M}(\bar{\mathfrak{c}}_{\mathfrak{i}})$. By Fact 4, $tp(\bar{d}/\bar{c}_n) \perp^a q'(y_i, \bar{c}_n)$ because $tp(\bar{d}/\bar{c}_n)$ is isolated and every complete extension $q'(y_i, \bar{c}_n)$ of the type q over any realization $\bar{c}_n$ of the type p is non-isolated. In this way $\mathfrak{M}(\mathfrak{C})$ omits Q, since for each $\bar{d} \in \mathfrak{M}(\mathfrak{C}) \setminus C$ we have at the same time that $\bar{d} \in \mathfrak{M}(\bar{\mathfrak{c}}_{\mathfrak{n}})$ and $tp(\bar{d}/\bar{c}_n) \perp^a q'(y_i, \bar{c}_n)$.

□

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Байжанов Б.С., Умбетбаев О.А., Замбарная Т.С. ТОЛЫҚ ЕМЕС ТЕОРИЯДАҒЫ ТИПТЕРДІҢ САНАЛЫМДЫ ЖИЫНЫНЫҢ ТҮСІРІЛІМДІГІНІҢ БІР КРИТЕРИЙІ ТУРАЛЫ

$\langle p_i : i < \omega \rangle$ және $\langle q_i : i < \omega \rangle$ – шағын теорияның оқшауланбаған типтерінің екі үйірі болсын, мұнда әрбір n үшін, бірінші n тал p_i типтерін жүзеге асыратын және бірінші n тал q_i типтерін түсіретін модель бар. Бұл мақалада толық емес теорияда $\langle p_i : i < \omega \rangle$ үйірі және $\langle q_i : i < \omega \rangle$ үйірі типтерін қатар түсіру және жүзеге асыру мүмкіндігі туралы сұрақ зерттеледі. Толық емес теорияның саналымды жиын типтерінің түсірілімдігін сипаттайтын критерий беріледі.

Кілттік сөздер. Шағын теория, саналымды модель, типтерді түсіру, ақырғы жиындағы жәй модель, толық емес теория.

Байжанов Б.С., Умбетбаев О.А., Замбарная Т.С. ОБ ОДНОМ КРИТЕРИИ ОПУСКАЕМОСТИ СЧЕТНОГО МНОЖЕСТВА ТИПОВ В НЕПОЛНОЙ ТЕОРИИ

Пусть $\langle p_i : i < \omega \rangle$ и $\langle q_i : i < \omega \rangle$ – два семейства неизолированных типов малой теории такие, что для каждого n существует модель, реализующая первые n типов p_i и опускающая первые n типов q_i. В данной статье исследуется вопрос о возможности одновременной реализации семейства $\langle p_i : i < \omega \rangle$ и опускании семейства $\langle q_i : i < \omega \rangle$ в неполной теории. Дается критерий, характеризующий опускаемость счетного множества типов неполной теории.

Ключевые слова. Малая теория, счётная модель, опускание типов, простая модель над конечным множеством, неполная теория.

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Solution of nonregular multidimensional two-phase problem for parabolic equations with time derivative in conjugation condition

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Abstract. Linearized multidimensional two-phase free boundary problem for the parabolic equations with two small parameters $\varepsilon > 0$ and $\kappa > 0$ at the principle derivatives in the conjugation condition is studied. The estimates of the solution and the perturbed term with respect to ε are derived in the Hölder space.

Keywords. Multidimensional two-phase boundary value problem, parabolic equations, small parameters in conjugation condition, solution in the explicit form, estimates of the solution and perturbed term, Hölder space.

1 Statement of the problem. Main results

Let $D_1 := \mathbb{R}^n_- = \{x : x' \in \mathbb{R}^{n-1},\ x_n < 0\}$, $D_2 := \mathbb{R}^n_+ = \{x : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$, $n \geq 2$, $R := \{x : x' \in \mathbb{R}^{n-1},\ x_n = 0\}$, $D_{pT} := D_p \times (0,T)$, $p = 1,2$, $R_T := R \times [0,T]$, $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$, $\varepsilon > 0$, $\kappa > 0$ are small parameters.

Consider problem with the unknown functions $u_1(x,t)$ and $u_2(x,t)$

$$\partial_t u_p - \sum_{i,j=1}^{n} a_{ij}^{(p)}\, \partial^2_{x_i x_j} u_p = f_p(x,t) \text{ in } D_{pT},\ p = 1,2, \tag{1.1}$$

$$u_p|_{t=0} = 0 \text{ in } D_p,\ p = 1,2, \tag{1.2}$$

$$\big(u_1 - u_2\big)|_{x_n=0} = \varphi_0(x',t),\ t \in (0,T), \tag{1.3}$$

$$\big(\varepsilon\, \partial_t u_1 + \kappa b \nabla^T u_1 - c \nabla^T u_2\big)|_{x_n=0} \tag{1.4}$$

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$$= \varepsilon\varphi_1(x',t) + \kappa\varphi_2(x',t) + \varphi_3(x',t) =: \Phi_{\varepsilon,\kappa}(x',t),\ t \in (0,T),$$

where all coefficients are constant, $b = (b', b_n)$, $b' = (b_1, \dots, b_{n-1})$, $c = (c', c_n)$, $c' = (c_1, \dots, c_{n-1})$, $\nabla^T = \mathrm{colon}(\partial_{x_1}, \dots, \partial_{x_n})$ is a column-vector, $b\,c^T = b_1c_1 + \dots + b_nc_n$ is a scalar product, $\Delta = \partial^2_{x_1} + \dots + \partial^2_{x_n}$, $\partial_t = \partial/\partial t$, $\partial_{x_i} = \partial/\partial x_i$,

$$b_n > 0,\ c_n > 0. \tag{1.5}$$

By C_1, $C_2, \dots$ we shall denote positive constants, their numeration begins from 1 in each chapter.

The coefficients of the equations (1.1) satisfy the conditions of the ellipticity

$$a^{(p)}_{i\,j} = a^{(p)}_{j\,i},\ i,\, j = 1, \dots, n,\ \sum_{i,j=1}^{n} a^{(p)}_{i\,j}\, \xi_i\xi_j \geq a_0\xi^2,\ \xi \in \mathbb{R}^n,\ p = 1, 2,$$

$a_0 = \mathrm{const} > 0$.

(1.1)–(1.4) is a linearized model two-phase free boundary problem with two small parameters $\kappa > 0$, $\varepsilon > 0$ at the principle terms in the boundary condition (1.4). Such problems arise in the theory of combustion, phase transition (melting, solidification of substance), theory of the filtration of liquid and gas in porous medium.

This problem with $\kappa = 1$, $\varepsilon = 1$ was studied in the Hölder space by B.V. Bazaliy [1], E.V. Radkevich [2], G.I. Bizhanova [3], [4], G.I. Bizhanova, V.A. Solonnikov [5]. In [6], [7] there was considered the problem (1.1)–(1.4) for the heat equations with $\kappa = 1$, $\varepsilon > 0$, the estimates of the solutions with the constants independent on ε were derived in the classical and weighted Hölder spaces. In [8] the two-phase problem for the heat equations with a small parameter at the time derivative $\varepsilon\partial_t u_1$ on the boundary $x_n = 0$ was considered, an estimate of the solution of the problem with the constant independent on ε was obtained in the Hölder space. The linear one-phase boundary value problem for the heat equation with a small parameter at the time derivative $\varepsilon\partial_t u$ on the boundary $x_n = 0$ was studied in [9]. The estimates of the solution and of $\varepsilon\partial_t u|_{x_n=0}$ with respect to ε were derived.

In [10] there were constructed the solution and obtained an estimate of the Green function of the two-phase boundary value problem for the parabolic equations with two small parameters at the principal derivatives in the conjugation condition.

The results of an article permit us to find the solutions of the partially or fully unperturbed problems with $\kappa = 0$, $\varepsilon > 0$; $\kappa > 0$, $\varepsilon = 0$, $\kappa = 0$, $\varepsilon = 0$ without solving them. The Theorems 1.1, 1.2 show that in the problem (1.1)–(1.4) there are not appeared boundary layer as small parameters go to zero, although the small parameters κ and ε are at the principle derivatives in the condition (1.4).

This model problem is on the basis of the establishment of the unique solvability of the linear and free boundary problems for the parabolic equation with two small parameters in the boundary condition. With the help of Theorems 1.1 and 1.2 we can justify the convergence of the solution of the perturbed problem to the solution of the partially or fully unperturbed ones, to obtain the existence, uniqueness and estimates of the partially or fully unperturbed problems without loss of the smoothness of the given functions.

We also point out that the problem (1.1)–(1.4) is conventionally correct. The condition (1.5) provides the solvability of it.

We shall study the problem (1.1)–(1.4) in the Hölder space $C_{x\ t}^{l,\,l/2}(\overline{\Omega}_T)$, l is a positive non-integer, of the functions $u(x,t)$ with the norm [11]

$$|u|_{\Omega_T}^{(l)} = \sum_{2m_0+|m|=0}^{[l]} |\partial_t^{m_0}\partial_x^m u|_{\Omega_T} + \sum_{2m_0+|m|=[l]} \Big([\partial_t^{m_0}\partial_x^m u]_{x,\Omega_T}^{(\alpha)} + [\partial_t^{m_0}\partial_x^m u]_{t,\Omega_T}^{(\alpha/2)} \Big)$$

$$+ \begin{cases} 0, & [l]=0, \\ \sum_{2j_0+j=[l]-1} \big[\partial_t^{j_0}\partial_x^j u\big]_{t,\Omega_T}^{(\frac{1+\alpha}{2})}, & [l]\geq 1, \end{cases} \tag{1.6}$$

where $\alpha = l-[l]\in(0,1)$, $\Omega_T := \Omega\times(0,T)$, Ω is a domain in $\mathbb{R}^n$, $n\geq 2$, $m=(m_1,\ldots,m_n)$, m_i are the non-negative integers, $i=0,\,1,\ldots,n$, $|m|=m_1+\ldots+m_n$,

$$|v|_{\Omega_T} = \max_{(x,t)\in\overline{\Omega}_T} |\,v|,$$

$$[v]_{x,\Omega_T}^{(\alpha)} = \max_{(x,t),(z,t)\in\overline{\Omega}_T} \frac{\big|v(x,t)-v(z,t)\big|}{|x-z|^{\alpha}},\ [v]_{t,\Omega_T}^{(\alpha)} = \max_{(x,t),(x,t_1)\in\overline{\Omega}_T} \frac{\big|v(x,t)-v(x,t_1)\big|}{|t-t_1|^{\alpha}}.$$

By $\overset{\circ}{C}{}_{x\ t}^{l,\,l/2}(\overline{\Omega}_T)$ we designate the subset of the functions $u(x,t)\in C_{x\ t}^{l,\,l/2}(\overline{\Omega}_T)$ such that $\partial_t^k u\big|_{t=0}=0,\ k=0,\ldots,[l/2]$.

We formulate the main theorems of the present work.

Theorem 1.1. *Let $b_n>0,\ c_n>0,\ \kappa\in(0,\,\kappa_0],\ \varepsilon\in(0,\,\varepsilon_0],\ l$ be a positive non-integer.*

For every functions $f_p(x,t)\in\overset{\circ}{C}{}_{x\ t}^{l,\,l/2}(\overline{D}_{pT})$, $p=1,2$, $\varphi_0(x',t)\in\overset{\circ}{C}{}_{x'\ \ t}^{2+l,1+l/2}(R_T)$, $\varphi_k(x',t)\in\overset{\circ}{C}{}_{x'\ \ t}^{1+l,\frac{1+l}{2}}(R_T)$, $k=1,2,3$, the problem (1.1)–(1.4) has a unique solution $u_p(x,t)\in\overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{D}_{pT})$, $p=1,2$, $\varepsilon\,\partial_t u_1|_{x_n=0}\in\overset{\circ}{C}{}_{x'\ \ t}^{1+l,\frac{1+l}{2}}(R_T)$ and it satisfies the estimate

$$\sum_{p=1}^{2} |u_p|_{D_{pT}}^{(2+l)} + |\varepsilon\partial_t u_1|_{R_T}^{(1+l)} \leq C_1 M_l, \tag{1.7}$$

$$M_l := \sum_{p=1}^{2} |f_p|_{D_{pT}}^{(l)} + |\varphi_0|_{R_T}^{(2+l)} + \varepsilon|\varphi_1|_{R_T}^{(1+l)} + \kappa|\varphi_2|_{R_T}^{(1+l)} + |\varphi_3|_{R_T}^{(1+l)},$$

where the constant C_1 does not depend on κ and ε.

Theorem 1.2. *Let $b_n > 0$, $c_n > 0$, $\kappa \in (0, \kappa_0]$, $\varepsilon \in (0, \varepsilon_0]$. Let all the conditions of the Theorem 1.1 with $l = k+\alpha$, $k = 0, 1, \dots,$ $\alpha \in (0,1)$, be fulfilled.*

Then the time derivative $\varepsilon\partial_t u(x,t)|_{x_n=0}$ in the condition (1.4) of the problem (1.1)–(1.4) satisfies estimate with $\beta \in (0, \alpha/2)$

$$|\varepsilon\partial_t u|_{C_{x'}^{1+k+\beta, \frac{1+k+\beta}{2}}{}_{t}(R_T)} \le C_2 t^{\alpha/4} M_{k+\alpha}, \tag{1.8}$$

where the constant C_2 is independent on κ and ε.

2 Auxiliary problems

We reduce the problem (1.1) – (1.4) to the more suitable form. For this we construct the auxiliary functions $U_1(x,t)$ and $U_2(x,t)$ as solutions of the first boundary value problems for the parabolic equations

$$\partial_t U_1 - \sum_{i,j=1}^{n} a_{ij}^{(1)} \partial_{x_i x_j}^2 U_1 = f_1(x,t) \text{ in } D_{1T},$$

$$U_1|_{t=0} = 0 \text{ in } D_1,\ U_1|_{x_n=0} = 0,\ t \in (0,T), \tag{2.1}$$

$$\partial_t U_2 - \sum_{i,j=1}^{n} a_{ij}^{(2)} \partial_{x_i x_j}^2 U_2 = f_2(x,t) \text{ in } D_{2T},$$

$$U_2|_{t=0} = 0 \text{ in } D_2,\ U_2|_{x_n=0} = -\varphi_0(x',t),\ t \in (0,T). \tag{2.2}$$

Lemma 2.1 [11]. *Let all the conditions of Theorem 1.1 be fulfilled.*

Then each of the problems (2.1) and (2.2) has unique solution $U_p(x,t) \in \overset{\circ}{C}{}_{x}^{2+l,1+l/2}{}_{t}(D_{pT})$, $p = 1, 2$, and the following estimates for them are fulfilled:

$$|U_1|_{D_{1T}}^{(2+l)} \le C_1 |f_1|_{D_{1T}}^{(l)}, \quad |U_2|_{D_{2T}}^{(2+l)} \le C_2 \Big(|f_2|_{D_{1T}}^{(l)} + |\varphi_0|_{R_T}^{(2+l)} \Big). \tag{2.3}$$

We point out that

$$\partial_t U_1|_{x_n=0} = 0 \tag{2.4}$$

due to the boundary condition of the problem (2.1).

Now we make the substitution

$$u_p(x,t) = U_p(x,t) + \widehat{u}_p(x,t),\ p = 1, 2, \tag{2.5}$$

in the problem (1.1)–(1.4) and obtain the problem for new unknown functions $\widehat{u}_1(x,t)$ and $\widehat{u}_2(x,t)$

$$\partial_t \widehat{u}_p - \sum_{i,j=1}^{n} a_{ij}^{(p)}\, \partial^2_{x_i x_j} \widehat{u}_p = 0 \text{ in } D_{pT},\ p=1,2, \tag{2.6}$$

$$\widehat{u}_p|_{t=0} = 0 \text{ in } D_p, p=1,2, \tag{2.7}$$

$$\big(\widehat{u}_1 - \widehat{u}_2\big)|_{x_n=0} = 0,\ t\in(0,T), \tag{2.8}$$

$$\big(\varepsilon\, \partial_t \widehat{u}_1 + \kappa b \nabla^T \widehat{u}_1 - c\nabla^T \widehat{u}_2\big)|_{x_n=0} = \Psi_{\varepsilon,\kappa}(x',t),\ t\in(0,T), \tag{2.9}$$

where

$$\Psi_{\varepsilon,\kappa}(x',t) = \varepsilon\varphi_1(x',t) + \kappa(\varphi_2(x',t) - b_n\partial_{x_n} U_1|_{x_n=0}) + \varphi_3(x',t) + c\nabla^T U_2|_{x_n=0}.$$

Here $\Psi_{\varepsilon,\kappa}(x',t)$ belongs to $\overset{\circ}{C}{}^{1+l,\frac{1+l}{2}}_{x'\ \ t}(R_T)$ and satisfies the estimate

$$|\Psi_{\varepsilon,\kappa}|^{(1+l)}_{R_T}$$

$$\le C_3\Big(\sum_{p=1}^{2} |f_p|^{(l)}_{D_{pT}} + |\varphi_0|^{(2+l)}_{R_T} + \varepsilon|\varphi_1|^{(1+l)}_{R_T} + \kappa|\varphi_2|^{(1+l)}_{R_T} + |\varphi_3|^{(1+l)}_{R_T}\Big). \tag{2.10}$$

To reduce the first equation (2.6) to the heat one we apply the coordinate mapping consisting of the orthogonal transform to reduce the original matrix $\mathcal{A}^{(1)} := \{a_{ij}^{(1)}\}_{ij=1}^n$ to the diagonal matrix $\mathcal{D}$ with positive eigenvalues of the matrix $\mathcal{A}^{(1)}$ and then we make use the contraction mapping defined by the inverse matrix D^{-1} multiplied by $a>0$ to obtain an operator $a\Delta$. After this we again apply the orthogonal transformation of the coordinates to rotate around the coordinate origin the plane R separating two domains D_1 and D_2 to express R by the equation $y_n = 0$ and to have $D_1 = \mathbb{R}^n_-$ and $D_2 = \mathbb{R}^n_+$. After the second orthogonal transform heat equation is not changed.

We denote this nondegenerate coordinate transformation by the formula

$$x = Ay. \tag{2.11}$$

After this mapping the problem (2.6)–(2.9) is reduced to the problem in the new coordinates $\{y\}$ with the unknown functions

$$v_p(y,t) = \widehat{u}_p(x,t)|_{x=Ay},\ p=1,2. \tag{2.12}$$

Remark 2.1. *For the sake of convenience we preserve the notation of the original coordinates $\{x\}$ instead of $\{y\}$ and write problem for the functions $v_p(x,t),\ p=1,2$ (instead of $v_p(y,t),\ p=1,2$). When we return to the origin coordinates $\{x\}$ by the formula $y=A^{-1}x$ and* *to the problem* (1.1)–(1.4)*, we shall remember that the functions v_p depend on the coordinates $\{y\}$.*

Thus, for the functions $v_1(x,t), v_2(x,t)$ we shall have the problem

$$\partial_t v_1 - a\,\Delta v_1 = 0 \text{ in } D_{1T}, \tag{2.13}$$

$$\partial_t v_2 - \sum_{i,j=1}^{n} a_{i\,j}\,\partial^2_{x_i x_j} v_2 = 0 \text{ in } D_{2T}, \tag{2.14}$$

$$v_p|_{t=0} = 0 \text{ in } D_p,\ p = 1,2, \tag{2.15}$$

$$\big(v_1 - v_2\big)|_{x_n=0} = 0,\ t \in (0,T), \tag{2.16}$$

$$\big(\varepsilon\,\partial_t v_1 + \kappa d\nabla^T v_1 - h\nabla^T v_2\big)|_{x_n=0} = \psi(x',t),\ t \in (0,T), \tag{2.17}$$

where all coefficients are constant, $a > 0$, $d = (d', d_n)$, $d' = (d_1, \dots, d_{n-1})$, $h = (h', h_n)$, $h' = (h_1, \dots, h_{n-1})$,

$$\psi(y',t) = \Psi_{\varepsilon,\kappa}(x',t)|_{x=Ay,\ y_n=0} \tag{2.18}$$

(in the problem (2.13)–(2.17) we have written $\psi(x',t)$ – see Remark 1.1. and omitted indexes ε, κ for the simplicity).

After the coordinate transformations (2.11) the equation (2.14) remains parabolic. Really, orthogonal mappings do not change the eigenvalues of the original matrix $\{a^{(2)}_{ij}\}^n_{i,j=1}$. Contraction mapping transforms the diagonal matrix D with positive eigenvalues to the identity matrix multiplied by $a > 0$ to obtain the operator $a\Delta$ in the equation (2.13). The eigenvalues of the matrix $\{a_{ij}\}^n_{i,j=1}$ in the equation (2.14) remain positive and this equation is parabolic.

Let ν_0 be a unit normal to the plain $R : x_n = 0$ in the problem (1.1)–(1.4) directed into D_2 (i.e. ν_0 is on the direction of the coordinate axis x_n). The scalar products $b\,\nu_0^T = b_n > 0$, $c\,\nu_0^T = c_n > 0$ do not change the signs after orthogonal coordinate mappings, after contraction the scalar products $b\,\nu_0^T = b_n > 0$, $c\,\nu_0^T = c_n > 0$ remain positive and take the forms $d\,\nu_0^T = d_n > 0$, $h\,\nu_0^T = h_n > 0$, here ν_0 is a unit normal to the plane $x_n = 0$ in the problem (2.13)–(2.17).

So, we have the problem (2.13)–(2.17) for the parabolic equations. The conditions

$$d_n > 0,\ h_n > 0$$

guarantee the solvability of this problem.

Consider the problem (2.13)–(2.17) with the unknown functions $v_1(x,t)$ and $v_2(x,t)$ for the parabolic equations.

Theorem 2.1. *Let $d_n > 0,\ h_n > 0,\ \kappa \in (0, \kappa_0],\ \varepsilon \in (0, \varepsilon_0]$, l be a positive non-integer.*

For every function $\psi(x', t) \in \overset{\circ}{C}{}^{1+l, \frac{1+l}{2}}_{x'\ \ t}(R_T)$ the problem has a unique solution $v_p(x, t) \in \overset{\circ}{C}{}^{2+l, 1+l/2}_{x\ \ t}(\overline{D}_{pT})$, $p = 1, 2$, $\varepsilon\, \partial_t v_1|_{x_n=0} \in \overset{\circ}{C}{}^{1+l, \frac{1+l}{2}}_{x'\ \ t}(R_T)$ and it satisfies estimate

$$\sum_{p=1}^{2} |v_p|^{(2+l)}_{D_{pT}} + |\varepsilon \partial_t v_1|^{(1+l)}_{R_T} \le C_4 |\psi|^{(1+l)}_{R_T}, \tag{2.19}$$

where the constant C_4 does not depend on κ and ε.

Theorem 2.2. *Let $d_n > 0,\ h_n > 0,\ \kappa \in (0, \kappa_0],\ \varepsilon \in (0, \varepsilon_0],\ l = k + \alpha,\ k = 0, 1, \dots,$ $\alpha \in (0, 1)$.*

For every function $\psi(x', t) \in \overset{\circ}{C}{}^{1+k+\alpha, \frac{1+k+\alpha}{2}}_{x'\ \ t}(R_T)$ the time derivative $\varepsilon \partial_t v_1(x, t)|_{x_n=0}$ in the condition (2.17) of the problem (2.13)–(2.17) satisfies estimate

$$|\varepsilon \partial_t v_1|_{C^{1+k+\beta, \frac{1+k+\beta}{2}}_{x'\ \ t}(R_T)} \le C_5 \varepsilon^{\alpha/4} |\psi|_{C^{1+k+\alpha, \frac{1+k+\alpha}{2}}_{x'\ \ t}(R_T)},\ \beta \in (0, \alpha/2), \tag{2.20}$$

where the constant C_5 is independent on κ and ε.

In [10] under the conditions $d_n > 0,\ h_n > 0,\ 0 < \varepsilon \le \varepsilon_0,\ 0 < \kappa \le \kappa_0,\ \psi(x', t) \in \overset{\circ}{C}{}^{1+\alpha, \frac{1+\alpha}{2}}_{x'\ \ t}(R_T)$, $\alpha \in (0, 1)$, there was find the solution of the problem (2.13)–(2.17) in the explicit form

$$v_p(x, t) = \frac{1}{\varepsilon} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \psi(y', \tau) G_p(x' - y', x_n, t - \tau)\, dy', \tag{2.21}$$

where

$$G_p(x, t) = \int_0^t K_p(x, \sigma, t - \sigma) d\sigma,$$

$$K_1(x, \sigma, t) = \partial_{x_n} g_1(x, \sigma, t), \quad K_2(x, \sigma, t) = \sum_{k=1}^{n} a_{kn} \partial_{x_k} g_2(x, \sigma, t), \tag{2.22}$$

$$g_1(x, \sigma, t) = -4a \int_0^t d\tau_1 \int_{\mathbb{R}^{n-1}} \Gamma_1(x - \eta - \frac{\kappa\sigma}{\varepsilon} d, t - \tau_1)$$

$$\times \frac{1}{\sqrt{|A_n|}} \sum_{k=1}^{n} a_{kn} \partial_{\eta_k} \Gamma_2(\eta + \frac{\sigma}{\varepsilon} h, \tau_1)\big|_{\eta_n=0}\, d\eta',$$

$$K_1(x, \sigma, t - \sigma) = -4a \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{x_n} \Gamma_1(x - \eta - \frac{\kappa\, \sigma}{\varepsilon} d, t - \sigma - \tau_1)$$

$$\times \frac{1}{\sqrt{|A_n|}} \sum_{k=1}^{n} a_{kn} \partial_{\eta_k} \Gamma_2(\eta + \frac{\sigma}{\varepsilon} h, \tau_1)\big|_{\eta_n=0} \, d\eta'$$

$$\equiv \int_0^{t-\sigma} d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{-x_n + \eta_n + \kappa d_n \sigma/\varepsilon}{(2\sqrt{a\pi(t-\sigma-\tau_1)})^n (t-\sigma-\tau_1)} e^{-\frac{(x-\eta-\kappa d\,\sigma/\varepsilon)^2}{4a(t-\sigma-\tau_1)}}$$

$$\times \frac{h_n \sigma/\varepsilon}{(2\sqrt{\pi\tau_1})^n \tau_1} \frac{1}{\sqrt{|A_n|}} e^{-\frac{\sum_{i,j=1}^n a^{ij}(\eta_i + h_i \sigma/\varepsilon)(\eta_j + h_j \sigma/\varepsilon)}{4\tau_1}} \big|_{\eta_n=0} \, d\,\eta', \;\; x_n < 0, \tag{2.23}$$

$$g_2(x, \sigma, t-\sigma) = -4a \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{\eta_n} \Gamma_1(\eta - \frac{\kappa\sigma}{\varepsilon} d, \tau_1)$$

$$\times \frac{1}{\sqrt{|A_n|}} \Gamma_2(x - \eta + \frac{\sigma}{\varepsilon} h, t - \tau_1)\big|_{\eta_n=0} \, d\eta',$$

$$K_2(x, \sigma, t) = -4a \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{\eta_n} \Gamma_1(\eta - \frac{\kappa\sigma}{\varepsilon} d, \tau_1)$$

$$\times \frac{1}{\sqrt{|A_n|}} \sum_{k=1}^{n} a_{kn} \partial_{x_k} \Gamma_2(x - \eta + \frac{\sigma}{\varepsilon} h, t - \sigma - \tau_1)\big|_{\eta_n=0} \, d\eta'$$

$$\equiv \int_0^{t-\sigma} d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{\kappa d_n \sigma/\varepsilon}{(2\sqrt{a\pi\tau_1})^n \tau_1} e^{-\frac{(\eta - \kappa d \sigma/\varepsilon)^2}{4a\tau_1}} \frac{x_n - \eta_n + h_n \sigma/\varepsilon}{(2\sqrt{\pi(t-\sigma-\tau_1)})^n (t-\sigma-\tau_1)}$$

$$\times \frac{1}{\sqrt{|A_n|}} e^{-\frac{\sum_{i,j=1}^n a^{ij}(x_i - \eta_i + h_i \sigma/\varepsilon)(x_j - \eta_j + h_j \sigma/\varepsilon)}{4(t-\sigma-\tau_1)}} \big|_{\eta_n=0} \, d\,\eta', \;\; x_n > 0, \tag{2.24}$$

$$\Gamma_1(x,t) = \frac{1}{(2\sqrt{a\pi t})^n} e^{-\frac{x^2}{4at}}, \quad \Gamma_2(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{\sum_{i,j=1}^n a^{ij} x_i x_j}{4t}}, \tag{2.25}$$

$|A_n| > 0$ is a determinant of the matrix $A_n = \{a_{ij}\}_{i,j=1}^n$, a^{ij}, $i, j = 1, \dots, n$, are the elements of the inverse matrix A_n^{-1}.

The solution (2.21) of the problem (2.13)–(2.17) was constructed with the help of Laplace on t and Fourier on x' integral transforms as in [3] and applying of the property of co - normal derivative. Moreover, in the integrals (2.23), (2.24) we made use of the formulas

$$\sum_{k=1}^{n} a_{kn} \partial_{x_k} e^{-\frac{\sum_{i,j=1}^n a^{ij} x_i x_j}{4t}} = -\sum_{k=1}^{n} a_{kn} \frac{1}{4t} \Big(\sum_{j=1}^{n} a^{kj} x_j + \sum_{i=1}^{n} a^{ki} x_i \Big) e^{-\frac{\sum_{i,j=1}^n a^{ij} x_i x_j}{4t}}$$

$$= -\frac{x_n}{2t} e^{-\frac{\sum_{i,j=1}^n a^{ij} x_i x_j}{4t}},$$

$$\sum_{k=1}^{n} a_{kn} a^{kj} = \begin{cases} 1, & j = n, \\ 0, & j \neq n. \end{cases}$$

In [10] under the conditions $d_n > 0,\ h_n > 0$, $\kappa \in (0, \kappa_0]$, $\varepsilon \in (0, \varepsilon_0]$ there were obtained the following estimates of the kernels $K_1(x, \sigma, t),\ K_2(x, \sigma, t)$ defined by the formulas (2.22)–(2.24):

$$|\partial_t^k \partial_x^m \, K_p(x, \sigma, t)| \leq C_6 \frac{1}{t^{\frac{n+2k+|m|+1}{2}}} \, e^{-\frac{q_1^2 x^2}{t} - \frac{q_2^2 \sigma^2}{\varepsilon^2 t}}, \ p = 1, 2, \tag{2.26}$$

where

$$q_1^2 = \frac{c_0^2 h_n^2}{2(h^2 + \kappa_0^2 d^2 + 2\kappa_0 |d' h'|)}, \quad q_2^2 = \frac{c_0^2 h_n^2}{2},$$

the constants $C_6,\ q_1,\ q_2$ do not depend on ε and κ, the constant c_0^2 is determined in the estimates (2.27) of the fundamental solutions (2.25) of the equations (2.13), (2.14):

$$|\partial_t^k \partial_x^m \, \Gamma_p(x, t)| \leq C_7 \frac{1}{t^{\frac{n+2k+|m|}{2}}} \, e^{-c_0^2 \frac{x^2}{t}}, \ p = 1, 2, \ c_0 = \text{const} > 0. \tag{2.27}$$

Remark 2.2. *We can see that the estimates of the Green functions of the problem (2.13)–(2.17) do not depend on the small parameter κ due to the derivative $h_n \partial_{x_n} v_2,\ h_n > 0$, in the boundary condition (2.17), that is the solution (2.21) does not possess the singularity on κ, but it is singular with respect to ε (see formula (2.21)).*

In [6] there was considered the problem, which can be reduced to the following one after excluding of the third unknown function[1)]

$$\partial_t u_p - a_p \, \Delta u_p = 0 \text{ in } D_{pT}, \ p = 1, 2, \tag{2.28}$$

$$u_p|_{t=0} = 0 \text{ in } D_p, \ p = 1, 2, \tag{2.29}$$

$$\big(u_1 - u_2\big)|_{x_n=0} = 0 \text{ on } R_T, \tag{2.30}$$

$$\big(\varepsilon \, \partial_t u_1 + d \nabla^T u_1 - h \nabla^T u_2\big)|_{x_n=0} = \Phi(x', t) \text{ on } R_T, \tag{2.31}$$

where $d_n > 0,\ h_n > 0,\ \varepsilon$ is a small parameter.

Under the conditions $d_n > 0,\ h_n > 0,\ \varepsilon \in (0, \varepsilon_0],\ \Phi(x', t) \in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha, \frac{1+\alpha}{2}}(R_T),\ \alpha \in (0, 1).$ the solution of the problem (2.28)–(2.31) in the explicit form was found

$$u_p(x, t) = \frac{1}{\varepsilon} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \Phi(y', \tau) \widetilde{G}_p(x' - y', x_n, t - \tau) \, dy' \tag{2.32}$$

$$=: \big(\frac{1}{\varepsilon} \Phi(y', \tau) * \widetilde{G}_2\big),$$

1): We have changed some notations of an article [6] for a simplicity.

where

$$\widetilde{G}_p(x,t)=\int_0^t \widetilde{K}_p(x,\sigma,t-\sigma)d\sigma,\ p=1,2,$$

the kernels $\widetilde{K}_p(x,\sigma,t),\ \ p=1,2$, satisfy the estimates

$$|\partial_t^k\partial_x^m\,\widetilde{K}_p(x,\sigma,t)|\leq C_8\frac{1}{t^{\frac{n+2k+|m|+1}{2}}}\,e^{-\frac{q_3^2x^2}{t}-\frac{q_4^2\sigma^2}{\varepsilon^2 t}},\tag{2.33}$$

the constants $q_3,\ q_4,\ C_8$ do not depend on ε.

In [6] it was proved that the solution of the problem (2.28)–(2.31) $u_p(x,t)$ belongs to the space $\overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\alpha/2}(\overline{D}_{pT})$, $p=1,2$, $\varepsilon\,\partial_t u_1(x,t)\in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T)$, and the estimate for the solution takes place

$$\sum_{p=1}^{2}|u_p|_{D_{pT}}^{(2+\alpha)}+|\varepsilon\partial_t u_1|_{R_T}^{(1+l)}\leq C_9|\Phi|_{R_T}^{(1+\alpha)},\tag{2.34}$$

where a constant C_9 does not depend on ε.

This inequality was obtained by direct evaluation of the solution (2.32) with the help of the estimates (2.33) of the Green functions.

To obtain Theorem 2.1 we prove the following theorem.

Theorem 2.3. *Let* $d_n>0,\ h_n>0,\ 0<\varepsilon\leq\varepsilon_0,\ \ k=0,1,\ldots,\ \alpha\in(0,1)$.

For every function $\Phi(x',t)\in \overset{\circ}{C}{}_{x'\ \ t}^{1+k+\alpha,\frac{1+k+\alpha}{2}}(R_T)$ *the problem (2.28) — (2.31) has a unique solution* $u_p(x,t)\in \overset{\circ}{C}{}_{x\ \ t}^{2+k+\alpha,1+\frac{k+\alpha}{2}}(\overline{D}_{pT})$, $p=1,2$, $\varepsilon\,\partial_t u_1(x,t)\in \overset{\circ}{C}{}_{x'\ \ t}^{1+k+\alpha,\frac{1+k+\alpha}{2}}(R_T)$, *and it satisfies the estimate*

$$\sum_{p=1}^{2}|u_p|_{D_{pT}}^{(2+k+\alpha)}+|\varepsilon\partial_t u_1|_{R_T}^{(1+k+\alpha)}\leq C_{10}|\Phi|_{R_T}^{(1+k+\alpha)},\tag{2.35}$$

where the constant C_{10} *does not depend on* ε.

Proof. We consider the trace of the function $u_2(x,t)$ on the plane $x_n=0:$

$$z_2(x',t):=u_2(x,t)|_{x_n=0}.\tag{2.36}$$

We shall prove that $z_2(x',t)\in \overset{\circ}{C}{}_{x\ \ t}^{2+k+\alpha,1+\frac{k+\alpha}{2}}(R_T)$. For this we estimate the norm

$$\begin{aligned}||z_2||_{\overset{\circ}{C}{}_{x\ \ t}^{2+k+\alpha,1+\frac{k+\alpha}{2}}(R_T)}&:=\sup_{(x',t)\in R_T}t^{-\frac{2+k+\alpha}{2}}|z_2|\\&+\sum_{2m_0+|m'|=2+k}\Big([\partial_t^{m_0}\partial_{x'}^{m'}z_2]_{x',R_T}^{(\alpha)}+[\partial_t^{m_0}\partial_{x'}^{m'}z_2]_{t,R_T}^{(\alpha/2)}\Big)\\&+\sum_{2m_0+|m'|=1+k}[\partial_t^{m_0}\partial_{x'}^{m'}z_2]_{t,R_T}^{(\frac{1+\alpha}{2})}.\end{aligned}\tag{2.37}$$

This norm is equivalent to the norm (1.6) [12].

First, we evaluate a modulo $|z_2| = |u_2(x', 0, t)|$ of the trace of the function $u_2(x,t)$ defined by (2.32). We make the substitution $\tau = \tau_1 - \sigma$ in the solution (2.32)

$$z_2(x',t) = \frac{1}{\varepsilon}\int_0^t d\sigma \int_{\mathbb{R}^{n-1}} dy' \int_0^{t-\tau} \Phi(y',\tau)\widetilde{K}_2(x'-y',0,\sigma,t-\tau-\sigma)\,d\tau$$

$$= \frac{1}{\varepsilon}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} dy' \int_0^{\tau} \Phi(y',\tau-\sigma)\widetilde{K}_2(x'-y',0,\sigma,t-\tau)\,d\sigma.$$

We apply the estimates (2.33) of $\widetilde{K}_2$ and the following one:

$$|\Phi(y',\tau)| \le |\Phi|_{R_T}^{(1+k+\alpha)} t^{\frac{1+k+\alpha}{2}},$$

integrate with respect to y'

$$|z_2(x',t)| \le C_{11}\frac{1}{\varepsilon}|\Phi|_{R_T}^{(1+k+\alpha)} \int_0^t d\tau \int_0^{\tau} \frac{(\tau-\sigma)^{\frac{1+k+\alpha}{2}}}{t-\tau} e^{-\frac{q_4^2\sigma^2}{\varepsilon^2(t-\tau)}}\,d\sigma,$$

integrate with respect to σ

$$|z_2(x',t)| \le C_{12}\frac{1}{\varepsilon}\varepsilon|\Phi|_{R_T}^{(1+k+\alpha)} t^{\frac{1+k+\alpha}{2}} \int_0^t \frac{d\tau}{\sqrt{t-\tau}},$$

then we shall have

$$|z_2(x',t)| \le C_{13}|\Phi|_{R_T}^{(1+k+\alpha)} t^{\frac{2+k+\alpha}{2}}. \tag{2.38}$$

We consider the derivatives $\partial_t^{m_0}\partial_{x'}^{m'} z_2(x',t)$.

Let $2m_0 + |m'| = 2 + k$. We have

$$\partial_t^{m_0}\partial_{x'}^{m'} z_2(x',t) \tag{2.39}$$

$$= \begin{cases} 1/\varepsilon\big(\partial_\tau^{m_0-1}\Phi(y',\tau) * \partial_t\widetilde{G}_2\big), & 2m_0 = 2+k,\ |m'| = 0, \\ 1/\varepsilon\big(\partial_\tau^{m_0}\partial_{y'}^{s'}\Phi * \partial_{x'}\widetilde{G}_2\big), & 2m_0 + |m'| = 2+k,\ |m'| \ge 1, |s'| = |m'|-1, \end{cases}$$

$s' = (s_1, \ldots, s_{n-1})$.

Assume $2m_0 + |m'| = 1 + k$, then

$$\partial_t^{m_0}\partial_{x'}^{m'} z_2(x',t) = \frac{1}{\varepsilon}\big(\partial_\tau^{m_0}\partial_{y'}^{m'}\Phi(y',\tau) * \widetilde{G}_2\big). \tag{2.40}$$

We should estimate the Hölder constants in according to the formulas (2.37) and (2.39), (2.40)

$$i_1 = \frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0-1}\Phi(y',\tau) * \partial_t\widetilde{G}_2\big)\big]_{x',R_T}^{(\alpha)},\ 2m_0 = 2+k,\ \partial_\tau^{m_0-1}\Phi \in \overset{\circ}{C}{}_{x'\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(R_T),$$

$$i_2=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0-1}\Phi(y',\tau)*\partial_t\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T},\ 2m_0=2+k,\ \partial_\tau^{m_0-1}\Phi\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad\ \ t}(R_T),$$

$$i_3=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{s'}\Phi*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha)}_{x',\,R_T},\ 2m_0+|s'|=1+k,\ \partial_\tau^{m_0}\partial_{y'}^{s'}\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\quad t}(R_T),$$

$$i_4=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{s'}\Phi*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T},\ 2m_0+|s'|=1+k,\ \partial_\tau^{m_0}\partial_{y'}^{s'}\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\quad t}(R_T),$$

$$i_5=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{m'}\Phi*\widetilde{G}_2\big)\big]^{(\frac{1+\alpha}{2})}_{t,\,R_T},\ 2m_0+|m'|=1+k,\ \partial_\tau^{m_0}\partial_{y'}^{m'}\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\quad t}(R_T).$$

In [6] there were obtained the following estimates of the Hölder constants in the case $\Phi\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad\ \ t}(R_T)$:

$$\big[\big(\Phi(y',\tau)*\partial_t\widetilde{G}_2\big)\big]^{(\alpha)}_{x',\,R_T}\le C_{14}|\Phi|^{(1+\alpha)}_{R_T},\tag{2.41}$$

$$\big[\big(\Phi(y',\tau)*\partial_t\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T}\le C_{15}|\Phi|^{(1+\alpha)}_{R_T},\tag{2.42}$$

$$\big[\big(\partial_{y'}\Phi(y',\tau)*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha)}_{x',\,R_T}\le C_{16}|\partial_{x'}\Phi|^{(\alpha)}_{R_T},\tag{2.43}$$

$$\big[\big(\partial_{y'}\Phi(y',\tau)*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T}\le C_{17}|\partial_{x'}\Phi|^{(\alpha)}_{R_T},\tag{2.44}$$

$$\big[\big(\partial_{y'}\Phi(y',\tau)*\widetilde{G}_2\big)\big]^{(\frac{1+\alpha}{2})}_{t,\,R_T}\le C_{18}|\partial_{x'}\Phi|^{(\alpha)}_{R_T}.\tag{2.45}$$

Comparing the estimates of the potentials with $\Phi\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad\ \ t}(R_T)$ (2.41)–(2.45) and i_1–i_5 with $\Phi\in\overset{\circ}{C}{}^{1+k+\alpha,\frac{1+k+\alpha}{2}}_{x'\qquad\ \ t}(R_T)$ we can see that the Hölder constants in i_1–i_5 and in (2.41)–(2.45) are one and the same, respectively, that is the Hölder constants i_1–i_5 can be estimated as in (2.41)–(2.45)

$$i_1+i_2=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0-1}\Phi(y',\tau)*\partial_t\widetilde{G}_2\big)\big]^{(\alpha)}_{x',\,R_T}+\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0-1}\Phi(y',\tau)*\partial_t\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T}$$

$$\le C_{19}|\partial_t^{m_0-1}\Phi|^{(1+\alpha)},\ 2m_0=2+k,\ \partial_\tau^{m_0-1}\Phi\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\quad\ \ t}(R_T),$$

$$i_3+i_4=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{s'}\Phi*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha)}_{x',\,R_T}+\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{s'}\Phi*\partial_{x'}\widetilde{G}_2\big)\big]^{(\alpha/2)}_{t,\,R_T},$$

$$\le C_{20}|\partial_t^{m_0}\partial_{y'}^{s'}\Phi|^{(\alpha)}_{R_T},\ 2m_0+|s'|=1+k,\ \partial_t^{m_0}\partial_{x'}^{s'}\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\quad t}(R_T),$$

$$i_5=\frac{1}{\varepsilon}\big[\big(\partial_\tau^{m_0}\partial_{y'}^{m'}\Phi*\widetilde{G}_2\big)\big]^{(\frac{1+\alpha}{2})}_{t,\,R_T}$$

$$\le C_{21}|\partial_t^{m_0}\partial_{x'}^{m'}\Phi|^{(\alpha)}_{R_T},\ 2m_0+|m'|=1+k,\ \partial_\tau^{m_0}\partial_{y'}^{m'}\Phi\in\overset{\circ}{C}{}^{\alpha,\alpha/2}_{x'\quad t}(R_T).$$

Applying obtained estimates of i_1–i_5 and (2.38) of the modulo $|z_2|$ in the norm (2.37) we shall have estimate for function $z_2(x',t) = u_2(x',0,t)$

$$|z_2|_{\overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(R_T)} \le C_{22}||z_2||_{\overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(R_T)} \le C_{23}|\Phi|_{R_T}^{(1+k+\alpha)}, \tag{2.46}$$

where the constant C_{23} does not depend on ε.

We point out that we can apply the estimates (2.41)–(2.45), because we consider the trace (2.36): $z_2(x',t) = u_2(x',0,t)$ on the plane $x_n = 0$. If we evaluate the solution $v_2(x,t)$, $x_n > 0$, then we should estimate the potentials containing the derivatives $\partial_t^{m_0}\partial_{x'}^{m'}\partial_{x_n}u_2(x,t)$, $2m_0 + |m'| + 1 = 2 + k, 1 + k$ additionally.

Further, the function $u_2(x,t)$ satisfies the heat equation (2.28), $p = 2$, on the plane $x_n = 0$ the trace $u_2|_{x_n=0} = z_2(x',t)$ belongs to the space $\overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(R_T)$ and is subjected to the estimate (2.46). We can consider the function $u_2(x,t)$ as a solution of the first boundary – value problem for the parabolic equation (2.28) with boundary function $z_2(x',t)$, then by [11] the function $u_2(x,t)$ belongs to the space $\overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(\overline{D}_{2T})$ and estimate is fulfilled

$$|u_2|_{D_{2T}}^{(2+k+\alpha)} \le C_{24}|z_2|_{R_T}^{(2+k+\alpha)} \le C_{25}|\Phi|_{R_T}^{(1+k+\alpha)}, \tag{2.47}$$

where the constant C_{25} does not depend on ε.

From the conjugation condition (2.30) we obtain $u_1|_{x_n=0} \in \overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(R_T)$ and an estimate takes place

$$|u_1|_{x_n=0}|_{R_T}^{(2+k+\alpha)} \le C_{26}|\Phi|_{R_T}^{(1+k+\alpha)}$$

with the constant independent on ε.

Then as above we shall have that $u_1(x,t) \in \overset{\circ}{C}{}_x^{2+k+\alpha,1+\frac{k+\alpha}{2}}_{\;t}(\overline{D}_{1T})$ and satisfies estimate

$$|u_1|_{D_{1T}}^{(2+k+\alpha)} \le C_{27}|\Phi|_{R_T}^{(1+k+\alpha)}. \tag{2.48}$$

From the conjugation condition (2.31) it follows that $\varepsilon\,\partial_t u_1(x,t) \in \overset{\circ}{C}{}_{x'}^{1+k+\alpha,\frac{1+k+\alpha}{2}}_{\;t}(R_T)$, and

$$|\varepsilon\partial_t u_1|_{R_T}^{(1+k+\alpha)} \le C_{28}|\Phi|_{R_T}^{(1+k+\alpha)}, \tag{2.49}$$

where the constant C_{28} does not depend on ε.

Gathering the estimates (2.47)–(2.49) we obtain the estimate (2.35) and Theorem 2.3. □

Proof of Theorem 2.1. We consider the problem (2.13)–(2.17). Compare its solution v_1, v_2 determined by the formula (2.21) with the solution u_1, u_2 of the problem (2.28)–(2.31) determined by (2.32). We can see that they have one and the same forms (2.21) and (2.32), and one and the same estimates (2.26) and (2.33) of the Green functions K_p and

$\widetilde{K}_p,\ p=1,2$, respectively. Moreover, the estimate (2.26) of K_p depends on ε as an estimate of $\widetilde{K}_p$ and does not depend on κ.

We should point out that the estimates of the solution of the problem are obtained only with the help of the estimate of the Green function.

Thus, we conclude that the solution $v_1,\ v_2$ of the problem (2.13)–(2.17) possesses the same properties as the solution $u_1,\ u_2$ of the problem (2.28)–(2.31), i.e. $v_p(x,t)\in \overset{\circ}{C}{}_{x\ \ t}^{2+k+\alpha,1+\frac{k+\alpha}{2}}(\overline{D}_{pT})$, $p=1,2$, and

$$\sum_{p=1}^{2}|v_p|_{D_{pT}}^{(2+k+\alpha)}\le C_{29}|\psi|_{R_T}^{(1+k+\alpha)}. \tag{2.50}$$

From the conjugation condition (2.17) we obtain that the time derivative $\varepsilon\,\partial_t v_1(x,t)|x_n=0$ belongs to the space $\overset{\circ}{C}{}_{x'\ \ t}^{1+k+\alpha,\frac{1+k+\alpha}{2}}(R_T)$, and it satisfies the estimate

$$|\varepsilon\partial_t v_1|_{R_T}^{(1+k+\alpha)}\le C_{30}|\psi|_{R_T}^{(1+k+\alpha)}. \tag{2.51}$$

The estimates (2.50) and (2.51) lead to the inequality (2.19) and Theorem 2.1. □

Proof of Theorem 2.2. Consider the time derivative $\varepsilon\partial_t v_1$ in the boundary condition (2.17) of the problem (2.13)–(2.17)

$$\varepsilon\partial_t u_1|_{x_n=0}=\psi(x',t)-\big(\kappa d\nabla^T v_1-h\nabla^T v_2\big)|_{x_n=0}. \tag{2.52}$$

We must obtain the estimate (2.20)

$$|\varepsilon\partial_t v_1|_{R_T}^{(1+k+\beta)}\le C_{31}\varepsilon^{\alpha/4}|\psi|_{R_T}^{1+k+\alpha},\ \beta\in(0,\alpha/2),$$

where constant C_{31} is independent on κ and ε.

We can see that the function $\psi(x',t)$ in (2.52) does not depend on ε fully and can not provide this estimate. So we should get rid of $\psi(x',t)$ in the identity (2.52) with the help of the formula of a jump of the heat potential of double layer, which we extract from the directional derivatives in (2.52).

In [13] it was proved that the time derivative $\varepsilon\partial_t v_1(x,t)|_{x_n=0}$ may be represented in the form

$$\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1|_{x_n=0}=-W_1^{(s)}(x',t)+W_2^{(s)}(x',t)-W_3^{(s)}(x',t), \tag{2.53}$$

$$\begin{aligned}W_1^{(s)}(x',t)=\frac{\kappa}{\varepsilon}\int_0^t d\tau\int_{\mathbb{R}^{n-1}}dy'\int_0^\tau\big(\psi_s(y',\tau-\sigma)-\psi_s(y',\tau)\big)\\ \times d\nabla_x^T K_1(x-y',\sigma,t-\tau)\,d\sigma\big|_{x_n=0},\end{aligned} \tag{2.54}$$

$$\begin{aligned}W_2^{(s)}(x',t)=\frac{1}{\varepsilon}\int_0^t d\tau\int_{\mathbb{R}^{n-1}}dy'\int_0^\tau\big(\psi_s(y',\tau-\sigma)-\psi_s(y',\tau)\big)\\ \times h\nabla_x^T K_2(x-y',\sigma,t-\tau)\,d\sigma\big|_{x_n=0},\end{aligned} \tag{2.55}$$

$$W_3^{(s)}(x',t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \psi_s(y',\tau) K_1(x-y',\tau,t-\tau)\, dy'\big|_{x_n=0}$$
$$\equiv \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \psi_s(y',t-\tau) K_1(x-y',t-\tau,\tau)\, dy'\big|_{x_n=0}, \tag{2.56}$$

where $\psi_s(x',t) = \partial_t^{m_0}\partial_{x'}^{m'}\psi(x',t)$, $W_i^{(s)}(x',t) = \partial_t^{m_0}\partial_{x'}^{m'} W_i(x',t)$, $s = 2m_0 + |m'|$; $s = 0,1,2, \ldots, k, 1+k$; $k = 0,\, 1,\, \ldots$; $m' = (m_1, \ldots, m_{n-1})$.

By direct evaluation of the functions (2.54) – (2.56) we obtain the estimates

$$|\varepsilon\partial_t v_1| \le C_{32}\, t^{\frac{1+k+\beta}{2}} \varepsilon^{\frac{\alpha-\beta}{2}} (1+\kappa_0)\, [\psi]_{t,R_T}^{(\frac{1+k+\alpha}{2})}, \tag{2.57}$$

$$\big[W_i^{(k+j)}\big]_{t,R_T}^{(\frac{1+\beta-j}{2})} \le C_{33}\, \varepsilon^{\alpha/4} \big[\partial_t^{m_0}\partial_{x'}^{m'}\psi\big]_{t,R_T}^{(\frac{1+\alpha-j}{2})}, \tag{2.58}$$

$$\big[W_3^{(1+k)}\big]_{x',R_T}^{(\beta)} \le C_{34}\kappa_0 \varepsilon^{\frac{\alpha-\beta}{2}} \big[\partial_t^{m_0}\partial_{x'}^{m'}\psi\big]_{t,R_T}^{(\alpha/2)}, \tag{2.59}$$

where $\beta \in (0, \alpha/2)$.

We remember the formulas (2.53)–(2.56), apply obtained estimates (2.57)–(2.59) in the norm

$$||\varepsilon\partial_t v_1||_{C_{x'}^{1+k+\beta,\frac{1+k+\beta}{2}}{}_t(R_T)} := \sup_{(x',t)\in R_T} t^{-(1+k+\beta)/2} |\varepsilon\partial_t v_1|$$
$$+ \sum_{2m_0+|m'|=1+k} \Big([\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]_{x',R_T}^{(\beta)} + [\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]_{t,R_T}^{(\beta/2)} \Big)$$
$$+ \sum_{2m_0+|m'|=k} [\varepsilon\partial_t^{m_0}\partial_{x'}^{m'}\partial_t v_1]_{t,R_T}^{(\frac{1+\beta}{2})},$$

which is equivalent to the norm (1.6) [12], and obtain the estimate

$$|\varepsilon\partial_t v_1|_{R_T}^{(1+k+\beta)} \le C_{35} ||\varepsilon\partial_t v_1||_{R_T}^{(1+k+\beta)} \le C_{36}\varepsilon^{\alpha/4} |\psi|_{R_T}^{(1+k+\alpha)},$$

$\beta \in (0, \alpha/2)$, and Theorem 2.2. □

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We have proved Theorems 2.1 and 2.2 for the solution $v_1(y,t)$, $v_2(y,t)$ of the problem (2.13)–(2.17).

Now we return to the original problem (1.1)–(1.4) with unknown functions $u_1(x,t)$, $u_2(x,t)$. We have made the substitution (2.5):

$$u_p(x,t) = U_p(x,t) + \widehat{u}_p(x,t),\ p = 1,2,$$

where the functions $U_p(x,t)$ due to Lemma 2.1 belong to $\overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(D_{pT})$, $p=1,2$, and satisfy the estimates (2.3). Then we have applied the nondegenerate coordinate mapping (2.11): $x = Ay$, notation (2.12): $\widehat{u}_p(x,t) = v_p(y,t)|_{y=A^{-1}x}$, $p = 1,2$. From here and Theorem 2.1 we shall have that $\widehat{u}_p(x,t) = v_p(A^{-1}x,t) \in \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{D}_{pT})$, $l = k+\alpha$, $p = 1,2$, $\varepsilon\partial_t\widehat{u}_1 \in \overset{\circ}{C}{}_{x'\ t}^{1+l,\frac{1+l}{2}}(R_T)$ and

$$\sum_{p=1}^{2}|\widehat{u}_p|_{D_{pT}}^{(2+l)} + |\varepsilon\partial_t\widehat{u}_1|_{R_T}^{(1+l)} \le C_1\sum_{p=1}^{2}|v_p(A^{-1}x,t)|_{D_{pT}}^{(2+l)} + |\varepsilon\partial_t v_1(A^{-1}x,t)|_{R_T}^{(1+l)}$$

$$\le C_2|\psi(A^{-1}x|_{x_n=0},t)|_{R_T}^{(1+l)} = C_2|\Psi_{\varepsilon,\kappa}|_{R_T}^{(1+l)}, \tag{3.1}$$

where by the notation (2.18)

$$\Psi_{\varepsilon,\kappa}(x',t) = \psi(A^{-1}x|_{x_n=0},t),$$

$$\Psi_{\varepsilon,\kappa}(x',t) = \varepsilon\psi_1(x',t) + \kappa(\psi_2(x',t) - b_n\partial_{x_n}U_1|_{x_n=0}) + \psi_3(x',t) + c\nabla^T U_2|_{x_n=0},$$

the function $\Psi_{\varepsilon,\kappa}(x',t)$ belongs to $\overset{\circ}{C}{}_{x'\ t}^{1+l,\frac{1+l}{2}}(R_T)$ and satisfies the estimate (2.10):

$$|\Psi_{\varepsilon,\kappa}|_{R_T}^{(1+l)} \le C_3\Big(\sum_{p=1}^{2}|f_p|_{D_{pT}}^{(l)} + |\psi_0|_{R_T}^{(2+l)} + \varepsilon|\psi_1|_{R_T}^{(1+l)} + \kappa|\psi_2|_{R_T}^{(1+l)} + |\psi_3|_{R_T}^{(1+l)}\Big).$$

From the substitution (2.5) with the help of (3.1), (2.4) and (2.3), and condition $U_1|_{x_n=0} = 0$ we obtain that $u_p(x,t) = U_p(x,t) + v_p(A^{-1}x,t) \in \overset{\circ}{C}{}_{x\ \ t}^{2+l,1+l/2}(\overline{D}_{pT})$, $p = 1,2$,

$$\varepsilon\partial_t u_1\big|_{x_n=0} = \varepsilon\Big(\partial_t U_1 + \partial_t v_1(A^{-1}x,t)\Big)\big|_{x_n=0} = \varepsilon\partial_t v_1(A^{-1}x,t)\big|_{x_n=0}, \tag{3.2}$$

$\varepsilon\partial_t u_1(x,t)|_{x_n=0} \in \overset{\circ}{C}{}_{x'\ t}^{1+l,\frac{1+l}{2}}(R_T)$ and

$$\sum_{p=1}^{2}|u_p|_{D_{pT}}^{(2+l)} + |\varepsilon\partial_t u_1|_{R_T}^{(1+l)} \le C_4\big(\sum_{p=1}^{2}|U_p|_{D_{pT}}^{(2+l)} + |\Psi_{\varepsilon,\kappa}|_{R_T}^{(1+l)}\big),$$

where the constant C_4 does not depend on ε and κ.

Applying the estimates (2.3) for U_1, U_2 and (2.10) for $\Psi_{\varepsilon,\kappa}$ we have got the estimate (1.7) and Theorem 1.1. □

Proof of Theorem 1.2. From the formulas (3.2) and the estimate (2.20) we obtain

$$|\varepsilon\partial_t u_1|_{C_{x'\ t}^{1+k+\beta,\frac{1+k+\beta}{2}}(R_T)} = |\partial_t v_1(A^{-1}x,t)|_{C_{x'\ t}^{1+k+\beta,\frac{1+k+\beta}{2}}(R_T)}$$

$$\le C_5\varepsilon^{\alpha/4}\,|\psi(A^{-1}x,t)|_{x_n=0}|_{C_{x'}^{1+k+\alpha,\frac{1+k+\alpha}{2}}_{t}(R_T)} \le C_6\varepsilon^{\alpha/4}|\Psi_{\varepsilon,\kappa}|_{C_{x'}^{1+k+\alpha,\frac{1+k+\alpha}{2}}_{t}(R_T)}.$$

Applying the inequality (2.10) for $\Psi_{\varepsilon,\kappa}$ we shall have the estimate (1.8) and Theorem 1.2. □

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Бижанова Г.И. ТҮЙІНДЕСУ ШАРТЫНДА УАҚЫТ БОЙЫНША ТУЫНДЫСЫ БАР ПАРАБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН РЕГУЛЯРЛЫ ЕМЕС КӨПӨЛШЕМДІ ЕКІФАЗАЛЫҚ ЕСЕПТІҢ ШЕШІМІ

Шекаралық шарттағы жоғары туындыларында $\varepsilon > 0$ және $\kappa > 0$ екі кіші параметрлері бар параболалық теңдеулер үшін еркін шекаралы сызықтандырылған көпөлшемді екіфазалық есеп зерттеледі. Гёльдер кеңістігінде шешімнің және қобалжыған мүшесінің ε бойынша бағалаулары алынды.

Кілттік сөздер. Көпөлшемді екі фазалық шеттік есеп, параболалық теңдеулер, түйіндесу шартындағы кіші параметрлер, шешімнің айқын түрі, шешімнің және қобалжыған мүшенің бағалаулары, Гёльдер кеңістігі.

Бижанова Г.И. РЕШЕНИЕ НЕРЕГУЛЯРНОЙ МНОГОМЕРНОЙ ДВУХФАЗНОЙ ЗАДАЧИ ДЛЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С ПРОИЗВОДНОЙ ПО ВРЕМЕНИ В УСЛОВИИ СОПРЯЖЕНИЯ

Изучается линеаризованная многомерная двухфазная задача со свободной границей для параболических уравнений с двумя малыми параметрами $\varepsilon > 0$ и $\kappa > 0$ при старших производных в граничном условии. Получены оценки решения и возмущенного члена по ε в пространстве Гёльдера.

Ключевые слова. Многомерная двухфазная краевая задача, параболические уравнения, малые параметры в условии сопряжения, решение в явном виде, оценки решения и возмущенного члена, пространство Гёльдера.

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Iterative method for solving special Cauchy problem for the system of integro-differential equations with nonlinear integral part

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Abstract. The paper deals with the special Cauchy problem for the system of nonlinear integro-differential equations. The problem is considered as a nonlinear operator equation. An iterative process for solving considered problem is constructed and its convergence conditions are established.

Keywords. Nonlinear Fredholm integro-differential equation, special Cauchy problem, parametrization's method, operator equation, iterative method.

On [0,T], we consider the nonlinear Fredholm integro-differential equation (FIDE)

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau) f_k(\tau, x(\tau)) d\tau, \quad t \in [0,T), \quad x \in R^n, \tag{1}$$

where $(n \times n)$-matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$ are continuous on $[0,T]$; $f_k : [0,T] \times R^n \to R^n$, $k = \overline{1,m}$ are continuous, $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$.

Various problems for FIDE have been studied by many authors [1]–[9]. In [10] a linear boundary value problem for FIDE is solved by parametrization's method. One of the important auxiliary problem in this method is a special Cauchy problem for the system of linear FIDEs. Let Δ_N be a partition of the interval $[0,T]$ into N parts with points $t_0 = 0 < t_1 < \ldots < t_N = T$.

If we denote by $x_r(t)$ the restriction of the function $x(t)$ to the subinterval $[t_{r-1}, t_r)$, introduce additional parameters $\lambda_r \widehat{=} x_r(t_{r-1})$ and make substitutions $u_r(t) = x_r(t) - \lambda_r, r =$

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$\overline{1,N}$, then we obtain the system of nonlinear integro-differential equations with parameters on subintervals

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \psi_k(\tau) f_k(\tau, u_j(\tau) + \lambda_j) d\tau,$$

$$t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{2}$$

and initial conditions at the left end points of subintervals

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{3}$$

The problem (2), (3) is a special Cauchy problem for the system of nonlinear integro-differential equations with parameters. Criteria of the solvability and unique solvability of the special Cauchy problems for the system of linear integro-differential equations with parameters have been established in [10]. The methods of finding solutions to the linear special Cauchy problems are also proposed there. The special Cauchy problem plays an important role by solving boundary value problems for FIDEs [10], [11] and by construction the Δ_N general solutions to the linear FIDEs [12].

In the present paper we consider the special Cauchy problem for the system of nonlinear integro-differential equations (IDEs) on the closed subintervals

$$\frac{dv_r}{dt} = A(t)(v_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) f_k(\tau, v_j(\tau) + \lambda_j) d\tau,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1,N}, \tag{4}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{5}$$

The aim of the paper is to establish conditions for the solvability of the problem (4), (5) and to construct an iterative process for finding its solution.

Denote by $\widetilde{C}([0,T], \Delta_N, R^{nN})$ the space of the function systems $v[t] = (v_1(t), v_2(t), ..., v_N(t))$, where functions $v_r : [t_{r-1}, t_r] \to R^n$ are continuous for all $r = \overline{1,N}$, with the norm $\|v[\cdot]\|_1 = \max\limits_{r=\overline{1,N}} \max\limits_{t\in[t_{r-1},t_r]} \|v_r(t)\|$.

A solution to the special Cauchy problem (4), (5) with $\lambda = \widehat{\lambda}$ is a function system $v[t, \widehat{\lambda}] = (v_1(t, \widehat{\lambda}), v_2(t, \widehat{\lambda}), ..., v_N(t, \widehat{\lambda})) \in \widetilde{C}([0,T], \Delta_N, R^{nN})$, which satisfies the system of integro-differential equations (4) with $\lambda = \widehat{\lambda}$ and initial conditions (5).

Choose the function system $\widehat{v}^{(0)}[t] = (\widehat{v}_1^{(0)}(t), \widehat{v}_2^{(0)}(t), ..., \widehat{v}_N^{(0)}(t)) \in \widetilde{C}([0,T], \Delta_N, R^{nN})$, the number $\rho_v > 0$ and the ball $S(\widehat{v}^{(0)}[t], \rho_v) = \{v[t] \in \widetilde{C}([0,T], \Delta_N, R^{nN}) : \|v[\cdot] - v^{(0)}[\cdot]\| < \rho_v\}$. To solve the problem (4), (5), we write it as an equivalent operator equation and use results from [12], [13].

We define the piecewise continuous function $\widehat{x}_0(t)$ by the equalities

$$\widehat{x}_0(t) = \widehat{\lambda}_r + \widehat{v}_r^{(0)}(t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N},$$

and introduce the set

$$G^0(\rho) = \{(t, x) : t \in [0, T], \|x - \widehat{x}_0(t)\| < \rho\}, \quad \rho > \rho_v.$$

Condition A. *The function $f(t, x)$ has uniformly continuous partial derivative $f'_x(t, x)$ in $G^0(\rho)$.*

We introduce the following spaces $X = \{v[t] = (v_1(t), v_2(t), ..., v_N(t)) \in \widetilde{C}([0, T], \triangle_N, R^{nN}) : v_r(t_{r-1}) = 0, \ r = \overline{1, N}\}$, $Y = \widetilde{C}([0, T], \triangle_N, R^{nN})$.

We consider the special Cauchy problem (4), (5) as nonlinear operator equation

$$Hv[t] + F(v[t], \widehat{\lambda}) = 0. \tag{6}$$

Here the linear operator $H : X \to Y$ is defined as

$$Hv[t] = \omega^{(1)}[t]$$

with

$$\omega^{(1)}[t] = (\omega_1^{(1)}(t), \omega_2^{(1)}(t), ..., \omega_N^{(1)}(t)),$$

$$\omega_r^{(1)}(t) = \dot{v}_r(t) - A(t)v_r(t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N}.$$

The domain of the operator H is $D(H) = \{v[t] = (v_1(t), v_2(t), \ldots, v_r(t)) \in X$, where $v_r(t)$ is continuously differentiable in $[t_{(r-1)}, t_r]$, $r = \overline{1, N}\}$ and nonlinear operator

$$F(v[t], \widehat{\lambda}) = \omega^{(2)}[t, \widehat{\lambda}],$$

where

$$\omega^{(2)}[t, \widehat{\lambda}] = (\omega_1^{(2)}(t, \widehat{\lambda}), \omega_2^{(2)}(t, \widehat{\lambda}), ..., \omega_N^{(2)}(t, \widehat{\lambda})),$$

$$\omega_r^{(2)}(t, \widehat{\lambda}) = -A(t)\widehat{\lambda}_r - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) f_k(\tau, v_j(\tau) + \widehat{\lambda}_j) d\tau, , \ t \in [t_{r-1}, t_r], \ r = \overline{1, N}.$$

Condition A provides the existence and uniform continuity of the Frechet derivative $F'_v(v[t], \widehat{\lambda})$ in $S(\widehat{v}^{(0)}[t], \rho_v)$ [14].

The Frechet differential has the following form

$$F'_v(\widetilde{v}[t], \widehat{\lambda})h = \omega^{(3)}[t, \widehat{\lambda}],$$

$$\omega^{(3)}[t, \widehat{\lambda}] = (\omega_1^{(3)}(t, \widehat{\lambda}), \omega_2^{(3)}(t, \widehat{\lambda}), ..., \omega_N^{(3)}(t, \widehat{\lambda})),$$

$$\omega_r^{(3)}(t,\widehat{\lambda}) = -\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)f'_{k,x}(\tau,v_j(\tau)+\widehat{\lambda}_j)h_j(\tau)d\tau, \quad t\in[t_{r-1},t_r], \quad r=\overline{1,N},$$

with the function system $h[t]=(h_1(t),h_2(t),...,h_N(t))\in X$, where $h_r:[t_{r-1},t_r]\longrightarrow R^n,\ r=\overline{1,N}$, are continuous.

The closed linear operator $H+F'_v(\widetilde{v}[t],\widehat{\lambda}):X\to Y$ has a bounded inverse iff the linear operator equation

$$(H+F'_v(\widetilde{v}[t],\widehat{\lambda}))h=g[t], \quad g[t]=(g_1(t),g_2(t),...,g_N(t))\in Y \tag{7}$$

is uniquely solvable. Equation (4) is equivalent to the special Cauchy problem for the system of linear integro-differential equations with parameters

$$\frac{dh_r(t)}{dt} = A(t)h_r(t)+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)f'_{k,x}(\tau,\widetilde{v}_j(\tau)+\widehat{\lambda}_j)h_j(\tau)d\tau+g_r(t),$$

$$t\in[t_{r-1},t_r], \quad r=\overline{1,N}, \tag{8}$$

$$h_r(t_{r-1})=0, r=\overline{1,N}. \tag{9}$$

Let $L(Y,X)$ be the space of linear bounded operators $\Lambda:Y\to X$ with some induced norm.

Definition 1. *The special Cauchy problem (8), (9) is called well-posed if for any $g(t)\in Y$ it has a unique solution $h[t]\in\widetilde{C}([0,T],\triangle_N,R^{nN})$, and the inequality $\|h[\cdot]\|_4\le\gamma\|g[\cdot]\|_4$ holds, where γ is a constant, independent of $g[t]$.*

The number γ is called the well-posedness constant of the problem (8), (9).

Theorem 1. *Suppose the following conditions be fulfilled:*

1) the Frechet derivative $F'_v(v[t],\widehat{\lambda})$ is uniformly continuous in $S(\widehat{v}^{(0)}[t],\widehat{\rho}_v)$;

2) the linear operator $H+F'_v(v[t],\widehat{\lambda}):X\to Y$ has a bounded inverse and

$$\|[H+F'_v(\widetilde{v}[t],\widehat{\lambda})]^{-1}\|_{L(Y,X)}\le\widehat{\gamma},\ \widehat{\gamma} \quad is \quad const,$$

for all $\widetilde{v}[t]\in S(\widehat{v}^{(0)}[t],\widehat{\rho}_v)$;

3) $\widehat{\gamma}\cdot\|H\widehat{v}^{(0)}[\cdot]+F\widehat{v}^{(0)}[\cdot],\widehat{\lambda}\|_1<\widehat{\rho}_v$.

Then there exist numbers $\alpha_k\ge 1,\ k=0,1,2,...$, such that the sequence of elements $\{\widehat{v}^{(k)}[t]\},\ k=0,1,2...$, generated by the iterative process

$$\widehat{v}^{(k+1)}[t]=\widehat{v}^{(k)}[t]-\frac{1}{\alpha_k}[H+F'_v(\widehat{v}^{(k)}[t],\widehat{\lambda})]^{-1}[H\widehat{v}^{(k)}[t]+F(\widehat{v}^{(k)}[t],\widehat{\lambda})],\ k=0,1,2,..., \tag{10}$$

converges to $v[t,\widehat{\lambda}]$, an isolated solution converges to Eq.(7) in $S(\widehat{v}^{(0)}[t],\widehat{\rho}_v)$, and the following estimate is valid:

$$\|v[\cdot,\widehat{\lambda}]-\widehat{v}^{(0)}[\cdot]\|_1\le\gamma\|H\widehat{v}^{(0)}[t]+F(\widehat{v}^{(0)}[t],\widehat{\lambda})\|_1. \tag{11}$$

The convergence of the sequence $\{\widehat{v}^{(k)}[t]\}$, $k=0,1,2,...,$ to $v[t,\widehat{\lambda}]$, an isolated solution to Eq.(6) in $S(\widehat{v}^{(0)}[t],\rho_v)$ is provided by Theorem 2 [13] and Theorem 3 [14]. Estimate (11) is established similarly to estimate (1.3) of Theorem 1 [16].

Theorem 2. *Let Condition A be fulfilled, let the special Cauchy problem (8), (9) be well-posed with constant* $\widehat{\gamma}$ *for all* $\widetilde{v}[t]\in S(v^{(0)}[t],\widehat{\rho}_v)$*, and let the following inequality be valid:*

$$\widehat{\gamma}\max_{r=\overline{1,N}}\max_{t\in[t_{r-1},t_r]}\|\dot{\widehat{v}}_r^{(0)}(t)-A(t)(\widehat{v}_r^{(0)}(t)+\lambda_r)$$

$$-\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)f_k(\tau,\widehat{v}_j^{(0)}(\tau)+\lambda_j)d\tau\|<\widehat{\rho}_v.$$

Then there exist numbers $\alpha_k\geq 1$, $k=0,1,2,\ldots,$ *such that the sequence of elements* $\{\widehat{v}_r^{(k)}[t]\}$, $k=0,1,2,\ldots,$ *generated by the iterative process*

$$\widehat{v}^{(k+1)}[t]=\widehat{v}^{(k)}[t]+\Delta v^{(k)}[t,\widehat{\lambda}],\quad k=0,1,2,\ldots, \tag{12}$$

where

$$\Delta v^{(k)}[t,\widehat{\lambda}]=(\Delta v_1^{(k)}(t,\widehat{\lambda}_1,\ldots,\widehat{\lambda}_N),\Delta v_2^{(k)}(t,\widehat{\lambda}_1,\ldots,\widehat{\lambda}_N),\ldots,\Delta v_N^{(k)}(t,\widehat{\lambda}_1,\ldots,\widehat{\lambda}_N)),$$

is the solution to the special Cauchy problem for the system of linear integro-differential equations with parameters

$$\frac{d\Delta v_r}{dt}=A(t)\Delta v_r+\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)f'_{k,x}(\tau,v_j^{(k)}(\tau)+\widehat{\lambda}_j)\Delta v_j(\tau)d\tau-$$

$$-\frac{1}{\alpha_k}\Big(\frac{dv_r^{(k)}(t)}{dt}-A(t)[v_r^{(k)}(t)+\widehat{\lambda}_r]-\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(\tau)f_k(\tau,v_j^{(k)}(\tau)+\widehat{\lambda}_j)d\tau\Big),$$

$$t\in[t_{r-1},t_r],\ r=\overline{1,N}, \tag{13}$$

$$\Delta v_r(t_{r-1})=0,\quad r=\overline{1,N},\quad k=0,1,2,..., \tag{14}$$

converges to $v[t,\widehat{\lambda}]$*, an isolated solution converges to the problem (4), (5), and*

$$\big\|v[\cdot,\widehat{\lambda}]-\widehat{v}^{(0)}[\cdot]\big\|_1$$

$$\leq\widehat{\gamma}\max_{r=\overline{1,N}}\max_{t\in[t_{r-1},t_r]}\Big\|\dot{\widehat{v}}_r^{(0)}(t)-A(t)(\widehat{v}_r^{(0)}(t)+\lambda_r)-\sum_{k=1}^{m}\varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi(\tau)f(\tau,\widehat{v}_j^{(0)}(\tau)+\lambda_j)d\tau\Big\|. \tag{15}$$

Example. On [0, 2] consider the nonlinear IDE

$$\frac{dx}{dt} = A(t)x(t) + \varphi(t)\int_0^T \psi(\tau)f(\tau, x(\tau))d\tau + f_0(t), \quad t \in [0,2], \tag{16}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ t^3-1 & t^2 \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} t & 1 \\ t^2 & t+3 \end{pmatrix}, \quad \psi(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f(\tau, x(\tau)) = \begin{pmatrix} \tau x_1^2(\tau) + \tau^2 x_2^3(\tau) \\ (\tau+1)x_1(\tau) + (\tau^3-1)x_2(\tau) \end{pmatrix},$$

$$f_0(t) = \begin{pmatrix} \frac{2\pi-12}{\pi^2} - \cos(\pi t) - t(\frac{28}{9\pi^2}+1) + \pi\cos(\pi t) \\ \frac{2(t+3)(\pi-6)}{\pi^2} - t^2(\frac{28}{9\pi^2}+1) - t^2\cos(\pi t) - \pi\sin(\pi t) - \sin(\pi t)(t^3-1) \end{pmatrix}.$$

The exact solution to equation (16) is

$$x^*(t) = \begin{pmatrix} \sin(\pi t) \\ \cos(\pi t) \end{pmatrix}.$$

We divide the interval [0, 2] into 2 parts and denote by Δ_2 the partition: $t_0 = 0,\ t_1 = 1,\ t_2 = 2$. By $x_{(r)}(t)$ let denote the restriction of the function $x(t)$ to the r-th closed interval $[t_{r-1}, t_r]$, i.e. $x_r(t) = x(t),\ t \in [t_{r-1}, t_r],\ r = \overline{1,2}$. We introduce parameters $\lambda_1 = x_1(t_0),\quad \lambda_2 = x_2(t_1)$ and make substitutions $v_1 = x_1(t) - \lambda_1,\quad v_2 = x_2(t) - \lambda_2$, then we get the special Cauchy problem for system of nonlinear integro-differential equations on closed intervals

$$\frac{dv_r}{dt} = A(t)(v_r + \lambda_r) + \varphi(t)\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j} \psi(\tau)f(\tau, v_j(\tau) + \lambda_j)d\tau + f_0(t),$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1,2}, \tag{17}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1,2}, \tag{18}$$

In order to solve the special Cauchy problem we use iterative process (12). As an initial approximation to the solution of the special Cauchy problem, we select the function system $v^{(0)}[t] = (0, 0)$.

We solve the following linear special Cauchy problem

$$\frac{d\Delta v_r}{dt} = A(t)\Delta v_r + \varphi(t)\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j} \psi(\tau)f'_x(\tau, \widehat{\lambda}_j)\Delta v(\tau)d\tau$$

$$+A(t)\widehat{\lambda}_r + \varphi(t)\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\psi(\tau)f(\tau,\widehat{\lambda}_j)d\tau, \quad t\in[t_{r-1},t_r], \quad r=\overline{1,2}, \quad (19)$$

$$\Delta v_r(t_{r-1}) = 0, \quad r=\overline{1,2}, \qquad (20)$$

and find its solution $\Delta v^{(0)}[t] = (\Delta v_1^{(0)}(t), \Delta v_2^{(0)}(t))$. We determine the first approximation of the solution to the special Cauchy problem (19), (20) by the equalities $v_r^{(1)}(t) = \Delta v_r^{(0)}(t), r = \overline{1,2}$.

We solve the linear special Cauchy problem

$$\frac{d\Delta v_r}{dt} = A(t)\Delta v_r + \varphi(t)\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\psi(\tau)f'_x(\tau, v_j^{(k)}(\tau)+\widehat{\lambda}_j)\Delta v_j(\tau)d\tau$$

$$+\frac{dv_r^{(k)}(t)}{dt} + A(t)[v_r^{(k)}(t)+\widehat{\lambda}_r] + \varphi(t)\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\psi(\tau)f(\tau, v_j^{(k)}(\tau)+\widehat{\lambda}_j)d\tau, t\in[t_{r-1},t_r], \quad (21)$$

$$\Delta v_r(t_{r-1}) = 0, \quad r=\overline{1,2}, \quad k=\overline{1,4}. \qquad (22)$$

and determine $(k+1)$-th approximation of the solution to the special Cauchy problem (17), (18) by the equality

$$v_r^{(k+1)}(t) = v_r^{(k)}(t) + \Delta v_r^{(k)}(t), \quad k=\overline{1,4}.$$

Table 1 – The numerical solution to the special Cauchy problem (17), (18) for the 1-st iteration

$\widehat{t}$	$v_{(1)1}^{(1)}(\widehat{t})$	$v_{(1)2}^{(1)}(\widehat{t})$	$\widehat{t}$	$v_{(2)1}^{(1)}(\widehat{t})$	$v_{(2)2}^{(1)}(\widehat{t})$
0.0000	0.0000000000	0.0000000000	1.0000	0.0000000000	0.0000000000
0.0625	0.1672906062	−0.0997288225	1.0625	−0.1714987527	−0.0370329738
0.1250	0.3252284900	−0.2367077874	1.1250	−0.3358124340	−0.0351124819
0.1875	0.4666428544	−0.4084286610	1.1875	−0.4856464522	0.0037983714
0.2500	0.5849438131	−0.6111592370	1.2500	−0.6142546543	0.0764650909
0.3125	0.6743685801	−0.8400913950	1.3125	−0.7156876587	0.1785581748
0.3750	0.7301953620	−1.0895378508	1.3750	−0.7850066757	0.3048934199
0.4375	0.7489162258	−1.3531717224	1.4375	−0.8184529171	0.4497378504
0.5000	0.7283616696	−1.6243017792	1.5000	−0.8135635486	0.6071863017
0.5625	0.6677713381	−1.8961754455	1.5625	−0.7692258749	0.7716203324
0.6250	0.5678072007	−2.1623014060	1.6250	−0.6856617536	0.9382731078
0.6875	0.4305074681	−2.4167841314	1.6875	−0.5643336223	1.1039446263
0.7500	0.2591814570	−2.6546638795	1.7500	−0.4077612307	1.2679470552
0.8125	0.0582474057	−2.8722580037	1.8125	−0.2192329201	1.4334204115
0.8750	−0.1669832195	−3.0675028285	1.8750	−0.0023849462	1.6092629011
0.9375	−0.4105702558	−3.2403003823	1.9375	0.2393968338	1.8131015559
1.0000	−0.6662529657	−3.3928813898	2.0000	0.5038331919	2.0760491787

Table 1 presents the numerical solution to the problem (17), (18) with proximity 1.393, i.e.:

$$\|v^{(1)}(\widehat{t}) - v^{*}(\widehat{t})\| \leq 1.393.$$

Table 2 – The numerical solution to the special Cauchy problem (17), (18) for the 2-d iteration

$\widehat{t}$	$v^{(2)}_{(1)1}(\widehat{t})$	$v^{(2)}_{(1)2}(\widehat{t})$	$\widehat{t}$	$v^{(2)}_{(2)1}(\widehat{t})$	$v^{(2)}_{(2)2}(\widehat{t})$
0.0000	0.0000000000	0.0000000000	1.0000	0.0000000000	0.0000000000
0.0625	0.1879059475	−0.0400222512	1.0625	−0.1889934728	0.0046784388
0.1250	0.3678351574	−0.1176215911	1.1250	−0.3705704014	0.0473741620
0.1875	0.5325884300	−0.2305281090	1.1875	−0.5374995934	0.1259581385
0.2500	0.6755357925	−0.3751437811	1.2500	−0.6831106914	0.2369609697
0.3750	0.8738250780	−0.7393543091	1.3750	−0.8879901498	0.5365760032
0.4375	0.9208625774	−0.9465988903	1.4375	−0.9388331781	0.7131214286
0.5000	0.9297995888	−1.1613404178	1.5000	−0.9518185967	0.8984838355
0.5625	0.8998920302	−1.3762741187	1.5625	−0.9261112673	1.0856515491
0.6250	0.8318585575	−1.5841627389	1.6250	−0.8623161059	1.2678329783
0.6875	0.7278476117	−1.7781366499	1.6875	−0.7624327546	1.4388551096
0.7500	0.5913479276	−1.9519871628	1.7500	−0.6297458977	1.5936132582
0.8125	0.4270454161	−2.1004440717	1.8125	−0.4686494428	1.7286004232
0.8750	0.2406312244	−2.2194299634	1.8750	−0.2844016064	1.8425721086
0.9375	0.0385674149	−2.3062860218	1.9375	−0.0828043328	1.9374502332
1.0000	−0.172182002	−2.3599670411	2.0000	0.1302073234	2.0196536879

Table 2 presents the numerical solution to the problem (17), (18) with proximity 0.36, i.e.:

$$\|v^{(2)}(\widehat{t}) - v^{*}(\widehat{t})\| \leq 0.36.$$

Table 3 – The numerical solution to the special Cauchy problem (17), (18) for the 5-th iteration

$\widehat{t}$	$v^{(5)}_{(1)1}(\widehat{t})$	$v^{(5)}_{(1)2}(\widehat{t})$	$\widehat{t}$	$v^{(5)}_{(2)1}(\widehat{t})$	$v^{(5)}_{(2)2}(\widehat{t})$
0.0000	0.0000000000	0.0000000000	1.0000	0.0000000000	0.0000000000
0.0625	0.1950903218	−0.0192147199	1.0625	−0.1950903216	0.0912147203
0.1250	0.3826834320	−0.0761204681	1.1250	−0.3826834315	0.0761204690
0.1875	0.5555702325	−0.1685303885	1.1875	−0.5555702318	0.1685303902
0.2500	0.7071067806	−0.2928932199	1.2500	−0.7071067796	0.2928932224
0.3125	0.8314696117	−0.4444297683	1.3125	−0.8314696104	0.4444297717
0.3750	0.9238795318	−0.6173165692	1.3750	−0.9238795304	0.6173165734
0.4375	0.9807852797	−0.8049096798	1.4375	−0.9807852782	0.8049096848
0.6250	0.9238795321	−1.3826834347	1.6250	−0.9238795301	1.3826834409
0.6875	0.8314696121	−1.5555702355	1.6875	−0.8314696097	1.5555702420
0.7500	0.7071067812	−1.7071067838	1.7500	−0.7071067781	1.7071067913
0.8125	0.5555702333	−1.8314696149	1.8125	−0.5555702289	1.8314696252
0.8750	0.3826834329	−1.9238795350	1.8750	−0.3826834264	1.9238795517
0.9375	0.1950903322	−1.9807852827	1.9375	−0.1950903129	1.9807853125
1.0000	0.0000000012	−2.0000000019	2.0000	0.0000000146	2.0000000566

Table 3 presents the numerical solution to the problem (17), (18) with proximity $5.656 \cdot 10^{-8}$, i.e.:

$$\|v^{(5)}(\widehat{t}) - v^{*}(\widehat{t})\| \leq 5.656 \cdot 10^{-8}.$$

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Джумабаев Д.С., Каракенова С.Г. ИНТЕГРАЛДЫҚ БӨЛІГІ СЫЗЫҚТЫ ЕМЕС ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН АРНАЙЫ КОШИ ЕСЕБІН ШЕШУДІҢ ИТЕРАЦИЯЛЫҚ ӘДІСІ

Мақалада сызықты емес интегралдық-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебі қарастырылады. Арнайы Коши есебі сызықтық емес операторлық теңдеу түрінде жазылады. Қарастырылып отырған есепті шешудің итерациялық әдісі құрылды және оның жинақтылық шарттары тағайындалды.

Кілттік сөздер. Сызықты емес Фредгольм интегралдық-дифференциалдық теңдеуі, арнайы Коши есебі, параметрлеу әдісі, операторлық теңдеу, итерациялық әдіс.

Джумабаев Д.С., Каракенова С.Г. ИТЕРАЦИОННЫЙ МЕТОД РЕШЕНИЯ СПЕЦИАЛЬНОЙ ЗАДАЧИ КОШИ ДЛЯ СИСТЕМЫ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С НЕЛИНЕЙНОЙ ИНТЕГРАЛЬНОЙ ЧАСТЬЮ

В этой статье рассматривается специальная задача Коши для системы нелинейных интегро-дифференциальных уравнений. Специальная задача Коши записывается как нелинейное операторное уравнение. Построен итерационный метод решения рассматриваемой задачи и установлены условия его сходимости.

Ключевые слова. Нелинейное интегро-дифференциальное уравнение Фредгольма, специальная задача Коши, метод параметризации, операторное уравнение, итерационный метод.

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Approximation of continuous functions by one class of regular splines

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Abstract. The class of regular splines generalizing such cases as cubic splines, rational splines with one pole, some classes of hyperbolic, trigonometric, parabolic and cubic splines with additional knots, and others is considered. The theorem of the existence and the uniqueness of interpolating regular spline is proved. The error estimates on the classes of continuous functions by interpolating regular splines are obtained.

Keywords. Regular splines, generalized splines, rational splines, error estimation.

1 Introduction

The term "regular spline" appeared first in Schaback's work [1] where the author considered an interpolation problem for general nonlinear classes of regular splines including some cases of exponential, trigonometric, rational functions and their combinations. Different properties of such regular splines and their applications were considered in the works of Arndt [2], [3], Werner [4], Werner and Loeb [5], and other authors. Later in the papers [6], [7] we considered regular splines (with another conditions of regularity) generalizing such splines as Späth's rational splines with two poles [8], splines in tension [9], (see also [10]), and others. In the paper [11] there was investigated a problem of approximation of continuous periodic functions by interpolatory regular parabolic splines, considered in [12].

In this paper we consider the problem of approximation of continuous functions by the class of regular splines generalizing such splines as Späth's rational splines with one pole [8], some classes of hyperbolic, trigonometric, parabolic and cubic splines with additional knots, and others. We prove the theorem of the existence and the uniqueness of interpolatory spline and error estimates on the classes of continuous functions. In [13] the local approximation of continuously differentiable functions by such regular splines was considered.

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2 Definition of Interpolatory Regular Spline (IRS)

Let C^r be the class of functions continuous together with their k-th derivatives on the closed interval [0, 1] ($r = 0, 1, ...;$ $C^0 = C$), $\widetilde{C}^r$ is the corresponding class of 1-periodic functions. Let $H_n = \{x_i\}_0^n$ with

$$0 = x_0 < x_1 < ... < x_n = 1$$

be a partition of the interval $[0, 1]$.

Let the function $S_n(x)$ have on each subinterval $[x_i, x_{i+1}]$ the form

$$S_n(x) = A_i + B_i t + C_i t^2 + D_i u_i(t), \tag{1}$$

where $t = (x - x_i)/h_i,$ $h_i = x_{i+1} - x_i,$ A_i, B_i, C_i, D_i are real coefficients, $u_i(t)$ are given functions.

Definition 1. *We shall call the function $S_n(x)$ regular spline if the functions $u_i(t)$ ($i = 0, 1, ..., n$) satisfy the following (regularity) conditions:*

a) second derivatives $u_i''(t)$ are monotone on $[0, 1]$,

b) $u_i''(t)$ are not identical constants on $[0, 1]$.

To every function $f \in C$ (or $f \in \widetilde{C}$) we assign its *interpolating regular spline (IRS)* $S_n(f, x)$, i.e. spline (1) such that

$$S_n(f, x_i) = f(x_i) =: f_i, \ i = 0, 1, ..., n, \tag{2}$$

with *boundary conditions*

$$S_n'(f, 0) = (f_1 - f_0)/h_0, \quad S_n'(f, 1) = (f_n - f_{n-1})/h_{n-1}; \tag{3}$$

and for $f \in \widetilde{C}$

$$S_n^{(j)}(f, 0) = S_n^{(j)}(f, 1), \ j = 1, 2. \tag{4}$$

3 Existence and Uniqueness of IRS

Denote

$$S_n'(f, x_i) = m_i, \ i = 0, 1, ..., n. \tag{5}$$

Then coefficients of IRS $S_n(f, x)$ can be found from interpolation conditions (2):

$$S(x_i) = A_i + D_i u_i(0) = f_i,$$

$$S(x_{i+1}) = A_i + B_i + C_i + D_i u_i(1) = f_{i+1},$$

and smoothness conditions (5):

$$S'(x_i) = \frac{B_i}{h_i} + \frac{D_i}{h_i} u_i'(0) = m_i,$$

$$S'(x_{i+1}) = \frac{B_i}{h_i} + \frac{2C_i}{h_i} + \frac{D_i}{h_i} u_i'(1) = m_{i+1},$$

from which

$$A_i \ = \ \frac{1}{\Delta_i} \Big[f_i \left(2u_i(1) - u_i^{'}(1) - u_i^{'}(0) \right) - u_i(0) \left(2f_{i+1} - m_{i+1}h_i - m_i h_i \right) \Big], \tag{6}$$

$$B_i \ = \ \frac{1}{\Delta_i} \Big[\left(2u_i(1) - 2u_i(0) - u_i^{'}(1) \right) m_i h_i + u_i^{'}(0) \left(2f_i + m_{i+1}h_i - 2f_{i+1} \right) \Big], \tag{7}$$

$$C_i \ = \ \frac{1}{\Delta_i} \Big[\left(f_{i+1} - f_i \right) \left(u_i^{'}(0) - u_i^{'}(1) \right)$$

$$+ \big(u_i(1) - u_i(0) \big) \left(m_{i+1} - m_i \right) h_i \ + \left(m_i u_i^{'}(1) - m_{i+1} u_i^{'}(0) \right) h_i \Big] \ , \tag{8}$$

$$D_i \ = \ \frac{1}{\Delta_i} \Big[2 \left(f_{i+1} - f_i \right) - m_i h_i - m_{i+1} h_i \Big], \tag{9}$$

where

$$\Delta_i = 2\big(u_i(1) - u_i(0)\big) - \big(u_i^{'}(1) + u_i^{'}(0)\big), \tag{10}$$

and spline $S_n(f,x)$ can be represented in the form:

$$S_n(f,x) = f_i \left(1 - \beta_i(t)\right) + f_{i+1} \, \beta_i(t) + m_i h_i \frac{\varphi_i(t)}{\Delta_i} + m_{i+1} h_i \frac{\psi_i(t)}{\Delta_i} \ , \tag{11}$$

where

$$\beta_i(t) = 1 + \frac{1}{\Delta_i} \Big[2(u_i(t) - u_i(1)) + u_i^{'}(0)(1-t)^2 + u_i^{'}(1)(1-t^2) \Big], \tag{12}$$

$$\varphi_i(t) = \big[u_i(1) - u_i(t)\big] + \big[u_i(0) - u_i(1)\big](1-t)^2 + u_i^{'}(1) t(t-1), \tag{13}$$

$$\psi_i(t) = \big[u_i(1) - u_i(t)\big] + \big[u_i(0) - u_i(1)\big](1-t^2) + u_i^{'}(0) t(1-t). \tag{14}$$

The values

$$m_i, \ i = 1, 2, ..., n-1,$$

can be found from condition of continuity of second derivatives

$$S_n^{''}(f, x_i + 0) = S_n^{''}(f, x_i - 0), \ i = 1, ..., n-1, \tag{15}$$

and boundary conditions (3)–(4).

Thereby we obtain the system:

$$\left[\frac{\varphi''_{i-1}(1)}{\Delta_{i-1}}\right]\lambda_i m_{i-1}+\left[\frac{\psi''_{i-1}(1)}{\Delta_{i-1}}\lambda_i-\frac{\varphi''_i(0)}{\Delta_i}\mu_i\right]m_i+\left[-\frac{\psi''_i(0)}{\Delta_i}\right]\mu_i m_{i+1}$$

$$=\frac{f_{i+1}-f_i}{h_i}\mu_i\beta''_i(0)-\frac{f_i-f_{i-1}}{h_{i-1}}\lambda_i\beta''_{i-1}(1),\ i=1,2,...,n-1, \tag{16}$$

where $\lambda_i = h_i/(h_{i-1}+h_i),\ \mu_i = 1-\lambda_i$.

Depending on boundary conditions we add to the system (16) the following equations: for the type (3)

$$m_0=(f_1-f_0)/h_0,\ m_n=(f_n-f_{n-1})/h_{n-1};$$

for the type (4) we add the equation with $i = n$ such that

$$\varphi''_{n+k}(t)=\varphi''_k(t),\ \psi''_{n+k}(t)=\psi''_k(t),\ \Delta_{n+k}=\Delta_k,\ \beta''_{n+k}(t)=\beta''_k(t),$$

$$f_{n+k}=f_k,\ m_{n+k}=m_k,\ \lambda_{n+k}=\lambda_k,\ \mu_{n+k}=\mu_k,\ k\ \text{ are integers.}$$

Now we establish some properties of the functions $\dfrac{\varphi_i(t)}{\Delta_i}$, $\dfrac{\psi_i(t)}{\Delta_i}$, $\beta_i(t)$, and their derivatives.

Property 1. *For all $i = 0, 1, ..., n-1$ the following relations hold:*

a) $\dfrac{\varphi_i(t)-\psi_i(t)}{\Delta_i} = t(1-t)$, *from which it follows*

b) $\dfrac{\psi''_i(t)}{\Delta_i}-\dfrac{\varphi''_i(t)}{\Delta_i}=2.$

Proof. The proof immediately follows from (13), (14), and (10).

Property 2. *For all $i = 0, 1, ..., n-1$ the following relations hold:*

a) $\dfrac{\varphi_i(0)}{\Delta_i}=0, \qquad \dfrac{\varphi_i(1)}{\Delta_i}=0;$

b) $\dfrac{\varphi'_i(0)}{\Delta_i}=1, \qquad \dfrac{\varphi'_i(1)}{\Delta_i}=0;$

c) *the functions* $\dfrac{\varphi''_i(t)}{\Delta_i}$ *are monotonically increasing and change its sign on the interval* $[0;1]$, *i.e.* $\dfrac{\varphi''_i(0)}{\Delta_i}<0,\ \dfrac{\varphi''_i(1)}{\Delta_i}>0;$

d) $0\leqslant\dfrac{\varphi_i(t)}{\Delta_i}\leqslant t(1-t).$

Proof. Statements a) and b) follow immediately from (13).

Let us prove c). From regularity conditions and (13) follow monotonicity of $\frac{\varphi_i^{''}(t)}{\Delta_i}$ and concavity of $\frac{\varphi_i^{'}(t)}{\Delta_i}$. Further note that

$$\int_0^1 \frac{\varphi_i^{'}(t)}{\Delta_i}dt = 0,$$

and since $\frac{\varphi_i^{'}(0)}{\Delta_i} > 0$, there are points on $(0;1)$ where function $\frac{\varphi_i^{'}(t)}{\Delta_i} < 0$, i.e. there exists a point $\xi \in (0;1)$ such that $\frac{\varphi_i^{'}(\xi)}{\Delta_i} = 0$. Then by Rolle's theorem there exists a point $\eta \in (\xi;1)$ such that $\frac{\varphi_i^{''}(\eta)}{\Delta_i} = 0$ and seeing that $\frac{\varphi_i^{'}(0)}{\Delta_i} = 1$, $\frac{\varphi_i^{'}(\eta)}{\Delta_i} < 0$, and $\frac{\varphi_i^{'}(1)}{\Delta_i} = 0$ the function $\frac{\varphi_i^{''}(t)}{\Delta_i}$ is monotonically increasing, $\frac{\varphi_i^{''}(0)}{\Delta_i} < 0$ and $\frac{\varphi_i^{''}(1)}{\Delta_i} > 0$.

To establish d) let us prove first that $\frac{\varphi_i(t)}{\Delta_i} \geqslant 0$. Indeed, if there exists a point $\xi_i \in (0,1)$ such that $\frac{\varphi_i(\xi_i)}{\Delta_i} < 0$, then there exists a point $\eta_i \in (0,1)$ such that $\frac{\varphi_i(\eta_i)}{\Delta_i} = 0$ and function $\frac{\varphi_i(t)}{\Delta_i}$ has at least four zeros considering multiplicity on $[0,1]$. Then the function $\frac{\varphi_i^{''}(t)}{\Delta_i}$ must have at least two variations of sign on $[0,1]$, which contradicts regularity conditions. Similarly, we can show that $\frac{\psi_i(t)}{\Delta_i} \leqslant 0$. Further, using last inequality and Property 1, we have $\frac{\varphi_i(t)}{\Delta_i} - t(1-t) = \frac{\psi_i(t)}{\Delta_i} \leqslant 0$, from which $\frac{\varphi_i(t)}{\Delta_i} \leqslant t(1-t)$.

The property is proved.

Similarly, we have

Property 3. *For all $i = 0, 1, ..., n-1$ the following relations hold:*

a) $\frac{\psi_i(0)}{\Delta_i} = 0$, $\frac{\psi_i(1)}{\Delta_i} = 0$;

b) $\frac{\psi_i^{'}(0)}{\Delta_i} = 0$, $\frac{\psi_i^{'}(1)}{\Delta_i} = 1$;

c) *the functions* $\dfrac{\psi_i''(t)}{\Delta_i}$ *are monotonically increasing and change its sign on the interval* $[0;1]$, *i.e.* $\dfrac{\psi_i''(0)}{\Delta_i} < 0$, $\dfrac{\psi_i''(1)}{\Delta_i} > 0$;

d) $t(t-1) \leqslant \dfrac{\psi_i(t)}{\Delta_i} \leqslant 0$.

Let us consider some properties of the function (12).

Property 4. *For all* $i = 0, 1, ..., n-1$ *the following relations hold:*

a) $\beta_i(0) = 0,\ \beta_i(1) = 1$;

b) $\beta_i'(0) = 0,\ \beta_i'(1) = 0$;

c) *the functions* $\beta_i(t)$ *are monotone on the interval* $[0;1]$ *and* $0 \leqslant \beta_i(t) \leqslant 1$.

Proof. Statements a) and b) follow from (12). To prove c) note that the functions $\beta_i(t)$ must be monotone, otherwise their first derivatives will have at least three different zeros on $[0;1]$ and therefore their second derivatives will have at least two variations of sign, which contradicts regularity conditions (see Definition 1).

We now can prove the theorem

Theorem 1. *For any function* $f \in C$ *(or* $f \in \widetilde{C}$*) there exists a unique regular spline* $S_n(x) \in C^2$ *(or* $S_n(x) \in \widetilde{C}^2$*) of the form (1) satisfying interpolation conditions (2) and boundary conditions (3) (or (4)).*

Proof. To prove this theorem it is sufficient to show that matrix of the system (16) is strictly diagonally dominant. We recall that matrix $A = (a_{ij})$ is strictly diagonally dominant if

$$r_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}| > 0 \ \forall i. \tag{17}$$

From Properties 1, 2, 3 and system (16) we have:

$$r_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}|$$

$$= \left| \frac{\psi_{i-1}''(1)}{\Delta_{i-1}} \lambda_i - \frac{\varphi_i''(0)}{\Delta_i} \mu_i \right| - \left| \frac{\varphi_{i-1}''(1)}{\Delta_{i-1}} \right| \lambda_i - \left| -\frac{\psi_i''(0)}{\Delta_i} \right| \mu_i$$

$$= \lambda_i \left[\frac{\psi_{i-1}''(1)}{\Delta_{i-1}} - \frac{\varphi_{i-1}''(1)}{\Delta_{i-1}} \right] + \mu_i \left[\left(-\frac{\varphi_i''(0)}{\Delta_i} \right) - \left(-\frac{\psi_i''(0)}{\Delta_i} \right) \right] = 2\left(\lambda_i + \mu_i\right) = 2, \tag{18}$$

i.e. the coefficient matrix of the system (16) is strictly diagonally dominant. This proves the theorem.

4 Error estimates for continuous functions by IRS

Theorem 2. *Let $f \in C$ (or $f \in \widetilde{C}$) and let IRS $S_n(f,x)$ of the form (1) satisfy interpolation conditions (2) and boundary conditions (3) (or (4)). Then on each of the segments $[x_i, x_{i+1}]$ the following inequality holds*

$$\Big|S_n(f,x) - f(x)\Big| \leqslant \Big[1 + \frac{\gamma}{8} \cdot \delta\Big]\, \omega(f, \bar{h}), \tag{19}$$

where $\gamma = \max_i \{ \Big|\beta_i''(0)\Big|, \Big|\beta_{i-1}''(1)\Big| \}$, *$\beta_i(t)$ are functions determined in (12),* $\delta = \bar{h}/\underline{h}$, $\bar{h} = \max_i h_i$, $\underline{h} = \min_i h_i$, *$\omega(f, \bar{h})$ is modulus of continuity of f.*

Proof. From (11) we obtain:

$$\Big|S_n(f,x) - f(x)\Big| = \Big|f_i(1-\beta_i(t)) + f_{i+1}\,\beta_i(t)$$

$$+ m_i h_i \,\frac{\varphi_i(t)}{\Delta_i} + m_{i+1} h_i \,\frac{\psi_i(t)}{\Delta_i} - f(x)\Big|$$

$$\leqslant \Big|f_i(1-\beta_i(t)) + f_{i+1}\,\beta_i(t) - f(x)\Big| + \left|\frac{\varphi_i(t) - \psi_i(t)}{\Delta_i}\right| h_i \;||\, m\,|| \,. \tag{20}$$

Since $0 \leqslant \beta_i(t) \leqslant 1$ (see Property 4), we have

$$\Big|f_i(1-\beta_i(t)) + f_{i+1}\,\beta_i(t) - f(x)\Big| \leqslant \omega(f, h_i). \tag{21}$$

As the system (16) is diagonally dominant, then its solution satisfies the inequality (see for example [14], p. 334)

$$||\, m\,|| := \max_i |m_i| \leqslant \max_i \frac{|d_i|}{r_i}, \tag{22}$$

where d_i is the right side of the system (16), r_i is defined in (17). Then from (22), (16), and (17) we have

$$||\, m\,|| \leqslant \max_i \frac{\Big(\mu_i \Big|\beta_i''(0)\Big| + \lambda_i \Big|\beta_{i-1}''(1)\Big|\Big)}{2} \cdot \frac{\omega(f, \bar{h})}{\underline{h}}, \tag{23}$$

where $\underline{h} = \min_i h_i$, $\bar{h} = \max_i h_i$.

In consideration of Property 1, (21), and (23) from (20) we obtain

$$\Big|S_n(f,x)-f(x)\Big| \leqslant \omega(f,\bar h)+\bar h\, t(1-t)\max_i \frac{\Big(\mu_i\Big|\beta_i^{''}(0)\Big|+\lambda_i\Big|\beta_{i-1}^{''}(1)\Big|\Big)}{2}\cdot\frac{\omega(f,\bar h)}{\underline{h}}$$

$$\leqslant \omega(f,\bar h)+\max_i \frac{\Big(\mu_i\Big|\beta_i^{''}(0)\Big|+\lambda_i\Big|\beta_{i-1}^{''}(1)\Big|\Big)}{8}\cdot\frac{\bar h\,\omega(f,\bar h)}{\underline{h}}$$

$$\leqslant \left[1+\max_i \frac{\Big(\mu_i\Big|\beta_i^{''}(0)\Big|+\lambda_i\Big|\beta_{i-1}^{''}(1)\Big|\Big)}{8}\cdot\delta\right]\omega(f,\bar h)$$

$$\leqslant \left[1+\frac{\max_i\{\Big|\beta_i^{''}(0)\Big|,\Big|\beta_{i-1}^{''}(1)\Big|\}}{8}\cdot\delta\right]\omega(f,\bar h)\leqslant\Big[1+\frac{\gamma}{8}\cdot\delta\Big]\omega(f,\bar h). \tag{24}$$

The theorem is proved.

We denote by W_p^1 the space of functions for which f is absolutely continuous and $f^{'}\in L_p$, $(1\leqslant p\leqslant\infty)$ on the interval $[0,1]$. $\widetilde{W_p^1}$ is the corresponding class of 1-periodic functions.

Theorem 3. *Let $f\in W_p^1$ (or $f\in\widetilde{W_p^1}$) $(1\leqslant p\leqslant\infty)$ and let IRS $S_n(f,x)$ of the form (1) satisfy interpolation conditions (2) and boundary conditions (3) (or (4)). Then on each of the segments $[x_i,x_{i+1}]$ the following inequality holds*

$$\Big|S_n(f,x)-f(x)\Big|\leqslant\Big[2^{-\frac{1}{q}}+\frac{\gamma}{8}\cdot\delta^{\frac{1}{p}}\Big]\,\bar h^{\frac{1}{q}}\,||\,f^{'}\,||_p\,, \tag{25}$$

where $\gamma=\max_i\{\Big|\beta_i^{''}(0)\Big|,\Big|\beta_{i-1}^{''}(1)\Big|\}$, $\beta_i(t)$ are the functions defined in (12), $\bar h=\max_i h_i$, $||\,f^{'}\,||_p=\max_i\left(\int\limits_{x_i}^{x_{i+1}}\Big|f^{'}(x)\Big|^p dx\right)^{\frac{1}{p}}$, $\delta=\max_i\{\frac{h_i}{h_{i-1}},1\}$, $1/q+1/p=1$.

Proof. From representation (11) and Property 1 we obtain:

$$\Big|S_n(f,x)-f(x)\Big|=\Big|f_i(1-\beta_i(t))+f_{i+1}\,\beta_i(t)$$

$$+m_ih_i\,\frac{\varphi_i(t)}{\varDelta_i}+m_{i+1}h_i\,\frac{\psi_i(t)}{\varDelta_i}-f(x)\Big|$$

$$\leqslant\Big|f_i(1-\beta_i(t))+f_{i+1}\,\beta_i(t)-f(x)\Big|+h_i\,\,||\,m\,||\,\Big|\frac{\varphi_i(t)-\psi_i(t)}{\varDelta_i}\Big|$$

$$= \left| f_i(1-\beta_i(t)) + f_{i+1}\,\beta_i(t) - f(x) \right| + t(1-t)\,h_i\,||\,m\,|| \,. \tag{26}$$

Using Taylor's expansion, Property 4, and Hölder's inequality for the first summand in (26), we have

$$\left| f_i(1-\beta_i(t)) + f_{i+1}\,\beta_i(t) - f(x) \right|$$

$$= \left| \left(f(x) - \int_{x_i}^{x} f^{'}(t)dt \right)(1-\beta_i(t)) + \left(f(x) - \int_{x_{i+1}}^{x} f^{'}(t)dt \right)\beta_i(t) - f(x) \right|$$

$$= \left| (1-\beta_i(t)) \int_{x_i}^{x} f^{'}(t)dt + \beta_i(t) \int_{x_{i+1}}^{x} f^{'}(t)dt \right|$$

$$\leqslant \left| (1-\beta_i(t)) t^{1/q} h_i^{1/q} ||f^{'}||_p + \beta_i(t)(1-t)^{1/q} h_i^{1/q} ||f^{'}||_p \right| \leqslant \frac{1}{2^{1/q}} \bar{h}^{1/q} ||f^{'}||_p. \tag{27}$$

For the second summand in (26) from (22) and (17) we obtain

$$\left| t(1-t)\,h_i\,||\,m\,|| \right| \leqslant \frac{h_i}{4} \max_i \frac{|d_i|}{r_i} \leqslant \frac{h_i}{8} \max_i \left| \frac{f_{i+1}-f_i}{h_i} \mu_i \beta_i^{''}(0) - \frac{f_i - f_{i-1}}{h_{i-1}} \lambda_i \beta_{i-1}^{''}(1) \right|$$

$$\leqslant \frac{h_i}{8} \max_i \left(||f^{'}||_p h_i^{1/q-1} \mu_i |\beta_i^{''}(0)| + ||f^{'}||_p h_{i-1}^{1/q-1} \lambda_i |\beta_{i-1}^{''}(1)| \right)$$

$$\leqslant \frac{1}{8} \max_i \left(\mu_i |\beta_i^{''}(0)| + \lambda_i |\beta_{i-1}^{''}(1)| \right) \delta^{1/p}\, \bar{h}^{1/q} ||f^{'}||_p \leqslant \frac{\gamma\,\delta^{1/p}}{8}\, \bar{h}^{1/q}\, ||f^{'}||_p, \tag{28}$$

where $\gamma = \max_i \{ \left| \beta_i^{''}(0) \right|, \left| \beta_{i-1}^{''}(1) \right| \}$.

From (27) and (28) follows (25). The theorem is proved.

5 Examples

1) Substituting the functions $u_i(t) = t^3$ $(i = 0, 1, ..., n-1)$ in representation (1), we obtain the usual polynomial cubic splines. Considering this case in Theorem 2, we obtain the known result for the cubic splines ([14], p. 102):

$$\left| S_n(f,x) - f(x) \right| \leqslant \left(1 + \frac{3}{4}\,\delta \right) \omega(f, \bar{h}). \tag{29}$$

2) For the functions

$$u_i(t) = \sum_{j=1}^{k} d_{ij} (t - \alpha_j)_+^2$$

we have parabolic splines with additional knots.

3) For the functions

$$u_i(t) = \sum_{j=1}^{k} d_{ij}(t-\alpha_j)_+^3$$

we have cubic splines with additional knots.

4) For the case

$$u_i(t) = \frac{t^3}{1+p_i t}$$

we obtain Späth's rational splines with one pole ([8], §6.3).

The following constructions and their combinations can also be considered:

5) $u_i(t) = \sqrt{t+\alpha_i}$, $\alpha_i > 0$

6) $u_i(t) = ln(t+\alpha_i)$, $\alpha_i > 0$

7) $u_i(t) = e^{\alpha_i t}$.

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Ысмағұлов М.Р. ҮЗІЛІССІЗ ФУНКЦИЯЛАРДЫ РЕГУЛЯРЛЫ СПЛАЙНДАРДЫҢ БІР КЛАСЫМЕН ЖУЫҚТАУ

Бұл жұмыста регулярлы сплайндардың бір класы қарастырылады. Олар сплайндардың келесі түрлерін: кубтық сплайндарды, бір полюсті рационал сплайндарды, қосымша түйіндері бар гиперболалық, тригонометриялық, параболалық және кубтық сплайндарды және басқаларын жалпылайды.

Кілттік сөздер. Регулярлы сплайндар, жалпыланған сплайндар, жуықтау бағалаулары.

Исмагулов М.Р. АППРОКСИМАЦИЯ НЕПРЕРЫВНЫХ ФУНКЦИЙ ОДНИМ КЛАССОМ РЕГУЛЯРНЫХ СПЛАЙНОВ

Рассматривается класс регулярных сплайнов, обобщающих, в частности, кубические, рациональные сплайны с одним полюсом, некоторые классы гиперболических, тригонометрических, параболических и кубических сплайнов с дополнительными узлами и другие.

Ключевые слова. Регулярные сплайны, обобщенные сплайны, оценки приближения.

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Blowing-up solutions of the shallow water equations

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Abstract. In the paper we study problems of global unsolvability of shallow water equations. For certain initial-boundary problems for the shallow water equations, we obtain necessary conditions of the existence of global solutions. The proof of the results is based on the nonlinear capacity method. In closing, we provide the examples.

Keywords. Shallow water, blow-up solution, Kawahara equation, Kaup-Kupershmidt equation, initial-boundary problem.

1 Introduction

In the paper, we study some shallow water equations as below:

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u + \mu u\partial_x u = 0,\ (x,t) \in \mathbb{R}\times(0,T), \tag{1}$$

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u - \mu\partial_x u\partial_x^2 u = 0,\ (x,t) \in \mathbb{R}\times(0,T), \tag{2}$$

with initial data

$$u(x,0) = u_0(x), \tag{3}$$

where $\alpha \neq 0$, β, γ and μ are real numbers.

The model (1) is also called the Kawahara equation [1]. It arises in study of the water waves with surface tension, in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime (see [2], [3]). Equation (2) is often called the Kaup-Kupershmidt equation [4]. The models have arisen in the study of capillary-gravity waves [5], [6].

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If $\alpha = 0$, the model (1) reduces to the well-known Korteweg-de Vries equation

$$\partial_t u + \beta \partial_x^3 u + \gamma \partial_x u + \mu u \partial_x u.$$

In [7] Huo proved the local and globall well-posedness of the problem (1)–(3) in Sobolev spaces by the Fourier restriction norm method. Note that the local well-posedness of the equations (1), (2) with initial data (3) in several Sobolev spaces was studied in [8], [9], [10].

This paper is devoted to blowing-up solutions of the above equations, more precisely, to solutions that blow up in a finite time. The approach to the problem is based on the Mitidieri-Pokhozhaev nonlinear capacity method [11], [12], more precisely, on the choice of test functions corresponding to the initial and boundary conditions under consideration.

2 Blow-up of solution of the Kawahara equation

We consider a test function $\varphi \in C^5([a,b])$ defined on the domain $a < x < b$ with arbitrary parameters $a, b \in \mathbb{R}$ and monotonically nondecreasing:

$$\varphi'(x) \geq 0 \ \text{ for } \ x \in [a,b], \tag{4}$$

and let the function φ satisfy the following properties:

$$\begin{cases} \theta_1 := \int\limits_a^b \frac{(\alpha\varphi^{(5)} + \beta\varphi''' + \gamma\varphi')^2}{\varphi'} dx < \infty; \\ \theta_2 := \int\limits_a^b \frac{\varphi^2}{\varphi'} dx < \infty. \end{cases} \tag{5}$$

Suppose the classical solution $u(x,t) \in C^{1,5}_{t,x}\left(\mathbb{R} \times (0,T)\right)$.

Multiplying the Kawahara shallow water equation (1) by the test function φ, we have

$$\int\limits_a^b \partial_t u(x,t)\varphi(x)dx = -\alpha \int\limits_a^b \partial_x^5 u(x,t)\varphi(x)dx$$

$$-\beta \int\limits_a^b \partial_x^3 u(x,t)\varphi(x)dx - \gamma \int\limits_a^b \partial_x u(x,t)\varphi(x)dx - \mu \int\limits_a^b u(x,t)\partial_x u(x,t)\varphi(x)dx.$$

Integrating by parts the last equation, we have

$$\int\limits_a^b u_t(x,t)\varphi(x)dx = \alpha \int\limits_a^b u(x,t)\varphi^{(5)}(x)dx + \beta \int\limits_a^b u(x,t)\varphi'''(x)dx$$

$$+\gamma \int_a^b u(x,t)\varphi'(x)dx + \frac{\mu}{2}\int_a^b u^2(x,t)\varphi'(x)dx + \mathcal{B}(u(x,t),\varphi(x))\Big|_{x=a}^{x=b}, \tag{6}$$

where

$$\mathcal{B}(u(x,t),\varphi(x)) = \alpha\left(\partial_x^4 u\varphi - \partial_x^3 u\varphi' + \partial_x^2 u\varphi'' - \partial_x u\varphi''' + u\varphi^{(4)}\right)$$
$$+\beta\left(\partial_x^2 u\varphi - \partial_x u\varphi' + u\varphi''\right) + \gamma u\varphi + \frac{\mu}{2}u^2\varphi.$$

Then, using properties (4), we find

$$\int_a^b \left(2u(x,t)\left(\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)\right) + \mu u^2(x,t)\varphi'(x)\right)dx$$

$$= \int_a^b \left(\sqrt{\mu}u(x,t) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\sqrt{\mu}\varphi'(x)}\right)^2 \varphi'(x)dx$$

$$-\frac{1}{\mu}\int_a^b \frac{\left(\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)\right)^2}{\varphi'(x)}dx.$$

We denote by $w(x,t)$ the following function

$$w(x,t) = \sqrt{\mu}u(x,t) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\sqrt{\mu}\varphi'(x)}.$$

We introduce the following functional:

$$F(t) = \int_a^b w(x,t)\varphi(x)dx.$$

By using the Hölder inequality for the functional $F(t)$, we obtain the following estimate

$$\left(\int_a^b w(x,t)\varphi(x)dx\right)^2 \leq \int_a^b w^2(x,t)\varphi'(x)dx \int_a^b \frac{\varphi^2(x)}{\varphi'(x)}dx.$$

Therefore, using the properties of the test of the function (5) for the expression (6), we obtain the following first order differential inequality

$$F'(t) \geq \frac{\theta_2^{-1}}{2}F^2(t) + \Phi(t) - \frac{\theta_1}{2\mu} \tag{7}$$

with initial conditions

$$F(0) = \int\limits_a^b \left(\sqrt{\mu} u_0(x) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\sqrt{\mu}\varphi'(x)} \right) \varphi(x) dx,$$

where $\Phi(t) = \mathcal{B}(u(b,t), \xi(b)) - \mathcal{B}(u(a,t), \xi(a))$. Then the following results are true

Theorem 1. *Let $u_0(x) \in L^1([a,b])$ and let the solution $u \in C^{1,5}_{t,x}(\mathbb{R} \times (0,T))$ of the Kawahara shallow water equation* (1) *be such that there exists a function φ satisfying conditions* (4), (5) *such that*

$$\Phi(t) \geq \sigma \text{ for all } t > 0,$$

where σ is a some constant.
Then

(A) *if $\sigma > \theta_2$, then $F(t) \to +\infty$ at $t \to T_1^*$, where*

$$T_1^* = \frac{2\sqrt{\alpha}}{\sqrt{\sigma - \theta_2}} \left(\frac{\pi}{2} - \arctan \frac{F(0)}{2\sqrt{\theta_1(\sigma - \theta_2)}} \right);$$

(B) *if $\sigma = \theta_2$ and $F(0) > 0$, then $F(t) \to +\infty$ at $t \to T_2^*$, where $T_2^* = \frac{4\theta_1}{F(0)}$;*

(C) *if $\sigma < \theta_2$ and $F(0) > 2\sqrt{\theta_1(\theta_2 - \sigma)}$, then $F(t) \to +\infty$ at $t \to T_3^*$, where*

$$T_3^* = \frac{\sqrt{\theta_1}}{\sqrt{\theta_2 - \sigma}} \ln \frac{F(0) + 2\sqrt{\theta_1(\theta_2 - \sigma)}}{F(0) - 2\sqrt{\theta_1(\theta_2 - \sigma)}}.$$

Applying the theory of ordinary differential inequalities, we can prove Theorem 1.

Example. Note that the trial function method has great practical convenience. For example, if in the problem (1), (3) with $\beta = 0$ on the interval $[0,1]$ there are given Dirichlet type boundary conditions

$$\begin{aligned} &u(0,t) = 0,\ u(1,t) = 0, \\ &\partial_x^2 u(0,t) = 0,\ \partial_x^2 u(1,t) = 0, \\ &\partial_x^4 u(0,t) + 4\partial_x^3 u(0,t) + 24\partial_x u(0,t) = 0,\ t \geq 0, \end{aligned}$$

then, if taking a function of the form $\varphi(x) = (1-x)^4$, we obtain

$$\theta_1 := \gamma^2,\ \theta_2 := \frac{1}{24}$$

and

$$\Phi(t) = 0 \text{ for all } \ t > 0.$$

Hence it follows from Theorem 1 that under condition

$$\int\limits_0^1 u_0(x)(1-x)^4 dx > \frac{\sqrt{6}\gamma}{6}$$

the solution of the problem (1), (3) is blowing up in finite time

$$T^* = 2\sqrt{6}\gamma \ln \frac{\sqrt{6}F(0)+\gamma}{\sqrt{6}F(0)-\gamma}.$$

3 Gradient blow-up of solution of the Kaup-Kupershmidt equation

In this section we obtain a result on the "soft blow-up" for the initial problem (2), (3) in the bounded domain. Suppose that there exists a smooth bounded classical solution. Differentiating equation (2) with respect to the space variable, we obtain

$$\partial^2_{tx}u + \alpha\partial^6_x u + \beta\partial^4_x u + \gamma\partial^2_x u - \mu\partial_x u\partial^3_x u - \mu\partial^2_x u\partial^2_x u = 0,\ a < x < b,\ t > 0. \tag{8}$$

We consider a test function $\phi \in C^5([a,b])$ defined on the domain $a < x < b$ with arbitrary parameters $a, b \in \mathbb{R}$ and nonconvex:

$$\phi''(x) \geq 0 \ \text{ for } \ x \in [a,b], \tag{9}$$

and let the function φ satisfy the following properties:

$$\begin{cases} \omega_1 := \int\limits_a^b \frac{(\alpha\phi^{(5)}+\beta\phi'''+\gamma\varphi')^2}{\phi''}dx < \infty; \\ \omega_2 := \int\limits_a^b \frac{\phi^2}{\phi''}dx < \infty. \end{cases} \tag{10}$$

Multiply the equation (8) by a test function $\phi(x)$. Let $\partial_x u = v$. Then, integrating by parts, we see that

$$\frac{d}{dt}\int_a^b v(x,t)\phi(x)dx = \alpha\int_a^b v(x,t)\phi^{(5)}(x)dx + \beta\int_a^b v(x,t)\phi'''(x)dx$$

$$+\gamma\int_a^b v(x,t)\phi'(x)dx + \frac{\mu}{2}\int_a^b v^2(x,t)\phi''(x)dx + \mathcal{M}(v(x,t),\phi(x))\Big|_{x=a}^{x=b}, \tag{11}$$

where

$$\mathcal{M}(v(x,t),\phi(x)) = -\alpha\partial_x^4 v(x,t)\phi(x) + \alpha\partial_x^3 v(x,t)\phi'(x) - \alpha\partial_x^2 v(x,t)\phi''(x)$$
$$+\alpha\partial_x v(x,t)\phi'''(x) - \alpha v(x,t)\phi^{(4)}(x) - \beta\partial_x^2 v(x,t)\phi(x) + \beta\partial_x v(x,t)\phi'(x) - \beta v(x,t)\phi''(x)$$
$$-\gamma v(x,t)\phi(x) + \frac{\mu}{2}\partial_x v^2(x,t)\phi(x) - \frac{\mu}{2}v^2(x,t)\phi'(x).$$

We denote by $w(x,t)$ the following function

$$w(x,t) = \sqrt{\mu}v(x,t) + \frac{\alpha\phi^{(5)}(x) + \beta\phi'''(x) + \gamma\phi'(x)}{\sqrt{\mu}\phi''(x)}.$$

,By using the Hölder inequality for the functional

$$H(t) = \int_a^b w(x,t)\phi(x)dx,$$

we obtain the following estimate

$$\left(\int_a^b w(x,t)\phi(x)dx\right)^2 \leq \int_a^b w^2(x,t)\phi''(x)dx \int_a^b \frac{\phi^2(x)}{\phi''(x)}dx.$$

We introduce the notation $\Phi(t) = \mathcal{M}(v(b,t),\phi(b)) - \mathcal{M}(v(a,t),\phi(a))$. Suppose that there exists a test function $\phi(x)$ for which $\Phi(t)$ is independent of time. If there is no such function, then $\Phi(t)$ must be considered separately, for example, assuming that the constant independent of t is bounded above.

Consequently, by properties (9) and (10) for the function $H(t)$ we obtain the following ordinary differential inequality

$$H'(t) \geq \frac{\omega_2^{-1}}{2}H^2(t) - \omega^2, \tag{12}$$

where $\omega = \frac{\omega_1}{2\mu} - \Phi(t)$.

Applying the theory of ordinary differential inequalities, we obtain the following result.

Theorem 2. *Let $u_0(x) \in H^1([a,b])$ and let the solution $u \in C^{1,6}_{t,x}(\mathbb{R}\times(0,T))$ of the equation* (8) *be such that there exists a function ϕ satisfying conditions* (9), (10) *such that*

$$H(0) = \int_a^b \left(\sqrt{\mu}u_0'(x) + \frac{\alpha\phi^{(5)}(x) + \beta\phi'''(x) + \gamma\phi'(x)}{\sqrt{\mu}\phi'(x)}\right)\phi(x)dx > \omega\sqrt{2\omega_2}.$$

Then the global (in time) gradient solution of the equation (2) *is blowing-up in finite time and the following estimate holds:*

$$H(t) \geq \omega\sqrt{2\omega_2}\frac{1+h_0\exp\left(2h_0\sqrt{2\omega_2}^{-1}t\right)}{1-h_0\exp\left(2h_0\sqrt{2\omega_2}^{-1}t\right)}, \quad h_0 = \frac{\sqrt{2\omega_2}^{-1}H(0)-\omega}{\sqrt{2\omega_2}^{-1}H(0)+\omega}$$

and hence

$$\lim_{t\to T^*} H(t) = +\infty, \quad T^* = -\frac{\sqrt{2\omega_2}}{2\omega}\ln|h_0|.$$

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Қошқарбаев Н.М., Төребек Б.Т. ТАЯЗ СУ ТЕҢДЕУЛЕРІНІҢ ШЕШІМІНІҢ ҚИРАУЫ

Мақалада таяз су теңдеулерінің глобалды шешілімсіздігі мәселелері қарастырылады. Таяз су теңдеулеріне қойылған кейбір бастапқы-шеттік есептер үшін глобалды шешімдердің бар болуының қажетті шарттары алынған. Нәтижелердің дәлелдеуі сызықтық емес сыйымдылық әдісіне негізделген. Қорытындыда мысалдар келтірілген.

Кілттік сөздер. Таяз су, шешімнің қирауы, Кавахара теңдеуі, Кауп-Купершмидт теңдеуі, бастапқы-шеттік есеп.

Кошкарбаев Н.М., Торебек Б.Т. РАЗРУШЕНИЕ РЕШЕНИЯ УРАВНЕНИЙ МЕЛКОЙ ВОДЫ

В статье рассматриваются проблемы глобальной неразрешимости уравнений мелкой воды. Для некоторых начально-краевых задач для уравнений мелкой воды получены необходимые условия существования глобальных решений. Доказательство результатов основано на методе нелинейной емкости. В заключение приведены примеры.

Ключевые слова. Мелкая вода, разрушение решения, уравнение Кавахары, уравнение Каупа-Купершмидта, начально-краевая задача.

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A technical prototype of the finite signature reduction procedure for the algebraic mode of definability

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Abstract. We prove that the predicate calculus of any finite rich signature has a Cartesian extension algebraically isomorphic to a finitely axiomatizable fragment of the predicate calculus of a pre-assigned finite rich signature. This is a technical result presenting a basis for a full-weight version of the finite signature reduction procedure; moreover, this statement is well integrated as a technical component into various constructions associated with the characterization of the expressive power of first-order logic.

Keywords. First order logic, incomplete theory, Tarski-Lindenbaum algebra, Cartesian extension of a theory, signature reduction procedure.

Laslo Kalmar in [1] first described the method of reducing signatures and applied it to transfer the well-known result of Alonzo Church [2] about the undecidability of the predicate logic of a certain signature to various other signatures. Robert Vaught used an improved version of the procedure of signature reduction to characterize [3] the algorithmic complexity of the theory of a class of constructive models of a finite rich signature, and William Hanf in [4] proposed a variant of the procedure for reducing signatures preserving isomorphism type of the Tarski-Lindenbaum algebra of the theory.

In this paper, we present a finite signature reduction method having the simplest form; however, the power of this procedure is maximum possible. We establish a Cartesian presentation of the predicate calculus $PC(\sigma_1)$ of a finite rich signature σ_1 in a finitely axiomatizable fragment of predicate calculus $PC(\sigma_2)$ of another finite rich signature σ_2. This is a technical result playing the role of a basis for a full-weight version of the finite signature reduction procedure; moreover, the obtained statement is well integrated as a technical component into various constructions related to the characterization of the expressive power of first-order logic.

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Preliminaries. We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, constructive models, and Boolean algebras found in [5], [6], and [7]. Generally, *incomplete theories* are considered. In the work, we consider only the signatures which admit Godel's numberings of the formulas. Such a signature is called *enumerable*.

In model theory, the notion of first-order definability is often used as a base. Along with this, there is a thinner concept of *first-order* $\exists\cap\forall$-*definability*, namely, presentability via formulas that are equivalent to both an $\exists$-formula and a $\forall$-formula in the theory. In this article, we systematically follow the algebraic approach manipulating with $\exists\cap\forall$-presentable formulas. As a $\exists\cap\forall$-formula $\varphi(\bar{x})$ of signature σ, we mean a pair of formulas $(\varphi^e(\bar{x}), \varphi^a(\bar{x}))$ of signature σ together with the *domain sentence* $DomEA(\varphi(\bar{x})) = (\forall\bar{x})[\varphi^e(\bar{x}) \leftrightarrow \varphi^a(\bar{x})]$, where $\varphi^e(\bar{x})$ is an $\exists$-formula, while $\varphi^a(\bar{x})$ is a $\forall$-formula of signature σ. The formula $\varphi(\bar{x})$ is said to be $\exists\cap\forall$-presentable in theory T if its signature matches T; moreover, $T \vdash DomEA(\varphi(\bar{x}))$. If $\psi(\bar{x})$ is a quantifier-free formula, $DomEA(\psi(\bar{x}))$ is supposed to be a generally true formula. If $\varkappa$ is a finite set (or a sequence) of $\exists\cap\forall$-formulas $\psi_i(\bar{x}_i)$, $i<k$, we denote by $DomEA(\varkappa)$ the conjunction $\bigwedge_{i<k} DomEA(\psi_i(\bar{x}_i))$.

By $\mathcal{L}(T)$, we denote the Tarski-Lindenbaum algebra of formulas of theory T without free variables considered together with a Godel numbering γ; thereby, the concept of a computable isomorphism is applicable to such objects. The following notations are used: $PC(\sigma)$ is predicate calculus of signature σ, i.e., a theory of signature σ defined by an empty set of axioms, $SL(\sigma)$ is the set of all sentences of signature σ, $FL(\sigma)$ is the set of all formulas of signature σ. When estimating types of quantifier prefixes, we suppose that any considered formula is reduced to a *prenex normal form*, cf. [8, Ch. 1, Ex. 1.36]. By $[\Sigma]^\sigma$ we denote a theory of signature σ generated by the set $\Sigma \subseteq SL(\sigma)$ as a set of its axioms. The sign $+$ is used to specify additional axioms for theories. For instance, $[T+\Phi]^\sigma$ means a theory T' obtained from the source theory T by adding a sentence Φ as an extra axiom. By *GR*, we denote *graph theory* of signature $\sigma_{GR} = \{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$, $(\forall x)(\forall y)[\Gamma(x,y) \leftrightarrow \Gamma(y,x)]$, while *GRE* is an extension of *GR* defined by the extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x \neq y) \wedge \neg\Gamma(x,y)\big]$. For theories, f.a. means *finitely axiomatizable*.

A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols. For two signatures σ_1 and σ_2, σ_1 is said to be *covered by* σ_2, written $\sigma_1 \leqslant \sigma_2$, if there is a mapping $\lambda : \sigma_1 \to \sigma_2$ such that for all $\mathfrak{s} \in \sigma_1$ the following conditions are satisfied: (a) $\mathfrak{s}$ and $\lambda(\mathfrak{s})$ are symbols of the same type (either predicates, or functions, or constants); (b) arity of $\mathfrak{s} \leqslant$ arity of $\lambda(\mathfrak{s})$ whenever $\mathfrak{s}$ is either a predicate or function symbol. By definition, the following relation takes place for an arbitrary finite signature σ:

$$\sigma \textit{ is rich } \Leftrightarrow \ \{P^2\} \leqslant \sigma \ \textit{ or } \ \{f^1, h^1\} \leqslant \sigma \ \textit{ or } \ \{g^2\} \leqslant \sigma. \tag{0.1}$$

A formula $\theta(x)$ is said to be *presenting a distinguished element* (*constant*) in a theory T if formulas $(\exists x)\theta(x)$ and $(\forall y)(\forall z)[\theta(y) \wedge \theta(z) \to (y = z)]$ are provable in T. For a constant symbol c that is not included in signature σ, the claim *on the universe set, all σ-symbols are defined c-trivially* means the following set of formulas:

$$
\begin{array}{ll}
\texttt{(a)} & (\forall x_1 \ldots x_n)\neg P(x_1,\ldots,x_n),\ P^n \in \sigma, \\
\texttt{(b)} & (\forall x_1 \ldots x_m)\big(f(x_1,\ldots,x_m)=x_1\big),\ f^m \in \sigma, \\
\texttt{(c)} & a=c,\quad \textit{for each constant symbol } a \in \sigma.
\end{array}
\tag{0.2}
$$

Let T be a theory of signature σ, $U^1 \notin \sigma$, and let $c_1, ..., c_k$ be new constants. We denote by $T\langle c_1, ..., c_k\rangle$ a theory of signature $\tau = \sigma \cup \{U(x)\} \cup \{c_1, ..., c_k\}$ defined by the following set of axioms:

1°. $\bigwedge_{0<i<j\leqslant k}(c_i \neq c_j)$,

2°. $U(x) \leftrightarrow (x \neq c_1 \wedge ... \wedge x \neq c_k)$,

3°. all axioms of T are satisfied in the domain $U(x)$,

4°. all σ-symbols are defined trivially outside the domain $U(x)$.

The theory $T\langle c_1, ..., c_k\rangle$ is said to be an *external constant extension* of T by constants $c_1, ..., c_k$.

Theories T and S of signatures τ and σ such that $\tau \cap \sigma = \varnothing$ are called *first-order* $\exists\cap\forall$-*equivalent* or *algebraically isomorphic*, written as $T \approx_a S$, if there is a theory H of signature $\tau \cup \sigma$ such that $T = H \upharpoonright \tau$, $S = H \upharpoonright \sigma$; moreover, σ-symbols are $\exists\cap\forall$-definable in H relative to τ-symbols via an effective scheme of expressions, while τ-symbols are $\exists\cap\forall$-definable in H relative to σ-symbols via an effective scheme of expressions. We turn to a general-model version of the concept. Theories T and S are called *first-order equivalent* or *isomorphic*, written as $T \approx S$, if similar relations are satisfied with the ordinary first-order definability instead of $\exists\cap\forall$-definability. It is obvious that $T \approx_a S \ \Rightarrow\ T \approx S$, for all theories T and S.

1 Interpretations for finitary first-order combinatorics

We accept a standard concept of an *interpretation* of theory T_0 in the domain $U(x)$ of theory T_1, cf. [9, Sec. 4.7]. An interpretation $I : T_0 \rightarrowtail T_1$ is defined by a mapping i, called the *basic assignment*, from signature symbols of theory T_0 into formulas of theory T_1. In a certain sense, the mapping i must preserve the number of free variables; moreover, these variables should be bounded in the domain $U(x)$. An n-ary predicate is mapped into a formula with n free variables, an n-ary function into a formula with $n{+}1$ free variables, while a constant into a formula with one free variable. Inductively, the mapping i is expanded up to a mapping $I : FL(\sigma_0) \to FL(\sigma_1)$. Any interpretation I has to satisfy the following conditions for all $\varphi \in SL(\sigma_0)$: `(a)` $T_1 \vdash (\exists x)U(x)$, `(b)` $T_0 \vdash \varphi \ \Rightarrow\ T_1 \vdash I(\varphi)$. An interpretation I is said to be *faithful* if the following extra condition takes place for all $\varphi \in SL(\sigma_0)$: $T_0 \vdash \varphi \Leftrightarrow T_1 \vdash I(\varphi)$. An interpretation I is said to be *effective* if transformation $\varphi \mapsto I(\varphi)$ is defined by a computable function with respect to the Godel numbers.

Given an interpretation I of theory T_0 of signature σ_0 in the domain $U(x)$ of theory T_1. Let $\mathfrak{M}$ be an arbitrary model of theory T_1. By virtue of the interpretation I, it is possible to define all predicates, functions, and constants of signature σ_0 within the first-order definable set $U(\mathfrak{M})$ obtaining a model $\mathfrak{N} = \langle U(\mathfrak{M}), \sigma_0\rangle$, that is said to be the *model kernel* of $\mathfrak{M}$ *with respect to* I, using the notation $\mathfrak{N} = \mathbb{K}_I(\mathfrak{M})$, or a short notation $\mathfrak{N} = \mathbb{K}(\mathfrak{M})$, when I is clear from context. An interpretation I is said to be *model-free* if $\mathrm{Mod}(T_0) = \{\mathbb{K}(\mathfrak{M}) \mid \mathfrak{M} \in \mathrm{Mod}(T_1)\}$ An interpretation I of theory T_0 in theory T_1 is said to be *isostone*, if it is model-free; moreover, the following condition is satisfied: $\mathbb{K}(\mathfrak{M}_0) \equiv \mathbb{K}(\mathfrak{M}_1) \;\Rightarrow\; \mathfrak{M}_0 \equiv \mathfrak{M}_1$, for all $\mathfrak{M}_0, \mathfrak{M}_1 \in \mathrm{Mod}(T_1)$. Consider principal properties of isostone interpretations.

Lemma 1.1. *Let I be an isostone interpretation of theory T_0 of signature σ_0 in theory T_1. Then, the mapping μ from $\mathcal{L}(T_0)$ into $\mathcal{L}(T_1)$ defined by the rule*

$$\mu([\varphi]_{T_0}) = [I(\varphi)]_{T_1}, \quad \varphi \in SL(\sigma_0), \tag{1.1}$$

is an isomorphism between these Tarski-Lindenbaum algebras. Moreover, if I is effective, the rule (1.1) *defines a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ between the Tarski-Lindenbaum algebras of theories T and S.*

Proof. Immediately, cf. [10]. □

An interpretation I of theory T_0 in theory T_1 is called *model-bijective*, if the following relations are satisfied:

`Reference_Block` (1.2)

`(a)` $\mathrm{Mod}(T_0) \;=\; \{\mathbb{K}(\mathfrak{M}) | \mathfrak{M} \in \mathrm{Mod}(T_1)\}$,

`(b)` $\mathbb{K}(\mathfrak{M}) \cong \mathbb{K}(\mathfrak{M}') \;\Leftrightarrow\; \mathfrak{M} \cong \mathfrak{M}'$, *for all* $\mathfrak{M}, \mathfrak{M}' \in \mathrm{Mod}(T_1)$.

`End_Ref`

Lemma 1.2. *Let I be a model-bijective interpretation of theory T_0 in theory T_1. Then, I is faithful, model-free and isostone. Moreover, the following relations are satisfied*:

(a) $||\mathbb{K}(\mathfrak{M})|| < \omega \;\Leftrightarrow\; ||\mathfrak{M}|| < \omega$, *for all* $\mathfrak{M} \in \mathrm{Mod}(T_1)$,

(b) $||\mathbb{K}(\mathfrak{M})|| = ||\mathfrak{M}||$, *for all infinite models* $\mathfrak{M} \in \mathrm{Mod}(T_1)$.

Proof. Immediately, [10]. □

2 Cartesian-type interpretations

Given a signature σ and a finite sequence of formulas of this signature of either of the following forms:

$$\begin{aligned} &\texttt{(a)}\ \varkappa = \langle \varphi_1^{m_1}/\varepsilon_1, \varphi_2^{m_2}/\varepsilon_2, \ldots, \varphi_s^{m_s}/\varepsilon_s\rangle, \\ &\texttt{(b)}\ \varkappa = \langle \varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_s^{m_s}\rangle, \end{aligned} \tag{2.1}$$

where φ_k is a formula with m_k free variables, $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ is a formula with $2m_k$ free variables such that $\mathrm{Len}\ \bar{y}_k = \mathrm{Len}\ \bar{z}_k = m_k$; moreover, (2.1)(b) is just a simpler notation instead of the common entry (2.1)(a) in the case when $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ coincides with $\bar{y}_k = \bar{z}_k$ for all $k \leqslant s$.

Starting from a model $\mathfrak{M}$ of signature σ together with a tuple $\varkappa$ of any of the forms (2.1)(a,b), we are going to construct a new model $\mathfrak{M}_1 = \mathfrak{M}\langle\varkappa\rangle$ of signature

$$\sigma_1 \;=\; \sigma \,\cup\, \{U^1, U_1^1, U_2^1, \ldots, U_s^1\} \,\cup \{K_1^{m_1+1}, \ldots, K_s^{m_s+1}\} \tag{2.2}$$

as follows. As the universe, we take $|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \ldots \cup A_s$, where all specified parts are pairwise disjoint sets. On the set $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in the remainder, they are defined trivially; predicate $U(x)$ distinguishes $|\mathfrak{M}|$; predicate $U_k(x)$ distinguishes A_k; the other predicates are defined by specific rules depending on the case. In the case (2.1)(b), each predicate K_k in (2.2) should be defined so that it would represent a one-to-one correspondence between the set of tuples $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}$ and the set $A_k = U_k(\mathfrak{M}_1)$. Turn to the most common case (2.1)(a). Denote by $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x})$ in $\mathfrak{M}$. In this case, (m_k+1)-ary predicate K_k should be defined so that it would represent a one-to-one correspondence between the quotient set $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}/\varepsilon'_k$ and the set $U_k(\mathfrak{M}_1)$, where

$$\varepsilon'_k(\bar{y}, \bar{z}) = \varepsilon_k(\bar{y}, \bar{z}) \vee \neg\, \mathrm{Equiv}(\varepsilon_k, \varphi_k). \tag{2.3}$$

The aim of replacement of ε_k by ε'_k using $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ is to provide total definiteness of the operation of an extension $\mathfrak{M}\langle\varkappa\rangle$ independently of whether the formulas ε_k, $k = 1, 2, .., s$, represent equivalence relations in corresponding domains or not. In the case (2.1)(a), $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian-quotient extension of* $\mathfrak{M}$, while in the case (2.1)(b), the model $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian extension of* $\mathfrak{M}$ *by a sequence of formulas* $\varkappa$.

Mention some kind of determinism for the operation under consideration.

Lemma 2.1. *Given a signature σ and a tuple $\varkappa$ of the form* (2.1)(*a*). *For a fixed choice of signature* (2.2), *Cartesian-quotient extension $\mathfrak{M}\langle\varkappa\rangle$ of the model $\mathfrak{M}$ is defined uniquely, up to an isomorphism over $\mathfrak{M}$. Moreover, any automorphism $\lambda : \mathfrak{M} \to \mathfrak{M}$ can be extended, by a unique way, up to an automorphism $\lambda^* : \mathfrak{M}\langle\varkappa\rangle \to \mathfrak{M}\langle\varkappa\rangle$.*

Proof. This statement is an immediate consequence of the construction of Cartesian-type extensions. Emphasize that, a basic idea of the construction as a whole is strictly subordinated to achieve this key property. □

We expand the operation of an extension (initially defined for models) on theories. Given a theory T and a tuple $\varkappa$ of the form (2.1). By using a fixed signature (2.2) for extensions of models, we define a new theory $T' = T\langle\varkappa\rangle$ as follows $T' = \mathrm{Th}(K)$, $K = \{\mathfrak{M}\langle\varkappa\rangle \mid \mathfrak{M} \in \mathrm{Mod}(T)\}$. In the case (2.1)(a), it is said to be a *Cartesian-quotient extension*, while in the case (2.1)(b), it is called a *Cartesian extension* of T *by a sequence* $\varkappa$. When using an entry $T\langle\varkappa\rangle$, we always suppose that theory T is applicable to the tuple $\varkappa$.

Lemma 2.2. *For any model $\mathfrak{M}$ of theory $T\langle\varkappa\rangle$, there is a model $\mathfrak{N}$ of theory T such that $\mathfrak{M} \cong \mathfrak{N}\langle\varkappa\rangle$.*

Proof. Immediately, from description of the operation $T \mapsto T\langle\varkappa\rangle$. □

In theory $T\langle\varkappa\rangle$, the domain $U(x)$ represents a model of theory T. Particularly, transformation $T \mapsto T\langle\varkappa\rangle$ defines a natural interpretation $I_{T,\varkappa}$ of the source theory T in the target theory $T\langle\varkappa\rangle$. In the common case (2.1)(a) it is called a *plain Cartesian-quotient* interpretation, while in the particular case (2.1)(b), it is called a *plain Cartesian* interpretation.

We study main properties of plain Cartesian-type interpretations.

Lemma 2.3. *Given a signature* σ *and a tuple* $\varkappa$ *of the form* (2.1)(*a*). *For a fixed choice of signature* (2.2), *Cartesian-quotient interpretation* $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ *has the following properties*:

(a) *the model-kernel passage is defined by rule* $\mathbb{K}(\mathfrak{N}\langle\varkappa\rangle) = \mathfrak{N}$, *for all* $\mathfrak{N} \in \mathrm{Mod}(T)$,

(b) $I_{T,\varkappa}$ *is* $\exists \cap \forall$*-presentable*,

(c) $I_{T,\varkappa}$ *is effective, faithful, auto-free, model-bijective, and isostone*,

(d) *interpretation* $I_{T,\varkappa}$ *determines in accordance with rule* (1.1) *a computable isomorphism* $\mu_{T,\varkappa} : \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ *between the Tarski-Lindenbaum algebras*,

(e) *for any model* $\mathfrak{N} \in \mathrm{Mod}(T)$, *isomorphism* $\mu_{T,\varkappa}$ *maps complete extension* $T' = \mathrm{Th}(\mathfrak{N})$ *of theory* T *into complete extension* $S' = \mathrm{Th}(\mathfrak{N}\langle\varkappa\rangle)$ *of theory* $T\langle\varkappa\rangle$.

Proof. (a), (b) Immediately, from construction. (c) Effectiveness of the interpretation is checked immediately. By Lemma 2.1 and Lemma 2.2, the mapping of passage to the model-kernel is a one-to-one correspondence between isomorphism types of models of the classes $\mathrm{Mod}(T\langle\varkappa\rangle)$ and $\mathrm{Mod}(T)$; thereby, interpretation $I_{T,\varkappa}$ is model bijective. By Lemma 1.2, the interpretation $I_{T,\varkappa}$ is faithful, model-free, and isostone. (d) By applying Lemma 1.1. (e) Immediately, from (a). □

Now, we pass to some class of interpretations of a more common form.

Definition 2.A. An interpretation J of a theory T in a theory S is called *Cartesian-quitient* or *Cartesian* if, up to an algebraic isomorphism of theories, it looks like $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ with a tuple of formulas $\varkappa$ of the form (2.1)(a) or, respectively, (2.1)(b); in other words, there is an algebraic isomorphism of theories $E : T\langle\varkappa\rangle \approx_a S$, such that J is presented by the following chain of passages:

$$J : T \overset{I_{T,\varkappa}}{\rightarrowtail} T\langle\varkappa\rangle \overset{E}{\rightarrowtail} S. \tag{2.4}$$

In the case, if we follow the general-model approach, a simple isomorphism $E : T\langle\varkappa\rangle \approx S$ should be applied in the chain (2.4).

Obviously, any plain Cartesian-quotient interpretation is a Cartesian-quotient interpretation, while any plain Cartesian interpretation is a Cartesian interpretation.

The algebraic approach we are systematically developing in this paper requeres to accept demands of $\exists \cap \forall$-presentability of all formulas in the sequence (2.1):

$$\begin{aligned} &\texttt{(a)}\ \varphi_k(\bar{x}_k) \textit{ is } \exists \cap \forall\textit{-presentable},\ 1 \leqslant k \leqslant s, \\ &\texttt{(b)}\ \varepsilon_k(\bar{y}_k\bar{z}_k) \textit{ is } \exists \cap \forall\textit{-presentable},\ 1 \leqslant k \leqslant s. \end{aligned} \tag{2.5}$$

Obviously, demand (2.5)(b) is automatically satisfied in the case we accept (2.5)(a) for a Cartesian extension with the tuple of the form (2.1)(b).

We denote by $\mathcal{KD}(\sigma)$ and $\mathcal{KC}(\sigma)$ the sets of tuples of formulas of signature σ of the forms, respectively, (2.1)(a) and (2.1)(b), while $\mathcal{KD}$ and $\mathcal{KC}$ are unions of these sets for all possible (enumerable) signatures σ. By $\mathcal{KC}_{\exists\cap\forall}$ we denote the set of all tuples (2.1)(b) satisfying (2.5)(a), while $\mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$ denotes the set of all tuples (2.1)(a) satisfying (2.5)(a,b). By applying an entry $T\langle\varkappa\rangle$, we always count that theory T is applicable to the tuple $\varkappa$, moreover, $T \vdash DomEA(\varkappa)$ is satisfied ensuring that T is in the domain of $\exists\cap\forall$-presentability of each of the formulas $\varphi_k(\bar{x}_k)$ and $\varepsilon_k(\bar{y}_k\bar{z}_k)$, $i = 1, ..., m$, in the tuple $\varkappa$.

In this paper, we systematically follow the *algebraic approach* accepting demands (2.5)(a,b) for the passage $T \mapsto T\langle\varkappa\rangle$ in all cases when the contrary is not mentioned. Moreover, we focus our attention on the case of a Cartesian extension (2.1)(b). As for the common case (2.1)(a) of a Cartesian-quotient extension, we concern this case of the operation for comparison purposes. Further in the paper, by applying an extra specifier *algebraic*, we point out explicitly that the algebraic approach is accepted. For instance, passage $T \mapsto T\langle\varkappa\rangle$ is called an *algebraic Cartesian-quotient extension* whenever $\varkappa \in \mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$, interpretation $I_{T,\varkappa}$ is called a *plain algebraic Cartesian interpretation* if $\varkappa \in \mathcal{KC}_{\exists\cap\forall}$, etc.

Notice that, the interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ is $\exists\cap\forall$-presentable for an arbitrary tuple $\varkappa \in \mathcal{KD}$. We obtain some extra properties of the interpretation whenever we additionally accept the demand of being algebraic for $\varkappa$.

Remark 2.4. An algebraic isomorphism of theories $T \approx_a S$ represents a particular case of an algebraic Cartesian interpretation with a tuple $\varkappa$ that is an empty sequence of formulas, cf. (2.4).

We study Cartesian-type extensions with respect to isomorphisms of theories.

Lemma 2.5. *The following assertions hold*:

(a) *Given a theory T and a tuple $\varkappa$ in $\mathcal{KD}$. Transformation $T \mapsto T\langle\varkappa\rangle$ is defined correctly on $\approx$-classes of theories; i.e., if $T \approx S$ and $\varkappa'$ is an image of $\varkappa$ relative to this isomorphism, then $\varkappa' \in \mathcal{KD}$; moreover, we have $T\langle\varkappa\rangle \approx S\langle\varkappa'\rangle$. The same is also true for the case of Cartesian extensions with $\varkappa, \varkappa' \in \mathcal{KC}$.*

(b) *Given a theory T and a tuple $\varkappa$ in $\mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$. Transformation $T \mapsto T\langle\varkappa\rangle$ is defined correctly on $\approx_a$-classes of theories; i.e., if $T \approx_a S$ and $\varkappa'$ is an image of $\varkappa$ relative to this algebraic isomorphism, then $\varkappa' \in \mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$; moreover, we have $T\langle\varkappa\rangle \approx_a S\langle\varkappa'\rangle$. The same is also true for the case of algebraic Cartesian extensions with $\varkappa, \varkappa' \in \mathcal{KC}_{\exists\cap\forall}$.*

Proof. (a) This part represents a simplified version of (b), cf. below.

(b) From $T \approx_a S$ we obtain $\varkappa' \in \mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$; moreover, length of $\varkappa'$ is the same as that of $\varkappa$ and dimensions m_i, $i = 1, 2, ..., s$ are common for these sequences. By virtue of $T \approx_a S$ all τ-symbols in $T\langle\varkappa\rangle$ are $\exists\cap\forall$-expressible via σ-symbols in $S\langle\varkappa'\rangle$, and vise versa, σ-symbols of $S\langle\varkappa'\rangle$ are $\exists\cap\forall$-expressible via τ-symbols in $T\langle\varkappa\rangle$. From (2.2), we see that all remainder signature symbols of $T\langle\varkappa\rangle$ are U^1, U^1_i and $K^{m_i}_i$, $i = 1, 2, ..., s$, as well as their mirror images used in theory $S\langle\varkappa'\rangle$. These symbols are obviously mutually $\exists\cap\forall$-expressible via each other. Notice that, quantifiers in formulas ε'_i, $i = 1, 2, ..., s$, due to the member $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ in (2.3) will have identical actions in both extensions $T\langle\varkappa\rangle$ and $S\langle\varkappa'\rangle$, thus, these quantifiers do not make obstacle within the algebraic isomorphism $T\langle\varkappa\rangle \approx_a S\langle\varkappa'\rangle$. Obviously, the case

$\varkappa, \varkappa' \in \mathcal{K}C_{\exists\cap\forall}$ is covered by the general case $\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$. $\square$

We study combinatorial properties of Cartesian-type interpretations.

Lemma 2.6 [Combinatorial Lemma for Cartesian-type extensions]. *Given a theory T of signature σ together with a sequence of formulas $\varkappa$. The following statements are satisfied, where all pointed out passages are effective with respect to Godel's numbers of tuples of formulas* (*it is supposed that the choice of tuples is limited with the condition of applicability to corresponding theories*):

(a) *Suppose that $\varkappa \in \mathcal{K}D$. For any $\varkappa'$ in $\mathcal{K}D$, there is a tuple $\varkappa''$ in $\mathcal{K}D$ such that an isomorphism*

$$T\langle \varkappa \,\hat{}\, \varkappa' \rangle \approx \big(T\langle \varkappa \rangle\big)\langle \varkappa'' \rangle \tag{2.6}$$

takes place; and vice versa, for any $\varkappa''$ in $\mathcal{K}D$, there is a tuple $\varkappa'$ in $\mathcal{K}D$ such that an isomorphism (2.6) *takes place.*

(b) *Suppose that $\varkappa \in \mathcal{K}C$. For any $\varkappa'$ in $\mathcal{K}C$, there is a tuple $\varkappa''$ in $\mathcal{K}C$ such that an isomorphism* (2.6) *takes place; and vice versa, for any $\varkappa''$ in $\mathcal{K}C$, there is a tuple $\varkappa'$ in $\mathcal{K}C$ such that an isomorphism* (2.6) *takes place.*

(c) *Suppose that $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$. For any $\varkappa'$ in $\mathcal{K}C_{\exists\cap\forall}$, there is a tuple $\varkappa''$ in $\mathcal{K}C_{\exists\cap\forall}$ such that an isomorphism*

$$T\langle \varkappa \,\hat{}\, \varkappa' \rangle \approx_a \big(T\langle \varkappa \rangle\big)\langle \varkappa'' \rangle \tag{2.7}$$

takes place; and vice versa, for any $\varkappa''$ in $\mathcal{K}C_{\exists\cap\forall}$, there is a tuple $\varkappa'$ in $\mathcal{K}C_{\exists\cap\forall}$ such that an isomorphism (2.7) *takes place.*

Proof. Validity of these statements can be checked by applying a routine construction based on expressive possibilities of first-order logic. $\square$

Consider a common statement concerning compositions.

Lemma 2.7. *Let $I : T \rightarrowtail S$ and $J : S \rightarrowtail R$ be interpretations. If both I and J are algebraic Cartesian interpretations, their composition $J \circ I : T \rightarrowtail R$ is also an algebraic Cartesian interpretation.*

Proof. Based on the scheme (2.4) for the case of an algebraic Cartesian interpretation, we obtain for some $\varkappa, \xi \in \mathcal{K}C_{\exists\cap\forall}$ the following chain of passages $T \mapsto T\langle\varkappa\rangle \approx_a S \mapsto S\langle\xi\rangle \approx_a R$. Denote theory $T\langle\varkappa\rangle$ by S^*. By applying Lemma 2.5(b) to the plain algebraic Cartesian extension $S \mapsto S\langle\xi\rangle$ together with an algebraic isomorphism $S \approx_a S^*$, we find a tuple $\xi' \in \mathcal{K}C_{\exists\cap\forall}$ together with an algebraic isomorphism $S\langle\xi\rangle \approx_a S^*\langle\xi'\rangle$. As a result, the following chain of relations arise: $T \mapsto T\langle\varkappa\rangle = S^* \mapsto S^*\langle\xi'\rangle \approx_a S\langle\xi\rangle \approx_a R$. By applying Lemma 2.6(c), we can join successive passages via tuples $\varkappa$ and ξ' in a single passage via a tuple $\xi'' \in \mathcal{K}C_{\exists\cap\forall}$, thus, eliminating an intermediate step. Thereby, we obtain that $J \circ I : T \rightarrowtail R$ is an algebraic Cartesian interpretation.

Lemma 2.7 is proved. $\square$

We pass to some principal applications of the combinatorial statement.

Lemma 2.8. *The following relation defined on the class of all theories*

$$T \approxeq_a S \Leftrightarrow_{dfn} (\exists \varkappa' \varkappa'' \in \mathcal{K}C_{\exists\cap\forall}) \left[\, T\langle \varkappa' \rangle \approx_a S\langle \varkappa'' \rangle \,\right] \tag{2.8}$$

is reflexive, symmetric, and transitive (that is, this is an equivalence relation). Besides, it is possible to define a new relation by the following rule

$$\begin{aligned} &T \mathrel{\mathring{\approxeq}}_a S \ \Leftrightarrow_{dfn} \big(\exists \text{ computable isomorphism } \mu : \mathcal{L}(T) \to \mathcal{L}(S)\big) \\ &(\forall \text{ complete extension } T' \supseteq T)\,(\forall \text{ complete extension } S' \supseteq S) \\ &[S' = \mu(T') \ \Rightarrow \ (\exists \varkappa' \varkappa'' \in \mathcal{K}C_{\exists\cap\forall})(T'\langle \varkappa \rangle \approx_a S'\langle \varkappa \rangle)]. \end{aligned} \tag{2.9}$$

It is also an equivalence relation on the class of all theories. Moreover, we have $T \approxeq_a S \Rightarrow T \mathrel{\mathring{\approxeq}}_a S$ *for all theories* T *and* S*, and* $T_1 \approxeq_a T_2 \Leftrightarrow T_1 \mathrel{\mathring{\approxeq}}_a T_2$ *for all complete theories* T_1 *and* T_2.

Proof. Obviously, $\approxeq_a$ it is reflexive and symmetric. Now, suppose that $T \approxeq_a H$ and $H \approxeq_a S$ are satisfied. By definition, there are tuples $\xi_i \in \mathcal{K}C_{\exists\cap\forall}$, $i = 1, 2, 3, 4$, such that $T\langle \xi_1 \rangle \approx_a H\langle \xi_2 \rangle$ and $H\langle \xi_3 \rangle \approx_a S\langle \xi_4 \rangle$. By applying Lemma 2.6(c), we can find tuples ξ_2' and ξ_3' in $\mathcal{K}C_{\exists\cap\forall}$ such that the following algebraic isomorphisms take place: $T\langle \xi_1 \hat{\ } \xi_3' \rangle \approx_a H\langle \xi_2 \hat{\ } \xi_3 \rangle \approx_a H\langle \xi_3 \hat{\ } \xi_2 \rangle \approx_a S\langle \xi_4 \hat{\ } \xi_2' \rangle$. Thus, we obtain $T \approxeq_a S$, ensuring the transitivity property. The fact that relation (2.9) is reflexive, symmetric, and transitive on the class of all theories, is checked immediately. As for the pointed out links between the relations $\approxeq_a$ and $\mathring{\approxeq}_a$, they are derived based on definitions (2.8) and (2.9) together with properties of the computable isomorphisms μ, cf. Lemma 2.3. □

Relation $\approxeq_a$ defined on the class of all theories by rule (2.8) is called the relation of *integrated type*. A relation $\approxeq_a \restriction \mathbb{C}$ obtained from $\approxeq_a$ by restriction on the class of all complete theories is called the *radical part* of $\approxeq_a$. As for relation $\mathring{\approxeq}_a$ that is defined on the class of all theories by rule (2.9), it is said to be the *regular extension* of the *radical part* $\approxeq_a \restriction \mathbb{C}$. Although the right-hand side of expression (2.9) refers to the full relation $\approxeq_a$, nevertheless, its radical part, $\approxeq_a \restriction \mathbb{C}$, is actually used because just the case of complete theories is involved in the condition at the last line in (2.9). As an alternative manner, when we wish to compare relation (2.8) as opposite to (2.9), the former is said to be a *monolithic-type* relation, while the latter is said to be a *pointwise-type* relation.

We study quantifier prefixes of formulas in theories we are considering.

We consider a theory T together with a sequence $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$. Cartesian-quotient extension $T \mapsto T\langle \varkappa \rangle$ defines an interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle \varkappa \rangle$ that is isostone, auto-free, and model-bijective, while the passage to the model-kernel is defined by rule $\mathbb{K}(\mathfrak{N}\langle \varkappa \rangle) = \mathfrak{N}$. It is easy to check that, for any model $\mathfrak{N}$ of signature σ, we have $\mathfrak{N} \models \psi \ \Leftrightarrow \ \mathfrak{N}\langle \varkappa \rangle \models (\psi)_U$, for $\psi \in SL(\sigma)$. Moreover, the following relation is satisfied for all $\psi \in SL(\sigma)$:

$$\textit{quantifier prefix of } (\psi)_U \textit{ is of the same type as that of } \psi. \tag{2.10}$$

We estimate quantifier prefixes of the key formulas.

Lemma 2.9. *Suppose that demands* (2.5)(a, b) *are satisfied for a tuple of formulas* $\varkappa$ *of the form* (2.1)*(a) used in the extension* $T \mapsto T\langle\varkappa\rangle$*. The following estimates for quantifier prefixes take place for all* k *satisfying* $1 \leqslant k \leqslant s$:

(a) *formula* $\varepsilon'_k(\bar{x}, \bar{y})$ *is an* $\exists$*-formula in* T,

(b) *formula* $\varepsilon'_k(\bar{x}, \bar{y})$ *is* $\exists \cap \forall$*-presentable in* T *whenever sentence* $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ *is either identically true or identically false in* T.

Proof. (a) By definition, $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ is a conjunction of three formulas:

$$\begin{aligned}
&\texttt{(a)}\quad (\forall \bar{x})[\varphi_k(\bar{x}) \to \varepsilon_k(\bar{x}, \bar{x})],\\
&\texttt{(b)}\quad (\forall \bar{x}\bar{y})[\varphi_k(\bar{x}) \wedge \varphi_k(\bar{y}) \wedge \varepsilon_k(\bar{x}, \bar{y}) \to \varepsilon_k(\bar{y}, \bar{x})],\\
&\texttt{(c)}\quad (\forall \bar{x})[\varphi_k(\bar{x}) \wedge \varphi_k(\bar{y}) \wedge \varphi_k(\bar{z}) \wedge \varepsilon_k(\bar{x}, \bar{y}) \wedge \varepsilon_k(\bar{y}, \bar{z}) \to \varepsilon_k(\bar{x}, \bar{z})].
\end{aligned}$$

According to (2.5)(a,b), both ε_k and φ_k are $\exists \cap \forall$-formulas in T. From this, we obtain that $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ is a $\forall$-sentence. From (2.3), we obtain that modified formula $\varepsilon'_k(\bar{x}, \bar{y})$ is an $\exists$-formula. Part (b) immediately follows from (2.3). $\square$

New domains $U_i(x)$, $i = 1, 2, ..., s$, are obtained by applying the standard quotient construction of first-order definable sets modulo definable equivalences in T. These relations are presented in $T\langle\varkappa\rangle$ by the following formulas acting in the model-kernel:

$$\begin{aligned}
&\texttt{(a)}\quad \breve{\varphi}_k(\bar{x}) = (\bar{x} \subseteq U) \wedge (\varphi_k(\bar{x}))_U\,, \qquad (2.11)\\
&\texttt{(b)}\quad \breve{\varepsilon}'_k(\bar{x}, \bar{y}) = (\bar{x} \subseteq U) \wedge (\bar{y} \subseteq U) \wedge (\varepsilon'_k(\bar{x}, \bar{y}))_U\,.
\end{aligned}$$

Lemma 2.10. *Formula* $\breve{\varepsilon}'_k$ *is* $\exists \cap \forall$*-presentable in* $T\langle\varkappa\rangle$ *whenever the formula* ε'_k *is* $\exists \cap \forall$*-presentable in* T*, for* $k = 1, ..., s$.

Proof. Immediately, from (2.10) and (2.11). $\square$

Now, we formalize the operation of a Cartesian-quotient extension $T \mapsto T\langle\varkappa\rangle$, $\varkappa \in \mathcal{K}D$, in accordance with the informal description given in Section 2.

System of axioms of theory $T\langle\varkappa\rangle$ includes the following sentences:

$\exists$: 1°. $(\exists x)\, U(x)$,

$\exists$: 2°. $(\exists x)\, U_i(x)$, $i = 1, 2, ..., s$,

$\forall$: 3°. $(\forall x)\, [\, U(x) \to \neg U_i(x)\,]$, $i = 1, 2, ..., s$,

$\forall$: 4°. $(\forall x)\, [\, U_i(x) \to \neg U_j(x)\,]$, $1 \leqslant i < j \leqslant s$,

$\forall$: 5°. All σ-predicates are defined trivially outside the domain $U(x)$,

$\forall$: 6°. All σ-functions are defined trivially outside the domain $U(x)$,

7°. $(\Phi)_U$, for all $\Phi \in SL(\sigma)$, such that $\Phi \in \Sigma$ (Σ is a set of axioms of T),

$\forall$: 8°. $(\forall x_1 ... x_{m_k} z)\, \big[\, K_k(x_1, ..., x_{m_k}, z) \to U(x_1) \wedge ... \wedge U(x_{m_k}) \wedge U_k(z)\, \big]$, $k = 1, ..., s$,

$\forall$: 9°. $(\forall \bar{x} z)\, \big[\, K_k(\bar{x}, z) \to \bar{x} \subseteq U \wedge \breve{\varphi}_k(\bar{x}) \wedge U_k(z)\, \big]$, $k = 1, ..., s$,

$\forall\exists$:10°. $(\forall\bar{x})\,\big[\,\bar{x}\subseteq U\wedge\check{\varphi}_k(\bar{x})\to(\exists z)K_k(\bar{x},z)\,\big]$, $k=1,...,s$,
$\forall\exists$:11°. $(\forall z)\,\big[U_k(z)\to(\exists\bar{x})\,\big(\,\bar{x}\subseteq U\wedge\check{\varphi}_k(\bar{x})\wedge K_k(\bar{x},z)\,\big)\,\big]$, $k=1,...,s$,
$\forall$: 12°. $(\forall\bar{x}\,\bar{y}zu)\,\big[\,\check{\varphi}_k(\bar{x})\wedge\check{\varphi}_k(\bar{y})\wedge\check{\varepsilon}'_k(\bar{x},\bar{y})\wedge K_k(\bar{x},z)\wedge K_k(\bar{y},u)\to z=u\,\big]$, $k=1,...,s$,
$\forall\exists$:13°. $(\forall\bar{x}\,\bar{y}\,z)\,\big[\check{\varphi}_k(\bar{x})\wedge\check{\varphi}_k(\bar{y})\wedge K_k(\bar{x},z)\wedge K_k(\bar{y},z)\to\check{\varepsilon}'_k(\bar{x},\bar{y})\,\big]$, $k=1,...,s$,
$\forall$: 14°. $(\forall\bar{x}\,\bar{y}\,z)\,\big[\,\check{\varphi}_k(\bar{x})\wedge\check{\varphi}_k(\bar{y})\wedge K_k(\bar{x},z)\wedge\check{\varepsilon}'(\bar{x},\bar{y})\to K_k(\bar{y},z)\,\big]$, $k=1,...,s$.

By $\mathit{FRM}(\varkappa)$ we denote the set of sentences included in Axioms 1°-6° and 7°-14°. The set $\mathit{FRM}(\varkappa)$ is called a *framework* of the operation $T\mapsto T\langle\varkappa\rangle$. This part of axioms participates in the operation with this tuple $\varkappa$ for different input theories. Actually, $\mathit{FRM}(\varkappa)$ depends not only on $\varkappa$, but also on the signature σ of theory T, and on a fixed signature (2.2) accepted for the construction $T\langle\varkappa\rangle$.

Lemma 2.11. *Consider a signature σ and a tuple $\varkappa\in\mathcal{K}D(\sigma)$. Let also σ_1 be a fixed signature connected with σ via relation* (2.2). *The following statements are satisfied for all theories T of signature σ:*

(a) *the set $\mathit{FRM}\langle\varkappa\rangle$ is finite,*

(b) *$I_{T,\varkappa}$ is an $\exists\cap\forall$-presentable interpretation,*

(c) *$I_{T,\varkappa}(\varphi)=(\varphi)_U$, for all $\varphi\in SL(\sigma)$,*

(d) *$T\langle\varkappa\rangle=\mathit{FRM}(\varkappa)+\{I_{T,\varkappa}(\varphi)\mid\varphi\in\Sigma\}$, where Σ is a system of axioms of T,*

(e) *T is finitely axiomatizable $\Leftrightarrow$ $T\langle\varkappa\rangle$ is finitely axiomatizable, whenever the signature of theory T is finite,*

(f) *T is computably axiomatizable $\Leftrightarrow$ $T\langle\varkappa\rangle$ is computably axiomatizable,*

(g) *if $\varkappa\in\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$, the set $\mathit{FRM}\langle\varkappa\rangle$ consists of formulas with quantifier prefixes of complexity not more than $\forall\exists$.*

Proof. Parts (a), (b), (c) and (d) are corollaries of axioms 11°-14°. An implication $\Rightarrow$ in (e) follows from (a) and (d). We prove the back implication. Assume that Δ is a finite system of axioms of theory $T\langle\varkappa\rangle$, while Σ a system of axioms of T. It follows from (c) and (d) that Δ is deduced from $\mathit{FRM}(\varkappa)+(\Sigma)_U$. Since Δ is finite, by Maltsev's Compactness Theorem, there is a finite subset $\Sigma_0\subseteq\Sigma$ such that Δ is deduced from $\mathit{FRM}(\varkappa)+(\Sigma_0)_U$. In view of (d) and the demand of being model-bijective for interpretation $I_{T,\varkappa}$, the set Σ_0 has to be a system of axioms of theory T. Part (f) is proved similarly to Part (e). Part (g) is checked based on the left-hand side comments to axioms 12°-14° pointing out estimates of quantifier prefixes of each of the axioms in the set $\mathit{FRM}(\varkappa)$ in the case $\varkappa\in\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$. □

3 Exact interpretations

In this section, we study a class of interpretations that is in close connection to the classes of Cartesian and Cartesian-quotient interpretations between theories.

Given an interpretation I of theory T in the domain $\mathcal{U}(x)$ of theory S. Interpretation I is said to be *capturing*, if we have

$$\mathrm{acl}(\mathcal{U}(\mathfrak{M}))=|\mathfrak{M}|,\ \textit{for all}\ \mathfrak{M}\in\mathrm{Mod}(S). \tag{3.1}$$

Demand (3.1) means that each element a in any model $\mathfrak{M}$ of theory S is first-order definable with constants in $\mathcal{U}(\mathfrak{M})$. For this, it is enough to focus our attention on the inclusion $\mathrm{acl}(\mathcal{U}(\mathfrak{M})) \supseteq |\mathfrak{M}| \smallsetminus \mathcal{U}(\mathfrak{M})$. By Maltsev's Compactness Theorem, we can choose a finite collection of formulas

$$\lambda_1(\bar{u}_1, x),\ \lambda_2(\bar{u}_2, x),\ \dots,\ \lambda_s(\bar{u}_s, x) \in FL(\sigma),\ \mathrm{Len}(\bar{u}_i) = m_i, \tag{3.2}$$

called a *finite realization* for (3.1), such that, for any model $\mathfrak{M}$ of theory S, each element a in $|\mathfrak{M}| \smallsetminus \mathcal{U}(\mathfrak{M})$ is first-order definable with constants in $\mathcal{U}(\mathfrak{M})$ by means of one of the formulas (3.2); moreover, the following restrictions are held:

$$\lambda_i(z_1, \dots, z_{m_i}, x) \to \neg\,\mathcal{U}(x)\ \wedge\ \mathcal{U}(z_1)\ \wedge\ \dots\ \wedge\ \mathcal{U}(z_{m_i}),\ i = 1, 2, ..., s. \tag{3.3}$$

A realization system (3.2) for (3.1) is said to be *normalized*, if each element $a \in |\mathfrak{M}| \smallsetminus \mathcal{U}(\mathfrak{M})$ in any model $\mathfrak{M}$ of theory S is first-order definable under $\mathcal{U}(\mathfrak{M})$ by means of at most one of the formulas $\lambda_i(\bar{u}_i, x)$, $i = 1, ..., s$. It is possible to apply a simple method to reduce an arbitrary realization system (3.2) for I to a normalized form. We say that the *inverse uniqueness property* is satisfied for realization (3.2) if each formula $\lambda_i(\bar{z}_i, x)$, $i = 1, ..., s$, has at most one solution $\bar{z}_i$ in the domain $\mathcal{U}(x)$ for any fixed element $x \in |\mathfrak{M}| \smallsetminus \mathcal{U}(\mathfrak{M})$ in any model $\mathfrak{M}$ of theory S.

Now, we turn to the principal classes of interpretations.

An interpretation I of theory T_0 in the domain $U(x)$ of theory T_1 is said to be *auto-free*, if the following condition is satisfied:

$$\big(\forall\, \mathfrak{M} \in \mathrm{Mod}\, T_1\,\big)\big(\forall \mu \in \mathrm{Aut}\, \mathbb{K}(\mathfrak{M})\,\big)\big(\exists \mu^* \in \mathrm{Aut}\, \mathfrak{M}\big)\ \big[\, \mu = \mu^* \restriction U(\mathfrak{M})\,\big]. \tag{3.4}$$

An interpretation I of theory T_0 in the domain $\mathcal{U}(x)$ of theory T_1 is said to be an *exact* interpretation if the following conditions are satisfied:

`Reference_Block` (3.5)

(a) *interpretation I is model-bijective*;

(b) *each element a in any model $\mathfrak{M} \in \mathrm{Mod}(T_1)$ is first-order definable with constants in the set $\mathcal{U}(\mathfrak{M})$*;

(c) *for any model $\mathfrak{M}$ of theory T_1, an arbitrary automorphism of the model-kernel μ: $\mathbb{K}(\mathfrak{M}) \to \mathbb{K}(\mathfrak{M})$ can be extended up to an automorphism $\mu^* : \mathfrak{M} \to \mathfrak{M}$ of the whole model.*

`End_Ref`

Lemma 3.1. *Any exact interpretation I of theory T_0 in theory T_1 is auto-free, model-bijective, and isostone.*

Proof. Condition (3.5)(a) ensures that I is a model-bijective interpretation. Lemma 1.2 provides that this interpretation is isostone. Condition (3.5)(c) exactly states that I is auto-free, cf. definition in (3.4).

Lemma 3.1 is proved. □

The following result establishes a close connection between the concepts of an exact interpretation and a Cartesian-quotient interpretation.

Lemma 3.2. *The following assertions hold*:

(a) *Any Cartesian-quotient interpretation is an exact interpretation.*

(b) *Let J be an exact interpretation of a theory T of signature τ in the domain $\mathcal{U}(x)$ of a theory S of signature σ. An arbitrary normalized finite realization* (3.2) *to the demand* (3.5)(b) *for J produces a sequence of formulas of signature τ*

$$\xi = \langle \varphi_1^{m_1}/_{\varepsilon_1}, \varphi_2^{m_2}/_{\varepsilon_2}, \dots, \varphi_s^{m_s}/_{\varepsilon_s} \rangle \tag{3.6}$$

together with an interpretation E from $T\langle\xi\rangle$ to S that is a general-model isomorphism of theories, such that the following diagram is commutative:

$$\begin{tikzcd} T \arrow[rr, "J"] \arrow[dr, "I_{T,\varkappa}"'] & & S \\ & T\langle\varkappa\rangle \arrow[ur, "E"'] & \end{tikzcd} \tag{3.7}$$

Moreover, the target tuple (3.6) *has the form* (2.1)(b) *whenever accepted realization* (3.2) *satisfies the inverse uniqueness property.*

Proof. By applying standard methods based on Beth's First-Order Definability Theorem, [5, Th. 5.5.4]. A sketch of the proof can be found in [11, Lem. 4.2]. □

We turn to an extra statement on exact interpretations by specifying environments available in the proof of Lemma 3.2 as well as in their yields:

Lemma 3.3. *Let J be an exact interpretation of a theory T of signature τ in the domain $\mathcal{U}(x)$ of a theory S of signature σ, with a tuple of formulas* (3.6) *yielded by applying Lemma* 3.2 *to a given normalized finite realization system* (3.2). *Let also the following conditions be satisfied*:

`Reference_Block` (3.8)

- (a) *the domain formula $\mathcal{U}(x)$ is $\exists\cap\forall$-presentable in S,*
- (b) *all formulas of basic assignments for J are $\exists\cap\forall$-presentable in S,*
- (c) *each formula $\lambda_i(\bar{z}_i, x)$ is $\exists\cap\forall$-presentable in S, $i = 1, ..., s$,*
- (d) *each formula $\varphi_i(\bar{x}_i)$ is $\exists\cap\forall$-presentable in T, $i = 1, ..., s$,*
- (e) *each formula $\varepsilon_i(\bar{x}_i, \bar{y}_i)$ is $\exists\cap\forall$-presentable in T, $i = 1, ..., s$,*
- (f) *all σ-symbols are $\exists\cap\forall$-presentable in S with respect to the formulas $\mathcal{U}(x)$, $\mathcal{U}_i(x)$, $\lambda_i(\bar{z}_i, x)$, and formulas of basic assignments for J.*

`End_Ref`

Then, J is an algebraic Cartesian-quotient interpretation with $\varkappa$ in $\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$. Moreover, J is an algebraic Cartesian interpretation with $\varkappa$ in $\mathcal{K}C_{\exists\cap\forall}$ whenever accepted realization (3.2) *satisfies the inverse uniqueness property.*

Proof. By applying standard methods of model theory. Having conditions listed in Lemma

3.3, it is possible to obtain that sequence (3.6) will have the form (2.1)(b) with $\varkappa$ in $\mathcal{K}C_{\exists\cap\forall}$, while isomorphism E in (3.7) is an algebraic isomorphism of theories.

Lemma 3.3 is proved. $\square$

4 Main Theorem

We formulate the main statement:

Theorem 4.1 [FINITE SIGNATURE REDUCTION PROCEDURE: PRIMITIVE FORM]. *Given two finite rich signatures τ and σ. Effectively in Godel's numbers of τ and σ, it is possible to find a sequence of formulas $\varkappa = \langle \varphi_1^{m_1}, ..., \varphi_s^{m_s} \rangle$ of signature τ satisfying* (2.5)(*a*) *and a $\forall\exists$-sentence Ψ of signature σ together with an algebraic isomorphism $\lambda :\ PC(\tau)\langle\varkappa\rangle \approx_a [PC(\sigma) + \Psi]^{\sigma}$.*

Proof. Our plan is to prove the following more common statement (supposing effectiveness of each choice):

$$\begin{array}{l} \textit{for two given finite rich signatures } \tau \textit{ and } \sigma, \\ \textit{we can find a tuple } \varkappa \in \mathcal{K}C_{\exists\cap\forall}, \textit{ such that}: \\ \big(\forall \textit{f.a. theory } T \supseteq PC(\tau)\big)\ \big(\exists \textit{f.a. theory } S \supseteq PC(\sigma)\big)\big[\,T\langle\varkappa\rangle \approx_a S\,\big]. \end{array} \tag{4.1}$$

Results of Section 2 establish technical properties of algebraic isomorphisms and Cartesian extensions of theories. Based on them, we can state that there is an interpretation $I : T \rightarrowtail S$ for the theories involved in (4.1) that is a composition of two following interpretations

$$T \overset{I'}{\rightarrowtail} T\langle\varkappa\rangle \overset{I''}{\rightarrowtail} S,\ \textit{where}: \tag{4.2}$$

(a) tuple $\varkappa$ has the form (2.1)(b) satisfying condition (2.5)(a),

(b) I' is a plain algebraic Cartesian interpretation,

(c) I'' is an algebraic isomorphism of theories.

It is possible to see that Theorem 4.1 is an immediate consequence of statement (4.1) because the former is a particular case of the latter with $T = PC(\tau)$, where, Ψ is the λ-image of a conjunction of sentences from $FRM(\varkappa)$, thus, Ψ must be a $\forall\exists$-sentence by virtue of Lemma 2.11(g).

Now, we focus our attention on the proof of statement (4.1).

Given two finite rich signatures τ and σ together with a finitely axiomatizable theory T of signature τ. Based on the property (0.1), we organize a signature reduction procedure consisting of two parts. In the first part, a reduction to any of three following "minimal" finite rich signatures

$$\rho' = \{P^2\},\ \rho'' = \{f^1, h^1\},\ \rho''' = \{g^2\} \tag{4.3}$$

is performed, while in the second part, a routine passage from either ρ' or ρ'' or ρ''' to the demanded finite rich signature σ is performed depending on which of the cases $\rho' \leqslant \sigma$ or $\rho'' \leqslant \sigma$ or $\rho''' \leqslant \sigma$ takes place.

We use a collection of transformations of theories consisting of the following five *elementary stages*:

$$\begin{aligned}&\textsf{finsig-to-fP},\\&\textsf{fP-to-Graph},\ \textsf{Graph-to-2u},\ \textsf{Graph-to-1b},\\&\textsf{Enrich}.\end{aligned} \tag{4.4}$$

Scheme in Fig. 4.1 represents interaction of elementary transformations (4.4). Circled digits and letters are used in the scheme to highlight intermediate points in order to have a possibility to explain different paths through the schemeAs an input, the scheme requires a theory T of a finite signature τAs an output, the scheme yields a theory S of the pre-specified finite rich signature σ.

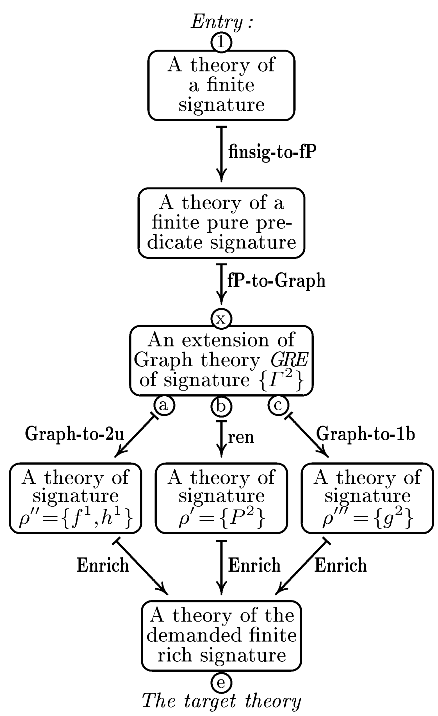

Figure 4.1 – A scheme for the finite signature reduction procedure

Now, we turn to the description of the elementary transformations.

We begin to describe the stage finsig-to-fP.

Given an arbitrary theory T of a finite signature τ. Our aim is to construct a theory S of a finite pure predicate signature τ' consisting of predicates of arities $\geqslant 1$ together with an interpretation $I : T \rightarrowtail S$ that is an algebraic isomorphism of theories.

Transformation finsig-to-fP is a standard transformation performing replacement of non-predicate symbols in a predicate form. An n-ary function $f(x_1, ..., x_n)$ is replaced by a $(n+1)$-

ary predicate presenting graphic of the function; a constant c is replaced by a unary predicate distinguishing an element presenting the value of the constant. Additionally, we should replace each nulary predicate X via a unary predicate $V(x)$ satisfying $(\forall x)[V(x)] \vee (\forall x)[\neg V(x)]$ by the rule $X \leftrightarrow (\exists z)V(z) \leftrightarrow (\forall z)V(z)$. Obviously, this transformation represents an algebraic isomorphism of theories. Thereby, specifications (4.2)(a,b,c) are indeed satisfied.

Description of the stage $\mathsf{finsig\text{-}to\text{-}fP}$ is complete.

We begin to describe the stage Enrich.

Given a theory T of a finite signature σ. We assume that T has a first-order $\exists \cap \forall$-definable element. Let also σ' be a finite signature such that $\sigma \leqslant \sigma'$, cf. Preliminaries for definition of the elation $\leqslant$ on signatures. We are going to construct a theory T' of signature σ' together with an interpretation $I : T \rightarrowtail S$ that is an algebraic isomorphism of theories. Notation $T' = \mathsf{Enrich}(T, \sigma')$ is used for the transformation; a short entry $T' = \mathsf{Enrich}(T)$ is also possible counting that the parameter σ' is omitted in context.

Fix a mapping $\delta : \sigma \to \sigma'$ preserving types of signature symbols and not lowering their arities. We include in T' symbols $\mathfrak{s} \in \sigma' \smallsetminus \delta(\sigma)$ as trivially defined. Namely, we apply rules (0.2)(a,b) for predicates and functions, and use an available $\exists \cap \forall$-definable element as a trivial value for the constants, cf. (0.2)(c). If $\mathfrak{s}$ is a predicate or function symbol of arity $\geqslant 1$, we replace s by its presentation in $\delta(\mathfrak{s})$, adding to axioms of T' a trivial definition in the part of $\delta(\mathfrak{s})$ that is not involved in the presentation. For example, it is possible to present a binary predicate $P(x, y)$ in a ternary predicate $R(x, y, z)$ by rule $P(x, y) \leftrightarrow R(x, y, y)$ adding the requirement $(\forall xyz)[(y \neq z) \to \neg R(x, y, z)]$ to axioms of T'; similarly, it is possible to present a binary function $f(x, y)$ in ternary function $h(x, y, z)$ by $f(x, y) = h(x, y, y)$ adding the requirement $(\forall xyz)[(y \neq z) \to h(x, y, z) = x]$ to the axioms. Notice that, actually, construction of theory $T' = \mathsf{Enrich}(T, \sigma')$ depends not only on T and σ', but also on the mapping δ. Obviously, the transformation Enrich is an algebraic isomorphism of theories. Thereby, specifications (4.2)(a,b,c) are indeed satisfied.

Description of the stage Enrich is complete.

We begin to describe the stage $\mathsf{fP\text{-}to\text{-}Graph}$.

Given a theory T of a finite pure predicate signature

$$\sigma = \{P_0^{n_0}, P_1^{n_1}, \dots, P_k^{n_k}\}, \tag{4.5}$$

whose predicate symbols have arities $n_i > 0$, $i = 0, 1, ..., k$. Our aim is to construct a theory S extending graph theory GRE of signature $\sigma_{GR} = \{\Gamma^2\}$ together with an algebraic Cartesian interpretation $I : T \rightarrowtail S$.

Starting from a model $\mathfrak{N}$ of signature σ, we construct some new model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of graph theory GRE of signature $\{\Gamma^2\}$, inside which the source model $\mathfrak{N}$ is $\exists \cap \forall$-presentable in some regular manner. For this, the following procedure is applied (schematically presented in Fig. 4.2). Define $|\mathfrak{M}|$ to be equal to the union of three following pairwise disjoint sets $U \cup C \cup D$, such that $D = \{d_1, d_2, \dots, d_7\}$ is a seven-element set, U is a set of the same cardinality as $|\mathfrak{N}|$ and $h : |\mathfrak{N}| \to U$ is a fixed bijection, while the set C is specified later. The construction provides that these parts U, C, and D are first-order definable by formulas $U(x)$, $C(x)$, $D(x)$, which are specified later.

Proceed to the definition of Γ-links on the set $|\mathfrak{M}|$.

At first stage, define Γ-links on $U \cup C$. According to the construction, predicates of signature (4.5) are interpreted in the domain U, while the truth values of the predicates are encoded by means of special *configurations* in an extra domain $C(x)$ as it is schematically shown in Fig. 4.2. The notation

$$Cnf(n, m, v), \quad 1 \leqslant n, \ 0 \leqslant m, \ v \in \{\mathtt{t}, \mathtt{f}\},$$

stands for the form of a configuration corresponding to predicate $P_n(x_1, ..., x_m)$ of signature (4.5), where n is an index of the predicate; the value $n+2$ represents lengths of all prime chains in the configurations for this predicate, m is the arity of the predicate; this value is equal to the number of foot elements (supports) of a configuration for this predicate, $v \in \{\mathtt{t}, \mathtt{f}\}$ is a parameter that represents a truth value of the predicate on a tuple; the value of this parameter causes in two different structure forms of a configuration.

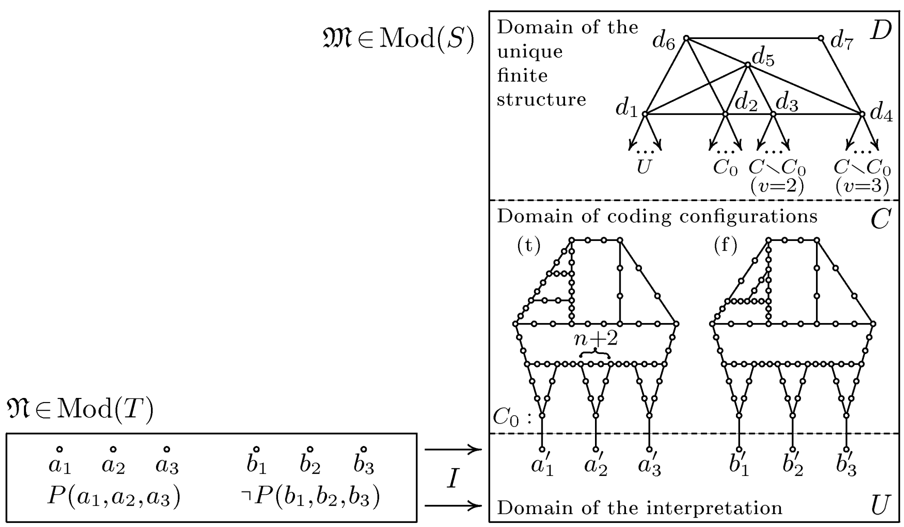

Figure 4.2 – Reduction of a finite pure predicate signature to graphs

By means of the given bijection $h : |\mathfrak{N}| \to U$, temporarily transfer the predicate structure of $\mathfrak{N}$ on the set U. Consider an m_n-ary predicate $P_n^{m_n}$ in σ and get any tuple of elements $a_1, a_2, \ldots, a_{m_n} \in U$. In the case when $P_n(a_1, a_2, \ldots, a_{m_n})$ is true, with the help of elements from C and Γ-links, we construct over the tuple $(a_1, a_2, \ldots, a_{m_n})$ a configuration of the form $Cnf(n, m_n, \mathrm{t})$. In the other case, when $P_n(a_1, a_2, \ldots, a_{m_n})$ is false, we construct over this tuple a configuration of the form $Cnf(n, m_n, \mathrm{f})$. Configurations for different predicates, as well as configurations for the same predicate but for different tuples, should not have common elements in C, but they may have shared elements in U. For each predicate $P_n \in \sigma$ and each tuple of elements from U of length m_n, there should be exactly one configuration, either of the form (t), or of the form (f), in accordance with the truth value of this predicate in this tuple. Now, we define the domain C as the set of elements of all coding configurations, excluding U-elements from them.

Now, we define Γ-links on the set D and between the sets D and $U \cup C$. On the set D, determine the structure of a seven-element graph which form is presented in the upper part in Fig. 4.2. Denote by $\Phi(z_1, z_2 \ldots, z_7)$ a primitive quantifier-free formula of signature $\sigma_{GR} = \{\Gamma^2\}$ which describes an atomic diagram of this Γ-structure under the assumption that the value of z_i is assigned to d_i. Then, we introduce notations for the following formulas:

$$\begin{aligned} &\texttt{(a)}\ D_i(x) = (\exists z_1 \ldots z_7)[\Phi(z_1, \ldots, z_7) \wedge (x = z_i)], \quad 1 \leqslant i \leqslant 7, \\ &\texttt{(b)}\ D(x) = D_1(x) \vee \ldots \vee D_7(x). \end{aligned} \tag{4.6}$$

Now, we are in a position to define Γ-links between the domains D and $U \cup C$ on the universe set $|\mathfrak{M}|$ as follows. For an arbitrary element x in the domain $U \cup C$, we put

$$\begin{aligned} &\Gamma(d_1, x) \Leftrightarrow x \in U, \\ &\Gamma(d_2, x) \Leftrightarrow (x \in C) \wedge (\exists y)\big(\Gamma(x, y) \wedge U(y)\big), \\ &\Gamma(d_3, x) \Leftrightarrow \neg\Gamma(d_2, x) \ \wedge \ (x \in C) \wedge (x \textit{ has valency } 2 \textit{ within the domain } C), \\ &\Gamma(d_4, x) \Leftrightarrow \neg\Gamma(d_2, x) \ \wedge \ (x \in C) \wedge (x \textit{ has valency } 3 \textit{ within the domain } C). \end{aligned}$$

Finally, we define Γ to be false on all those pairs of elements in $|\mathfrak{M}|$, which were not involved in the description above.

By $\mathfrak{M}^* = \mathbb{E}^*(\mathfrak{N})$ we denote the model of signature $\sigma^* = \sigma \cup \{\Gamma^2\}$ we have obtained. By $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ we denote the model $\mathfrak{M}^* \restriction \{\Gamma^2\}$ obtained from $\mathfrak{M}^*$ by eliminating temporarily defined predicates of signature σ on the set U. On this, construction of the model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma_{GR} = \{\Gamma^2\}$ is complete.

Study properties of models $\mathbb{E}(\mathfrak{N})$ and $\mathbb{E}^*(\mathfrak{N})$ obtained by the construction.

The construction guarantees uniqueness of realization of the formula Φ presenting the atomic diagram of a special seven-element graph, which we denote by G_7. Valency of each element among d_1, d_2, d_3, d_4, d_5, and d_6 is 4 or larger, while none elements can have such a valency in other parts of the model $\mathbb{E}(\mathfrak{N})$. As a result, each of the formulas $D_i(x)$, $i \in \{1, 2, ..., 7\}$, distinguishes exactly one element in the model $\mathbb{E}(\mathfrak{N})$, $\mathfrak{N} \in \mathrm{Mod}(\sigma)$; moreover, these seven elements are different; thereby, the formula $D(x)$ distinguishes exactly 7 elements in this model.

Using elements of the domain $D(x)$ as pivots, one can distinguish the other domains in the model. The following formulas

$$
\begin{aligned}
&U(x) = \neg D(x) \wedge (\exists z)[D_1(z) \wedge \Gamma(z,x)], \\
&C(x) = \neg D(x) \wedge (\exists z)[(D_2(z) \vee D_3(z) \vee D_4(z)) \wedge \Gamma(z,x)], \\
&C_0(x) = C(x) \wedge (\exists y)\big[\Gamma(x,y) \wedge U(y)\big],
\end{aligned}
\tag{4.7}
$$

distinguish the domains U and C involved in the description above, while an extra formula $C_0(x)$ distinguishes the elements of configurations immediately Γ-linked with elements of U.

Coding configurations of the transformation fP-to-Graph for an n_i-ary predicate $P_i \in \sigma$, sf. (4.5), are presented by the following atomic diagrams (i.e., quantifier-free primitive formulas):

$$
\mathfrak{C}^{\alpha}_{P_i}(\bar{u}_i, x_0, x_1, ..., x_{e_i-1}),\ \alpha \in \{\mathtt{t}, \mathtt{f}\},\ \mathrm{len}(\bar{u}_i) = n_i,\ i = 0, 1, ..., k,
\tag{4.8}
$$

where number e_i is equal to the quantity of element of the configurations for P_i in the domain $U(x)$ (by construction, configurations for P_i of types $(\mathtt{t})$ and $(\mathtt{f})$ consist of the same number of elements).

Construction of $\mathbb{E}^*(\mathfrak{N})$ provides that the following key properties of coding configurations are satisfied for all tuples $\bar{u}_i$ of appropriate arities in the model:

$$
\begin{aligned}
P_i(\bar{u}_i)\ &\leftrightarrow (\exists x_0, x_1, ..., x_{e_i-1})\, \mathfrak{C}^{\mathtt{t}}_{P_i}(\bar{u}_i, x_0, x_1, ..., x_{e_i-1}),\ i = 0, 1, ..., k, \\
\neg P_i(\bar{u}_i)\ &\leftrightarrow (\exists x_0, x_1, ..., x_{e_i-1})\, \mathfrak{C}^{\mathtt{f}}_{P_i}(\bar{u}_i, x_0, x_1, ..., x_{e_i-1}),\ i = 0, 1, ..., k.
\end{aligned}
\tag{4.9}
$$

Notice that the construction described above is not deterministic. It admits some ambiguity in the choice of the domain sets U, C, D, the bijective function h, etc. Nevertheless, the target model is defined, in some sense, uniquely up to an isomorphism. Namly, let $\mathfrak{N}$ be a model of signature σ and let two independent realizations of the construction be performed, which use domains and mappings U', C', D', $h' : \mathfrak{N} \to U'$, and U'', C'', D'', $h'' : \mathfrak{N} \to U''$, resulting in two models $\mathfrak{M}'$ and $\mathfrak{M}''$. Then, there is an isomorphism $\mu : \mathfrak{M}' \to \mathfrak{M}''$ such that the following diagram is commutative:

$$
\begin{tikzcd}
 & U' \subseteq \mathfrak{M}' \arrow[d, "\mu"'] \\
\mathfrak{N} \arrow[ur, "h'"] \arrow[dr, "h''"'] & \\
 & U'' \subseteq \mathfrak{M}''
\end{tikzcd}
$$

Moreover, the mapping μ is uniquely determined, and $\mu(U') = U''$, $\mu(C') = C''$, $\mu(D') = D''$ takes place.

Now, we turn to the definition of a theory S and an interpretation I.

We define theories S^* and S as follows:

$$
S^* = \mathrm{Th}\{\mathbb{E}^*(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\}, \quad S = \mathrm{Th}\{\mathbb{E}(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\},
$$

Domain $U(x)$ of an interpretation I of theory T in theory S is defined by rule (4.7). The equality relation $(x = y)$ of T is interpreted as an ordinary equality relation within the domain:

$$\varepsilon(x,y) = U(x) \wedge U(y) \wedge (x = y). \tag{4.10}$$

Atomic formula $P_n(x_1, x_2, \ldots, x_{m_n})$ of theory T is mapped onto a formula of signature $\sigma_{GR} = \{\Gamma^2\}$ asserting inclusions $U(x_i)$, $1 \leqslant i \leqslant m_n$, and existence over the tuple $(x_1, x_2, \ldots, x_{m_n})$ a configuration of the form (t) corresponding to the predicate P_n. By induction, we extend this mapping up to a transformation $I : FL(\sigma) \to FL(\sigma_{GR})$, which is just the required interpretation I of theory T in theory S.

As a result of the construction, starting from an input theory T, we have obtained a pair of objects: a theory S and an interpretation $I : T \rightarrowtail S$. We denote by $\mathbb{K}(\mathfrak{M})$ the model kernel of a model $\mathfrak{M} \in \mathrm{Mod}(S)$ with respect to the interpretation I, in accordance with definition in Section 1. Relations (4.9) show that predicate symbols (4.5) are $\exists\cap\forall$-presentable relative to $\{\Gamma(x,y)\}$ ensuring an algebraic isomorphism $S \approx_a S^*$.

We list axioms of theory S^*. They include the following requirements:

1°. $(\forall x)\neg\Gamma(x,x)$, and $(\forall x)(\forall y)[\Gamma(x,y) \leftrightarrow \Gamma(y,x)]$.

2°. Special formula $\Phi(z_1, \ldots, z_7)$ distinguishes exactly one seven-element tuple; moreover, formulas $U(x)$, $C(x)$, $C_0(x)$, and $D(x)$ are defined via the formula $\Phi(z_1, \ldots, z_7)$ by rules (4.6) and (4.7).

3°. Formulas $U(x)$, $C(x)$, and $D(x)$ determine a partition of the universe set in three nonempty parts.

4°. $U(x) \wedge U(y) \to \neg\Gamma(x,y)$.

5°. For any predicate $P \in \sigma$ and any tuple of elements in U of corresponding length, in the domain C, there is exactly one coding configuration of the form either (`t`) or (`f`) in accordance with the truth-value of this predicate in this tuple including also positions of elements of the configuration with respect to the domains U, C_0, and $C \setminus C_0$.

6°. Any two different coding configurations (related to different predicates or to different tuples of the same predicate) neither have shared elements in the domain C nor have any Γ-links between their elements in the domain C.

7°. Any element x in the domain C belongs to a coding configuration.

8°. $I(\varphi)$, for any sentence $\varphi \in SL(\sigma)$, which is an axiom of theory T.

System of axioms is complete.

It is possible to check that transformation $\varphi \mapsto I(\varphi)$ involved in Axiom 8° preserves types of quantifier prefixes of the sentences, while Axioms 1°–8° hold in $\mathbb{E}(\mathfrak{N})$ for any model $\mathfrak{N} \in \mathrm{Mod}(T)$. On the other hand, if $\mathfrak{M}$ is an arbitrary model of signature $\{\Gamma^2\}$, we have $\mathfrak{M} \models$1°–8° if and only if there is a model $\mathfrak{N} \in \mathrm{Mod}(T)$ such that $\mathfrak{M} \cong \mathbb{E}(\mathfrak{N})$.

We turn to checking statements of Lemma 3.2 and then those of Lemma 3.3. For this, as a finite realization system, we consider the set of formulas

$$\lambda_i(\bar{z}_i, x),\ i = 1, 2, ..., r, \tag{4.11}$$

that represents the union of two following collections

$$
\begin{aligned}
&\texttt{(a)}\ \ D_i(x),\ i = 1, 2, ..., 7, \\
&\texttt{(b)}\ \ \mathfrak{C}^{\alpha}_{P_i,j}(\bar{u}_i, x),\ \alpha \in \{\texttt{t},\texttt{f}\},\ \mathrm{Len}(\bar{u}_i) = n_i,\ 0 \leqslant j < e_i,\ i = 0, 1, ..., k,
\end{aligned}
\tag{4.12}
$$

where $D_i(x)$ is defined in (4.6), while $\mathfrak{C}^{\alpha}_{P_i,j}(\bar{u}_i, x)$ is obtained from the formula $\mathfrak{C}^{\alpha}_{P_i}(\bar{u}_i, x_0, x_1, ..., x_{e_i-1})$ in (4.8) by attaching an index j, renaming variable x_i into x, and bounding the other variables x_t, $t \neq j$, with $\exists$-quantifiers. The formula (4.12)(b) exactly states that element x is situated in a certain position within the configuration of the form $\alpha \in \{\texttt{t},\texttt{f}\}$ for the predicate P_i over a tuple $\bar{u}_i$ in the domain $U(x)$. Thus, we have $r = 7 + \Sigma^{k}_{i=0} e_i$. We also count that formulas $\lambda_i(\bar{z}_i, x)$, $i = 1, 2, ..., 7$, coincide with the part (4.12)(a), while the others correspond to the collection (4.12)(b).

We show that interpretation I is an exact interpretation of theory T in the domain $U(x)$ of theory S. For this, we have to check conditions in (3.5). Demand (3.5)(a) is a consequence of the deterministic property for the construction. Let a be an element in a model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$, $\mathfrak{N} \in \mathrm{Mod}(T)$. For (3.5)(b), we have to prove that $a \in \mathrm{acl}(U(\mathfrak{M}))$. If $U(a) \vee D(a)$ holds, $a \in \mathrm{acl}(U(\mathfrak{M}))$ is satisfied trivially. Consider the case when $C(a)$ is held. There is a configuration $\mathfrak{C}$ of some type $\mathfrak{T}$ over a tuple $\bar{c} \in U(\mathfrak{M})$ such that $a \in \mathfrak{C}$. Moreover, there is the only such configuration. Take a formula $\varphi(\bar{z}, x)$ stating that x belongs (in a particular position) to a configuration of type $\mathfrak{T}$ over a tuple $\bar{z}$ in $U(x)$. Then, $\varphi(\bar{c}, x)$ is held for $x = a$ ensuring that $a \in \mathrm{acl}(\bar{c})$, obtaining $a \in \mathrm{acl}(U(\mathfrak{M}))$. The condition (3.5)(c) is a consequence of the deterministic property for the construction ensuring that each automorphism $\mu : \mathbb{K}(\mathfrak{M}) \to \mathbb{K}(\mathfrak{M})$ can be extended upto an automorphism $\mu' : \mathfrak{M} \to \mathfrak{M}$.

It is shown that interpretation I is indeed exact. Then, by Lemma 3.2, there is a tuple of formulas $\varkappa$ of the form (2.1)(a) and a general-model isomorphism E such that the following diagram is commutative

$$
\begin{tikzcd}
T \arrow[rr, tail, "I"] \arrow[dr, tail, "I_{T,\varkappa}"'] & & S \\
& T\langle\varkappa\rangle \arrow[ur, tail, "E"'] &
\end{tikzcd}
$$

For the passage $\mathfrak{N} \mapsto \mathbb{E}^*(\mathfrak{N})$, a finite realization system for the interpretation $I : T \rightarrowtail S$ is defined in (4.11) and (4.12). Based on this, we obtain that the interpretation I is presented by a chain similar to that shown in the scheme (2.4) with the following tuple $\varkappa$:

$$
\begin{aligned}
&\varkappa = \varkappa' \hat{}\, \varkappa'',\ \textit{where}\ \ \varkappa' = \langle \underbrace{\varphi, ..., \varphi}_{7\ times} \rangle,\ \varphi = (\exists x)(x = x), \\
&\varkappa'' = \langle \underbrace{P_0(\bar{z}_0), P_0(\bar{z}_0), ...}_{e_0\ times}, \underbrace{\neg P_0(\bar{z}_0), \neg P_0(\bar{z}_0), ...}_{e_0\ times},\ ...\ , \underbrace{P_k(\bar{z}_k), ...}_{e_k\ times}, \underbrace{\neg P_k(\bar{z}_k), ...}_{e_k\ times} \rangle,
\end{aligned}
\tag{4.13}
$$

In (4.13), we can use any $\exists \cap \forall$-presentable identically true sentence instead of φ.

Based on details of our construction, it is possible to establish that all demands of Lemma 3.3 are as well satisfied for the finite realization system (4.11).

Thereby, we have obtained that the isomorphism E is actually an algebraic isomorphism of theories, thus, ensuring that I is an algebraic Cartesian interpretation. Thereby, specifications (4.2)(a,b,c) are indeed satisfied for the stage fP-to-Graph.

Description of the stage fP-to-Graph is complete.

We begin to describe the stage Graph-to-2u.

Given a theory T extending graph theory *GRE* of signature $\sigma_{GR} = \{\mathit{\Gamma}^2\}$ and a signature $\sigma = \{f^1, h^1\}$ consisting of two unary functions. Our aim is to construct a theory S of signature σ together with an $\exists \cap \forall$-presentable Cartesian interpretation of theories $I : T \rightarrowtail S$.

Starting from an arbitrary model $\mathfrak{N}$ of theory *GRE* of signature $\sigma_{GR} = \{\mathit{\Gamma}^2\}$, we build a model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma = \{f^1, h^1\}$, in which $\mathfrak{N}$ is first-order definable in some regular manner, as it is shown in Fig. 4.3. We put $|\mathfrak{M}|$ to be equal to the union of two disjoint sets $U \cup C$, where U is a set of the same cardinality as $|\mathfrak{N}|$, while the set C is specified later. Choose a bijection $h : |\mathfrak{N}| \to U$ and temporarily transfer all signature predicates of $\mathfrak{N}$ on the set U in accordance with h. Truth-values of predicates of signature $\sigma_{GR} = \{\mathit{\Gamma}^2\}$ are coded in the model $\mathfrak{M}$ by special *coding configurations* in the domain C available in two forms (t) and (f); i.e., `true` and `false`. Each configuration is defined over a pair of non-equal elements from U, and consists of two elements in the domain C, as it is shown in Fig. 4.3.

For such a configuration (whose elements are supposed to be not coincided with each other), the signature functions $f(x)$ and $h(x)$ are defined by the following rule:

$$\begin{array}{llll}
\texttt{(t)} & f : x \mapsto y \mapsto a' \mapsto a', & \texttt{(f)} & f : u \mapsto v \mapsto c' \mapsto c', \\
 & f : b' \mapsto b', & & f : d' \mapsto d', \\
 & h : x \mapsto y \mapsto b' \mapsto b', & & h : v \mapsto u \mapsto d' \mapsto d', \\
 & h : a' \mapsto a'; & & h : c' \mapsto c'.
\end{array} \tag{4.14}$$

Moreover, in any case, $f(z)$ and $h(z)$ must not belong to a configuration whenever z is not included in this configuration. Since $\mathit{\Gamma}(x, y)$ is a symmetric predicate, the pairs (a, b) and (b, a) with $a \neq b$ generate two separate configurations, both having the same form, either (t) or (f).

Different coding configurations must not intersect with each other in the domain C, but they may have shared elements in U. To each pair (a, b) of different elements from U, there should exist exactly one configuration, either of the form (t) or of the form (f), depending on the truth-value of the predicate $\mathit{\Gamma}$ on the pair. Now, we let C to be the set of elements of all coding configurations, excluding from them U-elements. Thus, C consists of $2 \cdot \alpha(\alpha - 1)$ elements, where α is the cardinality of $|\mathfrak{N}|$.

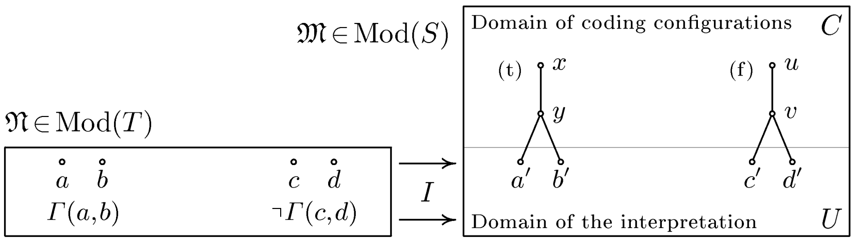

Figure 4.3 – Reduction of graphs to a couple of unary functions

By $\mathfrak{M}^* = \mathbb{E}^*(\mathfrak{N})$ we denote the model of signature $\sigma^* = \sigma \cup \{\Gamma^2\}$ we have obtained. By $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ we denote the model $\mathfrak{M}^* \upharpoonright \{f^1, h^1\}$ obtained from $\mathfrak{M}^*$ by eliminating temporarily defined predicates of signature σ_{GR} within the domain U. On this, construction of the model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma = \{f^1, h^1\}$ is complete.

Now, we turn to the definition of a theory S and an interpretation I. We define theories S^* and S as follows:

$$S^* = \mathrm{Th}\{\mathbb{E}^*(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\}, \quad S = \mathrm{Th}\{\mathbb{E}(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\},$$

The domain of the interpretation I is defined by the formula

$$U(x) = \left[\, f(x) = x \,\wedge\, h(x) = x \,\right],$$

while formula $C(x) = \neg U(x)$ determines domain of coding configurations in the theory S.

An atomic formula $\Gamma(x,y)$ of signature σ_{GR} is mapped into a formula of signature σ stating that $U(x) \wedge U(y)$ and there is a configuration of the form (t) over the pair (x,y). By induction, we expand this mapping up to a transformation $I : FL(\sigma_{GR}) \to FL(\sigma)$, which is just the required interpretation I of theory T in theory S. As a result of the construction, starting from an input theory T, we have obtained a pair of objects: a theory S and an interpretation $I : T \rightarrowtail S$.

By $\mathfrak{C}^{\mathfrak{t}}(a,b,x,y)$ and $\mathfrak{C}^{\mathfrak{f}}(c,d,u,v)$ we denote quantifier-free formulas of signature $\sigma = \{f^1, h^1\}$ presenting diagrams of configurations of types (t) and, respectively, (f), as it is defined in (4.14). Le also $\mathfrak{C}(c,d,u,v)$ denotes the disjunction $\mathfrak{C}^{\mathfrak{t}}(c,d,u,v) \vee \mathfrak{C}^{\mathfrak{f}}(c,d,u,v)$.

We list axioms of theory S^*. They include the following sentences:

1°. $(\forall xu)\left[\, u = f\!f(x) \vee u = hh(x) \;\to\; f(u) = u \,\wedge\, h(u) = u \,\right]$,
2°. $(\forall xyuv)\left[\, \mathfrak{C}(x,y,u,v) \to (x \neq y \wedge x \neq u \wedge x \neq v \wedge y \neq u \wedge y \neq v \wedge u \neq v) \,\right]$,
3°. $(\forall xyuvz)\left[\, \mathfrak{C}(x,y,u,v) \wedge z \notin \{x,y,u,v\} \to \{f(z), h(z)\} \cap \{x,y,u,v\} = \varnothing \,\right]$,
4°. $(\exists x) U(x)$,
5°. $U(x) \wedge U(y) \wedge (x \neq y) \leftrightarrow (\exists uv)\mathfrak{C}(x,y,u,v)$,
6°. $\mathfrak{C}(x,y,u,v) \wedge \mathfrak{C}(x,y,w,t) \to (u,v) = (w,t)$,
7°. $\mathfrak{C}(x,y,u,v) \wedge \mathfrak{C}(x',y',u',v') \wedge (x,y) \neq (x',y') \to \{u,v\} \cap \{u',v'\} = \varnothing$,

8°. $\neg U(z) \to (\exists xyuv)[\mathfrak{C}(x,y,u,v) \wedge z \in \{u,v\}]$,

9°. $I(\varphi)$, for any sentence $\varphi \in SL(\sigma)$ which is an axiom of theory T.

By construction, predicate $\Gamma(x,y)$ is $\exists\cap\forall$-presentable in S^* with respect to the functions $f(x)$ and $h(x)$ by means of the coding configurations. On the other hand, the passage $\mathfrak{N} \mapsto \mathbb{E}^*(\mathfrak{N})$ represents, up to an isomorphism, a Cartesian extension with the tuple

$$\varkappa = \langle \varphi(x,y), \varphi(x,y), \psi(x,y), \psi(x,y) \rangle,$$

where $\varphi(x,y) = \Gamma(x,y)$, and $\psi(x,y) = \neg\Gamma(x,y) \wedge (x \neq y)$. Moreover, it is possible to show that functions $f(x)$ and $h(x)$ are $\exists\cap\forall$-presentable via predicate $\Gamma(x,y)$ together with the relations involved in the structure of a Cartesian extension with this tuple $\varkappa$. This shows that $S^* \approx_a S$. The construction ensures that Lemma 3.2 together with Lemma 3.3 are applicable. From this, we obtain that transformation $I : T \rightarrowtail S$ is an algebraic Cartesian interpretation of theories.

Description of the stage Graph-to-2u is complete.

We begin to describe the stage Graph-to-1b.

Given a theory T extending graph theory *GRE* of signature $\sigma_{GR} = \{\Gamma^2\}$ and a signature $\sigma = \{g^2\}$ consisting of one binary function. Our aim is to construct a theory S of signature σ together with an $\exists\cap\forall$-presentable Cartesian interpretation of theories $I : T \rightarrowtail S$.

Starting from an arbitrary model $\mathfrak{N}$ of graph theory *GRE* of signature $\sigma = \{\Gamma^2\}$, we build a model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma = \{g^2\}$, in which this model $\mathfrak{M}$ is first-order definable in some regular manner. As a whole, a scheme of the transformation is similar to those earlier considered. We set $|\mathfrak{M}|$ to be equal to the union of two disjoint sets $U \cup D$, where U is a set of the same cardinality as $|\mathfrak{N}|$ and $h : |\mathbb{N}| \to U$ is a fixed bijection, while the set D consists of two elements d_0 and d_1. For the construction of $\mathfrak{M}$, by means of bijection h, we temporarily transfer the structure of $\mathfrak{N}$ onto the set U. Define a function $g(x,y)$ on the set $|\mathfrak{M}| = U \cup \{d_0, d_1\}$ as follows:

$$\begin{aligned}
&g(d_0,d_0) = d_1,\\
&g(d_0,d_1) = g(d_1,d_0) = g(d_1,d_1) = d_0,\\
&g(x,y) = d_1,\ \textit{for all } x,y \in U \textit{ such that } \Gamma(x,y),\\
&g(x,y) = d_0,\ \textit{for all } x,y \in U \textit{ such that } \neg\Gamma(x,y) \wedge (x \neq y),\\
&g(x,y) = d_0,\ \textit{for the other cases.}
\end{aligned}$$

After that, we erase temporarily defined structure of the model $\mathfrak{N}$ on the set U, and construction of the model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma = \{g^2\}$ is complete.

Consider the following quantifier free formulas of signature $\{g^2\}$:

$$\begin{aligned}
&D(x) = \big[\, x = g(g(x,x), g(x,x)) \,\big],\\
&D_0(x) = \big[\, x = g(x, g(x,x)) \,\big],\\
&D_1(y) = \big[\, x = g(g(x,x), g(x,x)) \ \wedge\ x \neq g(x, g(x,x)) \,\big].
\end{aligned}$$

We can check that in the model $\mathfrak{M}$ it is satisfied

$$D_0(x) \leftrightarrow (x = d_0), \quad D_1(x) \leftrightarrow (x = d_1), \quad D(x) \leftrightarrow D_0(x) \vee D_1(x).$$

Thus, the construction ensures uniqueness of realization of the formulas $D_0(x)$ and $D_1(y)$, distinguishing special elements d_0 and, respectively, d_1 in the model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$.

By $\mathfrak{M}^* = \mathbb{E}^*(\mathfrak{N})$ we denote the model of signature $\sigma^* = \sigma \cup \{\Gamma^2\}$ we have obtained. By $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ we denote the model $\mathfrak{M}^* \upharpoonright \{g^2\}$ obtained from $\mathfrak{M}^*$ by eliminating temporarily defined predicates of signature σ_{GR} within the domain U. On this, construction of the model $\mathfrak{M} = \mathbb{E}(\mathfrak{N})$ of signature $\sigma = \{g^2\}$ is complete.

Now, we turn to the definition of a theory S and an interpretation I. We define theories S^* and S as follows:

$$S^* = \mathrm{Th}\{\mathbb{E}^*(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\}, \quad S = \mathrm{Th}\{\mathbb{E}(\mathfrak{N}) \mid \mathfrak{N} \in \mathrm{Mod}(T)\},$$

The domain of the interpretation is defined by the formula

$$U(x) = \left[\, x \neq g(g(x,x), g(x,x)) \,\right],$$

which obviously satisfies the condition $U(x) \leftrightarrow \neg D(x)$. An atomic formula $\Gamma(x,y)$ of signature $\sigma_{GR} = \{\Gamma^2\}$ is mapped into the formula of signature σ of the form

$$U(x) \wedge U(y) \wedge (x \neq y) \wedge D_1(g(x,y)).$$

By induction, we expand this mapping up to a transformation $I : FL(\sigma_{GR}) \to FL(\sigma)$, which is just the required interpretation of theory T in theory S.

List axioms of theory S^*. They include the following series of sentences:

1°. $(\exists x) D_0(x) \;\wedge\; (\forall xy)\big[D_0(x) \wedge \big(D_0(y) \to (y = x)\big)\big]$,
2°. $(\exists x) D_1(x) \;\wedge\; (\forall xy)\big[D_1(x) \wedge \big(D_1(y) \to (y = x)\big)\big]$,
3°. $D_0(x)$, $D_1(x)$, and $U(x)$ represent a disjoint partition of the universe,
4°. $(\exists x)\, U(x)$,
5°. $(\forall xy)\, \big[g(x,y) = g(y,x)\,\big]$,
6°. $(\forall x)\, \big[D_0(x) \to D_1(g(x,x))\,\big]$,
7°. $(\forall xy)\, \big[\, \big(\neg D_0(x) \vee \neg D_1(y)\,\big) \;\wedge\; \big(\,\neg U(x) \vee \neg U(y)\,\big) \;\to\; D_0(g(x,y))\,\big]$,
8°. $(\forall xy)\, \big[\, U(x) \wedge U(y) \;\to\; D_0(g(x,y)) \vee D_1(g(x,y))\,\big]$,
9°. $I(\varphi)$, for any sentence $\varphi \in SL(\sigma_{GR})$ which is an axiom of theory T.

It is possible to check that this set of axioms indeed determines the theory S which was introduced earlier as the theory of a class of models. By construction, S is algebraically isomorphic to an external constant extension $T\langle c_1, c_2\rangle$ of theory T by a couple of constant elements. Thereby, I is a Cartesian interpretation because the operation of an external constant extension of a theory is a particular case of the operation of a Cartesian extension. It is possible to check that the transformation $T \mapsto S$ we have described is an algebraic Cartesian interpretation. Thus, specifications (4.2)(a,b,c) are indeed satisfied for the stage Graph-to-1b.

Description of the stage Graph-to-1b is complete.

Now, we turn immediately to the proof of the statement (4.1).

Given a theory T of a finite signature τ together with a finite rich signature σ for the output theory. By applying stage finsig-to-fP, we transform theory T upto a theory T_1 of a finite pure predicate signature τ^* with predicates of arity $\geqslant 1$. Then, by applying stage fP-to-Graph, we transform T_1 to a theory T_2 of signature $\{\Gamma^2\}$ that is an extension of special graph theory *GRE*. Passage x-e in the second part of scheme in Fig. 4.1 can be performed by one of the ways x-a-e, x-b-e, or x-c-e. Choice of the variant is defined by the member in (4.3) that is covered by the required finite rich signature σ for the output theory S. Thus, at least one of the branches 1-x-a-e, 1-x-b-e or 1-x-c-e must realize the output signature σ. Notice that, a distinguished element which is required for the procedure Enrich, is automatically provided by the previous stage fP-to-Graph.

Let us choose and fix a passage π in the scheme in Fig. 4.1 matching the output signature σ. As the procedure to be designed we take composition of the transformations along the path π. This procedure transform the input theory T to a theory S having the required signature σ. Each stage in list (4.4) represents a Cartesian interpretation; therefore, by Lemma 2.7, the complete transition along the path π is also a Cartesian interpretation. By construction, each of the stages (4.4) passes property of being a finitely axiomatizable theory from its input theory to the output one, thus, the target theory S will be finitely axiomatizable whenever starting theory T is finitely axiomatizable.

Statement (4.1) is proved.

Thereby, Theorem 4.1 is proved. □

An url `http://predicate-logic.org/expw/sigp/sigp.pdf` contains a paper where the proof of Theorem 4.1 is presented in more technical details.

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Перетятькин М.Г. КӨРСЕТІЛІМНІҢ АЛГЕБРАЛЫҚ ТҮРІ ҮШІН АҚЫРЛЫ ҚОЛТАҢБАЛАРДЫ АЗАЙТУ РӘСІМІНІҢ ТЕХНИКАЛЫҚ ПРОТОТИПІ

Жұмыста кез келген ақырлы бай сигнатураның предикаттар қисабының декарттық кеңейтуі бар екені дәлелденеді, ол осы ақырлы бай сигнатураның предикаттар қисабының белгілі бір ақырлы аксиоматталатын үзіндісіне алгебралық изоморфты болады. Осы техникалық сипаттағы нәтиже тірек тұжырым болып табылады, оның негізінде ақырлы сигнатураларды азайту рәсімінің толыққанды нұсқасы алынуы мүмкін. Сонымен бірге, көрсетілген тұжырым бірінші ретті логиканың көрсетілімдік күшін сипаттаумен байланысты әртүрлі құрылымдардың техникалық бөлігі ретінде жақсы бірлеседі.

Кілттік сөздер. Бірінші ретті логика, толық емес теория, Тарский-Линденбаум алгебрасы, теорияның декарттық кеңейтуі, сигнатураларды азайту рәсімі.

Перетятькин М.Г. ТЕХНИЧЕСКИЙ ПРОТОТИП ПРОЦЕДУРЫ РЕДУКЦИИ КОНЕЧНЫХ СИГНАТУР ДЛЯ АЛГЕБРАИЧЕСКОГО ТИПА ВЫРАЗИМОСТИ

В работе доказывается, что исчисление предикатов произвольной конечной богатой сигнатуры имеет декартово расширение, алгебраически изоморфное некоторому конечно аксиоматизируемому фрагменту исчисления предикатов данной конечной богатой сигнатуры. Этот результат технического характера представляет опорное утверждение, на основе которого может быть получена полновесная версия процедуры редукции конечных сигнатур, вместе с тем, указанное утверждение хорошо интегрируется в качестве технической компоненты в различные конструкции, связанные с характеризацией выразительной силы логики первого порядка.

Ключевые слова. Логика первого порядка, неполная теория, алгебра Тарского-Линденбаума, декартово расширение теории, процедура редукции сигнатур.

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On inverse problem of reconstructing a heat subdiffusion process with periodic data

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Abstract. In this work, we consider a problem of modeling the thermal diffusion process in a closed metal wire wrapped around a thin sheet of insulation material. The layer of insulation is assumed to be slightly permeable. Therefore, the temperature value from one side affects the diffusion process on the other side. For this reason, the standard heat equation is modified and an third term with an involution is added. Modeling of this process leads to the consideration of inverse problem for an one-dimensional fractional evolution equation with involution and with periodic boundary conditions with respect to a spatial variable. Such equations are also called nonlocal sub-diffusion equations or nonlocal heat equations. The inverse problem is the restoration (simultaneously with the solution) of the unknown right-hand side of the equation, which depends only on the spatial variable. The conditions for overdetermination are initial and final states. The existence and the uniqueness results for the given problem are obtained by the method of separation of variables.

Keywords. Inverse problem, fractional evolution equation, nonlocal sub-diffusion equation, nonlocal heat equation, equation with involution, periodic boundary conditions, method of separation of variables.

1 Introduction and statement of the problem

The problems that imply the determination of coefficients or the right-hand side of a differential equation (together with its solution) are commonly referred to inverse problems of mathematical physics. In this paper we consider one family of problems implying the determination of the density distribution and of heat sources from given values of initial and final distributions. The mathematical statement of such problems leads to the inverse

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problem for the diffusion equation, where it is required to find not only a solution of the problem, but also its right-hand side that depends only on a spatial variable.

In recent years, the phenomena of anomalous diffusion have been observed in many fields, such as turbulence, seepage in porous media, pollution control. The demand for appropriate mathematical models is high: from biomechanics to geophysics, to acoustics. A popular approach to depicting a variety of complex anomalous diffusion phenomena is nonlinear modeling. However, it is generally accepted that it is very difficult to analyze mathematically and very expensive to simulate computationally. In addition, nonlinear models often require some parameters unavailable from experiment or field measurements. As alternative approaches, in recent decades fractal and fractional derivatives have been found effective in modeling anomalous diffusion processes. The advantage of the fractal or fractional derivative models over the standard integer-order derivative models is that the former can describe very well the inherent abnormal-exponential or heavy tail decay processes.

Fractional powers in indicators also arise when describing fractal (multiscale, whole-like) media. In a fractal environment, unlike a continuous medium, a randomly wandering particle moves away from the launch site more slowly, since not all directions of motion become available for it. The slowing down of diffusion in the fractal media is so significant that the physical quantities begin to change more slowly than the first derivative and this effect can be taken into account only in the integral-differential equation containing the time derivative of the fractional order:

$$D_t^{\alpha} u(x,t) = A_x u(x,t) + F.$$

In this paper, we will consider an inverse problem close to that investigated in [1], [2]. Together with the solution it is necessary to find the unknown right-hand side of the equation. The equation contains a fractional derivative with respect to time and an involution with respect to a spatial variable. In contrast to [1], [2], we investigate the problem under nonlocal boundary conditions with respect to the spatial variable. The conditions for overdetermination are initial and final states.

The second of the main differences in the inverse problem under consideration is that the unknown function is included both in the right-hand side of the equation and in the conditions of the initial and final overdetermination.

Let us consider a problem of modeling the thermal diffusion process which is close to that described in the report of Cabada and Tojo [2], where the example that describes a concrete situation in physics, is given. Consider a closed metal wire (length 2π) wrapped around a thin sheet of insulation material in the manner shown in Fig. 1.

Assuming that the position $x = 0$ is the lowest of the wire, and the insulation goes up to the left at $-\pi$ and to the right up to π. Since the wire is closed, the points $-\pi$ and π coincide.

The layer of insulation is assumed to be slightly permeable. Therefore, the temperature value from one side affects the diffusion process on the other side. For this reason, the standard

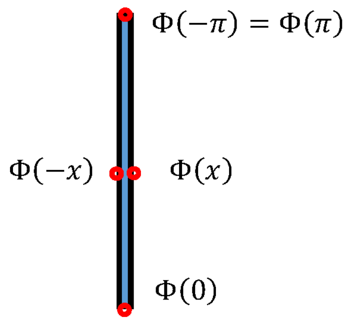

Figure 1 – Closed metal wire wrapped around thin sheet of insulation material

heat equation is modified and to its right-hand side $\frac{\partial^2\Phi}{\partial x^2}(x,t)$ the third term $\varepsilon\frac{\partial^2\Phi}{\partial x^2}(-x,t)$ (where $|\varepsilon| < 1$) is added. Here $\Phi(x,t)$ is the temperature at the point x of the wire at time t.

We will consider a process which is so slow that it is described by an evolutionary equation with a fractional time derivative. Thus, this process is described by the equation

$$t^{-\beta}D_t^{\alpha}\Phi\left(x,t\right) - \Phi_{xx}\left(x,t\right) + \varepsilon\Phi_{xx}\left(-x,t\right) = f\left(x\right) \tag{1}$$

in the domain $\Omega = \{(x,t) : \ -\pi < x < \pi, \ 0 < t < T\}$. Here $f(x)$ is the influence of an external source that does not change with time; $\alpha + \beta > 0$; $t = 0$ is an initial time point and $t = T$ is a final one; and the derivative D_t^{α} defined as

$$D_t^{\alpha}\varphi\left(t\right) = I^{1-\alpha}\left[\frac{d}{dt}\varphi\left(t\right)\right], \ 0 < \alpha < 1, \quad t \in [0,T],$$

is Caputo derivative for a differentiable function built on the Riemann-Liouville fractional integral

$$I^{1-\alpha}[\varphi(t)] = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\varphi(s)}{(t-s)^{\alpha}}ds, \ 0 < \alpha < 1, \quad t \in [0,T].$$

Such a Caputo derivative allows us to impose initial conditions in a natural way.

As the additional information, we take values of one initial and one final conditions of the temperature

$$\Phi\left(x,0\right) = \phi\left(x\right), \quad \Phi(x,T) = \psi(x), \quad x \in [-\pi,\pi]. \tag{2}$$

Since the wire is closed, it is natural to assume that the temperature at the ends of the wire is the same at all times:

$$\Phi\left(-\pi,t\right) = \Phi\left(\pi,t\right), \ t \in \left[0,T\right]. \tag{3}$$

Consider a process in which the temperature at one end at each time point t is proportional to the (fractional) rate of change of the average value of the temperature throughout the wire. Then,

$$\Phi(-\pi, t) = \gamma t^{-\beta} D_t^{\alpha} \int_{-\pi}^{\pi} \Phi(\xi, t)\, d\xi,\ t \in [0, T]. \tag{4}$$

Here γ is a proportionality coefficient.

Thus the process under investigation is reduced to the following mathematical inverse problem: *Find the right-hand side $f(x)$ of the subdiffusion equation* (1)*, and its solution $\Phi(x,t)$ subject to the initial and final conditions* (2)*, boundary condition* (3)*, and condition* (4)*.*

The case when $\alpha = 1$, $\beta = 0$, is examined in [3], [4]. The similar problem was considered by us in [5] for the case when $\gamma > 0$. In this paper, we consider the case $\gamma < 0$.

2 Reduction to mathematical problem

Condition (4) is significantly nonlocal. The integral along inner lines of the domain is presented in this condition. Using the idea of A.A. Samarskii, we transform this condition. Taking into account equation (1), from (4) we get

$$\Phi(-\pi, t) = \gamma \int_{-\pi}^{\pi} \big\{\Phi_{\xi\xi}(\xi, t) - \varepsilon \Phi_{\xi\xi}(-\xi, t) + f(\xi)\big\} d\xi,\ t \in [0, T].$$

Hence

$$\Phi(-\pi, t) = \gamma(1-\varepsilon)[\Phi_x(\pi, t) - \Phi_x(-\pi, t)] + \gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi,\ t \in [0, T].$$

Let us introduce the notations

$$u(x,t) = \Phi(x,t) - \gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi.$$

Then in terms of the new function $u(x,t)$ we get the following inverse problem: *In the domain $\Omega = \{(x,t):\ -\pi < x < \pi,\ 0 < t < T\}$ find a right-hand side $f(x)$ of the time fractional evolution equation with involution*

$$t^{-\beta} D_t^{\alpha} u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \tag{5}$$

and its solution $u(x,t)$ which satisfies one initial condition

$$u(x, 0) = \phi(x) - \gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi, \quad x \in [-\pi, \pi], \tag{6}$$

and one final condition

$$u(x, T) = \psi(x) - \gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi, \quad x \in [-\pi, \pi], \tag{7}$$

and the boundary conditions

$$\begin{cases} u_x(-\pi,t) - u_x(\pi,t) - au(\pi,t) = 0, \\ u(-\pi,t) - u(\pi,t) = 0, \end{cases} \qquad t \in [0,T]. \tag{8}$$

Where $\phi(x)$ and $\psi\,(x)$ are given sufficiently smooth functions; $0 < \alpha < 1$; $\alpha + \beta > 0$; ε is a nonzero real number such that $|\varepsilon| < 1$; and $a = \dfrac{1}{\gamma(\varepsilon - 1)}$.

In the physical sense, the second condition in (8) means the equality of the distribution density at the ends of the interval. And the first condition in (8) means the proportionality of the difference of fluxes across opposite boundaries to the density value at the boundary. We note that in [1] the Dirichlet boundary conditions $u(-\pi,t) = u(\pi,t) = 0$ were used instead of condition (8).

The well-posedness of the direct and the inverse problems for parabolic equations with involution was considered in [6]–[8].

The solvability of various inverse problems for parabolic equations was studied in papers of Anikonov Yu.E. and Belov Yu.Ya., Bubnov B.A., Prilepko A.I. and Kostin A.B., Monakhov V.N., Kozhanov A.I., Kaliev I.A., Sabitov K.B. and many others. These citations can be seen in [9], [10]. In [1] there are good references to publications on related issues. We note [11]-[32] as recent papers close to the theme of our article. In these papers different variants of the direct and inverse initial-boundary value problems for evolutionary equations are considered, including problems with nonlocal boundary conditions and problems for equations with fractional derivatives.

The mathematical problem (5)–(8) for $a = 0$ was considered in [30], and for $a = \beta = 0$ in [31].

We solve the problem by the Fourier method. Some new variants for solving nonlocal boundary value problems by the method of separation of variables were used in our papers [33]-[37]. In this paper we shall use a spectral problem for ordinary differential operators with involution. Such and similar spectral problems were considered in [38]-[49].

Definition. *By a regular solution of the inverse problem* (5)–(8)*, we mean a pair of functions* $(u(x,t), f(x))$ *of the class* $u(x,t) \in C^{2,1}_{x,t}(\overline{\Omega})$*,* $f(x) \in C[-\pi,\pi]$ *that inverts Eq.* (5) *and conditions* (6)–(8) *into an identity.*

Definition. *By a generalised solution of the inverse problem* (5)–(8)*, we mean a pair of functions* $(u(x,t), f(x))$ *of the class* $u(x,t) \in W^{2,1}_2(\Omega) \cap C\left(\overline{\Omega}\right)$*,* $f(x) \in L_2(-\pi,\pi)$ *that satisfy Eq.* (5) *and conditions* (6)–(8) *almost everywhere.*

When one uses the method of separation of variables to solve the problem, a spectral problem appears, which is mentioned in the next section.

3 Spectral problem

The use of the Fourier method for solving problem (5)–(8) leads to the spectral problem for the operator $\mathcal{L}$ given by the differential expression

$$\mathcal{L}X(x) \equiv -X''(x) + \varepsilon X''(-x) = \lambda X(x), \ -\pi < x < \pi, \tag{9}$$

and the boundary conditions

$$\begin{cases} X'(-\pi) - X'(\pi) - aX(\pi) = 0, \\ X(-\pi) - X(\pi) = 0, \end{cases} \tag{10}$$

where λ is a spectral parameter.

The spectral problems for Eq. (9) were first considered, apparently in [40]. There was considered a case of Dirichlet and Neumann boundary conditions, and cases of conditions of the form (10) for $a = 0$. Here we consider the case $a \neq 0$. We assume that $a > 0$.

A general solution of Eq. (9) we represent in the form:

$$X(x) = A\sin(\mu_1 x) + B\cos(\mu_2 x), \quad \mu_1 = \sqrt{\frac{\lambda}{1+\varepsilon}}, \quad \mu_2 = \sqrt{\frac{\lambda}{1-\varepsilon}},$$

where A and B are arbitrary complex numbers. Satisfying the boundary conditions (10) for finding eigenvalues, we obtain the equation

$$\sin(\mu_1 \pi) = 0, \quad \tan(\mu_2 \pi) = \frac{a}{2\mu_2}.$$

Therefore, the spectral problem (9)–(10) has two series of the eigenvalues

$$\lambda_{k,1} = (1+\varepsilon)k^2, \ k \in \mathbb{N};$$

$$\lambda_{k,2} = (1-\varepsilon)(k+\delta_k)^2, \ \delta_k = \frac{a}{k+1}O(1) < 0, \ k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\},$$

with corresponding normalized eigenfunctions given by

$$X_{k,1}(x) = \frac{1}{\sqrt{\pi}}\sin(kx), \quad k \in \mathbb{N}; \quad X_{k,2}(x) = \nu_k \cos((k+\delta_k)x), \ k \in \mathbb{N}_0. \tag{11}$$

Here ν_k is the normalization coefficient:

$$\nu_k^{-2} = \|\cos((k+\delta_k)x)\|^2 = \pi + \frac{a^2}{(k+\delta_k)\left[a^2 + (k+\delta_k)^2\pi^2\right]}.$$

It is easy to see that the system (11) is simultaneously a system of eigenfunctions for the Sturm-Liouville operator

$$\mathcal{L}_1 X(x) \equiv -X''(x) = \lambda X(x), \ -\pi < x < \pi,$$

with the self-adjoint boundary conditions (10) corresponding to the eigenvalues

$$\hat{\lambda}_{k,1} = k^2,\ k \in \mathbb{N};\quad \hat{\lambda}_{k,2} = (k+\delta_k)^2,\ k \in \mathbb{N}_0.$$

Consequently, the system (11) forms the complete orthonormal system in $L_2(-\pi,\pi)$, that is, it is the orthonormal basis of the space.

4 Uniqueness of the solution of the problem

Let the pair of functions $(u(x,t), f(x))$ be a solution of the inverse problem (5)–(8). Let us introduce notations

$$u_{k,i}(t) = \int_{-\pi}^{\pi} u(x,t)X_{k,i}(x)\,dx,\ f_{k,i} = \int_{-\pi}^{\pi} f(x)X_{k,i}(x)\,dx,\ i = 1,2. \tag{12}$$

We apply the operator $t^{-\beta}D^{\alpha}$ to $u_{k,i}(t)$. Then, using Eq. (5), by integrating by parts, we obtain the problem

$$t^{-\beta}D^{\alpha}u_{k,i}(t) + \lambda_{k,i}u_{k,i}(t) = f_{k,i},\quad 0<t<T,\quad i=1,2; \tag{13}$$

$$u_{k,i}(0) = \phi_{k,i} - \sigma_{k,i},\quad i=1,2; \tag{14}$$

$$u_{k,i}(T) = \psi_{k,i} - \sigma_{k,i},\quad i=1,2. \tag{15}$$

Here we use notations

$$\phi_{k,i} = \int_{-\pi}^{\pi}\phi(x)X_{k,i}(x)\,dx,\ \ \psi_{k,i} = \int_{-\pi}^{\pi}\psi(x)X_{k,i}(x)\,dx,\ \ \sigma_{k,i} = \gamma\int_{-\pi}^{\pi}f(\xi)\,d\xi\int_{-\pi}^{\pi}X_{k,i}(x)\,dx.$$

It is easy to see that the function $\tilde{u}_{k,1}(t) = (\lambda_{k,i})^{-1}f_{k,i}$ is a partial solution of the inhomogeneous equation (13). We use the general solution of the homogeneous equation (13), which is constructed in ([50], p. 233) for $\alpha+\beta>0$. Combining them, we get

$$u_{k,i}(t) = \frac{f_{k,i}}{\lambda_{k,i}} + C_{k,i}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ t^{\alpha+\beta}\right),\quad 0<t<T,\quad i=1,2,$$

where $E_{\alpha+\beta,1,1-\alpha}$ is the generalized Mittag-Leffler function ([49], p. 48):

$$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z) = \sum_{k=0}^{\infty}c_k z^k;\ \ c_0=1,\ c_k = \prod_{j=0}^{k-1}\frac{\Gamma(j(\alpha+\beta)+\beta+1)}{\Gamma(j(\alpha+\beta)+\alpha+\beta+1)},\ k\in\mathbb{N},$$

and the constants $C_{k,i}$ and $f_{k,i}$ are unknown.

To find these constants, we use conditions (14) and (15). From (14) we obtain a unique solution of the Cauchy problem (13)–(14) in the form

$$u_{k,i}(t) = \left[1 - E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ t^{\alpha+\beta}\right)\right]\frac{f_{k,i}}{\lambda_{k,i}} + (\phi_{k,i}-\sigma_{k,i})\,E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ t^{\alpha+\beta}\right). \tag{16}$$

Since $\lambda_{k,i} > 0$, then by virtue of the well-known asymptotics [50]:

$$\left|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(z)\right| \le \frac{M}{1+|z|}\ ,\ \arg(z)\ = \pi,\ |z| \to \infty,\ M = Const > 0, \tag{17}$$

under large enough T the estimate

$$1-E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ T^{\alpha+\beta}\right) \ge m^* > 0 \tag{18}$$

will hold, in which the constant m^* does not depend on values of the indexes k, i.

Therefore, using the condition of "final overdetermination" (15), we get

$$f_{k,i} = \lambda_{k,i}\frac{\psi_{k,i} - \phi_{k,i}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ T^{\alpha+\beta}\right)}{1 - E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\lambda_{k,i}\ T^{\alpha+\beta}\right)} - \lambda_{k,i}\sigma_{k,i}. \tag{19}$$

Lemma. *If* (18) *holds for all the values of the indexes* k, i*, then the solution* $(u(x,t), f(x))$ *of the inverse problem* (5)–(8) *is unique.*

Proof. Suppose that there are two solutions $(u_1(x,t), f_1(x))$ and $(u_2(x,t), f_2(x))$ of the inverse problem (5)–(8). Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)\,,\quad f(x) = f_1(x) - f_2(x)\,.$$

Then the functions $u(x,t)$ and $f(x)$ satisfy Eq. (5), the boundary conditions (8) and the homogeneous conditions (6) and (7):

$$u(x,0) = -\gamma\int_{-\pi}^{\pi} f(\xi)\,d\xi,\quad x \in [-\pi,\pi],$$

$$u(x,T) = -\gamma\int_{-\pi}^{\pi} f(\xi)\,d\xi,\quad x \in [-\pi,\pi].$$

Therefore, by using the notations (12), from (19) we find

$$f_{k,i} = -\lambda_{k,i}\sigma_{k,i} \equiv -\lambda_{k,i}\gamma\int_{-\pi}^{\pi} f(\xi)\,d\xi\int_{-\pi}^{\pi} X_{k,i}(x)\,dx.$$

Since

$$\lambda_{k,i}\int_{-\pi}^{\pi} X_{k,i}(x)\,dx = -\int_{-\pi}^{\pi} X''_{k,i}(x)\,dx = X'_{k,i}(-\pi) - X'_{k,i}(\pi) = aX_{k,i}(\pi)\,,$$

this gives

$$f_{k,i} = -a\left(\gamma\int_{-\pi}^{\pi} f(\xi)\,d\xi\right)X_{k,i}(\pi)\,. \tag{20}$$

Since $X_{k,1}(\pi) = 0$, then for $i = 1$ from Eq. (20) we have $f_{k,1} = 0$.
Since $a \neq 0$,

$$X_{k,2}(\pi) = \nu_k \cos\left((k+\delta_k)\pi\right), \quad \lim_{k\to\infty} \nu_k = \frac{1}{\sqrt{\pi}}, \quad \lim_{k\to\infty} \delta_k = 0,$$

then Eq. (20) is possible only if

$$\left(\gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi\right) = 0.$$

Hence we obtain $f_{k,2} = 0$.

Therefore, using this result, from (16) and (19) we find

$$u_{k,i}(t) \equiv \int_{-\pi}^{\pi} u(x,t) X_{k,i}(x)\, dx = 0, \;\; f_{k,i} \equiv \int_{-\pi}^{\pi} f(x) X_{k,i}(x)\, dx = 0$$

for all values of the indexes $k \in \mathbb{N}$ for $i = 1$ and $k \in \mathbb{N}_0$ for $i = 2$. Further, by the completeness of the system (11) in $L_2(-\pi, \pi)$ we obtain

$$u(x,t) \equiv 0, \;\; f(x) \equiv 0 \;\; \forall \;\; (x,t) \in \overline{\Omega}.$$

The uniqueness of the solution of the inverse problem (5)–(8) is proved.

5 Construction of formal solution of the problem

As the eigenfunctions of the system (11) forms an orthonormal basis in $L_2(-\pi, \pi)$, the unknown functions $u(x,t)$ and $f(x)$ can be represented as

$$u(x,t) = \sum_{k=1}^{\infty} u_{k,1}(t) X_{k,1}(x) + \sum_{k=0}^{\infty} u_{k,2}(t) X_{k,2}(x), \tag{21}$$

$$f(x) = \sum_{k=1}^{\infty} f_{k,1} X_{k,1}(x) + \sum_{k=0}^{\infty} f_{k,2} X_{k,2}(x), \tag{22}$$

where $u_{k,1}(t)$ and $u_{k,2}(t)$ are unknown functions; $f_{k,1}$ and $f_{k,2}$ are unknown constants.

Substituting (21) and (22) into equation (5), we obtain the inverse problems (13)–(15). If the constants $\sigma_{k,i}$ are assumed to be given, then the solutions of these inverse problems exist, are unique and represented by formulas (16) and (19). Substituting (16) and (19) into series (21) and (22), we obtain a formal solution of the inverse problem (5)–(8).

From the analysis of formula (19) it is easy to see that the formal solution (21) of the problem (5)–(8) will form a convergent series if and only if

$$\lim_{k\to\infty} \lambda_{k,i} \sigma_{k,i} = 0, \quad i = 1, 2. \tag{23}$$

As above, we calculate

$$\lambda_{k,i}\sigma_{k,i} \equiv a\left(\gamma \int_{-\pi}^{\pi} f(\xi)\, d\xi\right) X_{k,1}(\pi),$$

where $X_{k,1}(\pi) = 0$ and

$$X_{k,2}(\pi) = \nu_k \cos\left((k+\delta_k)\pi\right), \quad \lim_{k\to\infty} \nu_k = \frac{1}{\sqrt{\pi}}, \quad \lim_{k\to\infty} \delta_k = 0.$$

Thus, (23) holds if and only if $\sigma_{k,i} = 0$ for all values of the indexes k, i. This is possible only in the case

$$\int_{-\pi}^{\pi} f(\xi)\, d\xi = 0. \tag{24}$$

In this case, problems (5)–(8) and (1)–(4) coincide. Indeed, from (24) and Eq. (1) we have

$$0 = \int_{-\pi}^{\pi} f(\xi)\, d\xi = \int_{-\pi}^{\pi} t^{-\beta} D_t^{\alpha} \Phi(\xi, t)\, d\xi - \int_{-\pi}^{\pi} \Big\{\Phi_{\xi\xi}(\xi, t) + \varepsilon \Phi_{\xi\xi}(-\xi, t)\Big\} d\xi.$$

For the first integral, we apply condition (4), and calculate the second integral. Then we obtain

$$0 = (1-\varepsilon)\left[\Phi_x(-\pi, t) - \Phi_x(\pi, t) + \frac{1}{\gamma(1-\varepsilon)}\Phi(-\pi, t)\right].$$

This means that the boundary conditions (4) and (8) coincide. Hence, the problems (5)–(8) and (1)–(4) also coincide.

Thus, in what follows, we shall consider the problem (1)–(3) with the boundary condition

$$\Phi_x(-\pi, t) - \Phi_x(\pi, t) - a\Phi(-\pi, t) = 0. \tag{25}$$

Thus, in what follows, we will consider the inverse problem (1)–(3), (25).

Similarly, as before, the formal solution of this problem can be constructed in the form of series

$$\Phi(x, t) = \sum_{k=1}^{\infty} \Phi_{k,1}(t) X_{k,1}(x) + \sum_{k=0}^{\infty} \Phi_{k,2}(t) X_{k,2}(x), \tag{26}$$

$$f(x) = \sum_{k=1}^{\infty} f_{k,1} X_{k,1}(x) + \sum_{k=0}^{\infty} f_{k,2} X_{k,2}(x), \tag{27}$$

where

$$\Phi_{k,i}(t) = \left(\phi_{k,i} - \frac{f_{k,i}}{\lambda_{k,i}}\right) E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, t^{\alpha+\beta}\right) + \frac{f_{k,i}}{\lambda_{k,i}}\ , \tag{28}$$

$$f_{k,i} = \lambda_{k,i} \frac{\psi_{k,i} - \phi_{k,i} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, T^{\alpha+\beta}\right)}{1 - E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\lambda_{k,i}\, T^{\alpha+\beta}\right)}. \tag{29}$$

In order to complete our study, it is necessary, as in the Fourier method, to justify the smoothness of the resulting formal solutions and the convergence of all appearing series.

6 Main results

Here we present the existence and the uniqueness results for our inverse problem.

Theorem. *Let $a > 0$, $\alpha + \beta > 0$ and T be large enough that condition* (18) *holds for all values of the indexes k, i.*

(A) Let $\phi(x)$, $\psi(x) \in W_2^2(-\pi, \pi)$ and satisfy the boundary conditions (10). *Then for a real number ε such that $|\varepsilon| < 1$ the inverse problem* (1)–(3), (25) *has a unique generalized solution, which is stable in the norm:*

$$\left\| t^{-\beta} D_t^{\alpha} \Phi \right\|_{L_2(\Omega)}^2 + \|\Phi_{xx}\|_{L_2(\Omega)}^2 + \|f\|_{L_2(-\pi,\pi)}^2 \leq C \left\{ \|\phi\|_{W_2^2(-\pi,\pi)}^2 + \|\psi\|_{W_2^2(-\pi,\pi)}^2 \right\},$$

where the constant C does not depend on $\phi(x)$, $\psi(x)$.

(B) Let $\phi(x), \psi(x) \in C^4[-\pi, \pi]$ and let the functions $\phi(x), \psi(x)$ and $\phi''(x)$, $\psi''(x)$ satisfy the boundary conditions (10), *then for a real number ε such that $|\varepsilon| < 1$ the inverse problem* (1)–(3), (25) *has a unique regular solution.*

Proof. The generalized Mittag-Leffler function's estimates (17) and (18) are known. Therefore, from representations (16) and (19) we get estimates

$$|f_{k,i}| \leq C_1 |\lambda_{k,i}| \left\{ |\phi_{k,i}| + |\psi_{k,i}| \right\}, \tag{30}$$

$$|\Phi_{k,i}(t)| \leq C_1 \left\{ |\phi_{k,i}| + |\psi_{k,i}| \right\}, \tag{31}$$

where the constant C_1 does not depend on the indexes k, i and on the functions $\phi(x), \psi(x)$. Since the system of eigenfunctions (11) forms an orthonormal basis in $L_2(-\pi, \pi)$, then by virtue of the Parseval equality, from this it is easy to obtain estimates

$$\|f\|_{L_2(-\pi,\pi)}^2 \leq C \left\{ \left\|\phi''\right\|_{L_2(-\pi,\pi)}^2 + \left\|\psi''\right\|_{L_2(-\pi,\pi)}^2 \right\}, \tag{32}$$

$$\|\Phi_{xx}\|_{L_2(\Omega)}^2 \leq C \left\{ \left\|\phi''\right\|_{L_2(-\pi,\pi)}^2 + \left\|\psi''\right\|_{L_2(-\pi,\pi)}^2 \right\}. \tag{33}$$

In deriving these inequalities we have used the fact that the functions $\phi(x), \psi(x)$ satisfy the boundary conditions (10). Now we can easily obtain an estimate for $t^{-\beta} D_t^{\alpha} \Phi(x, t)$ from Eq. (5). This together with (32) and (33) gives the necessary estimate for the solution.

From the obtained estimates it also follows that in the formal solution of the inverse problem constructed by us all the series converge, they can be term-by-term differentiated, and the series obtained during differentiation also converge in the sense of the metrics L_2.

From (21) and (31), by using the Holder's inequality, it is easy to justify the inequality

$$\max_{(x,t)\in\overline{\Omega}} |\Phi(x,t)|^2 \leq C\left\{\|\phi''\|^2_{L_2(-\pi,\pi)} + \|\psi''\|^2_{L_2(-\pi,\pi)}\right\},$$

which justifies the continuity of $\Phi(x,t)$ in the closed domain $\overline{\Omega}$.

From the representation of the solution in the form of series (21), (22) and inequalities (30), (31) it is easy to justify estimates

$$|\Phi_{xx}(x,t)| + |\Phi_{tt}(x,t)| + |f(x)| \leq C\sum_{k=1}^{\infty} |\lambda_{k,i}|^2 \Big\{ |\phi_{k,i}| + |\psi_{k,i}| \Big\}. \tag{34}$$

Let $\phi(x), \psi(x) \in C^4[-\pi,\pi]$ and the functions $\phi(x), \psi(x)$ and $\phi''(x)$, $\psi''(x)$ satisfy the boundary conditions (10), then the number series in the right-hand side of (34) converges. Therefore, in this case the formal solution constructed by us gives the regular solution of the inverse problem (5)–(8). The Theorem is completely proved.

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Садыбеков М.А., Сәрсенбі А.А. ЖЫЛУ СУБДИФФУЗИЯСЫНЫҢ ҮДЕРІСІН ПЕРИОДТЫ БЕРІЛІМДЕРІ БОЙЫНША ҚАЛПЫНА КЕЛТІРУ КЕРІ ЕСЕБІ ТУРАЛЫ

Бұл мақалада жұқа қабатты оқшаулағыш материалға оратылған тұйықталған металл сымдағы термодиффузия үдерісін моделдеу есебі қарастырылған. Оқшаулағыш қабатының аздаған өткізгіштік қасиеті бар деп жорамалданады. Яғни, оның бір жағындағы температурасының мәні екінші жағындағы диффузиялық үдеріске әсер етеді. Сол себепті стандартты жылуөткізгіштік теңдеуі оған инволюциясы бар үшінші қосылғышты қосу арқылы түрлендіріледі. Аталған үдерісті моделдеу инволюциясы бар және кеңістіктік айнымалысы бойынша периодты шекаралық шарттары болатын бір өлшемді бөлшек ретті эволюциялық теңдеу үшін кері есепті қарастыруға алып келеді. Мұндай теңдеулерді бейлокал субдиффузия теңдеулері немесе бейлокал жылуөткізгіштік теңдеулері деп те атайды. Кері есеп теңдеудің оң жағындағы тек кеңістіктік айнымалыға ғана тәуелді болатын белгісіз функцияны (есептің шешімімен қатар) қалпына келтіру мәселесін қамтиды. Бастапқы күйі мен ақырғы күйі қайта анықтау шарттары болып табылады. Бұл есептің шешімінің бар болуы мен жалғыздығы айнымалыларды айыру тәсілімен дәлелденеді.

Кілттік сөздер. Кері есеп, бөлшек ретті эволюциялық теңдеу, бейлокал субдиффузия теңдеуі, бейлокал жылуөткізгіштік теңдеуі, инволюциясы бар теңдеу, периодты шекаралық шарттар, айнымалыларды айыру тәсілі.

Садыбеков М.А., Сарсенби А.А. ОБ ОБРАТНОЙ ЗАДАЧЕ ВОССТАНОВЛЕНИЯ ПРОЦЕССА ТЕПЛОВОЙ СУБДИФФУЗИИ ПО ПЕРИОДИЧЕСКИМ ДАННЫМ

В этой статье рассматривается задача моделирования процесса термодиффузии в замкнутой металлической проволоке, намотанной на тонкий лист изоляционного материала. Слой изоляции предполагается слегка проницаемым. Следовательно, значение температуры с одной стороны влияет на процесс диффузии на другой стороне. По этой причине стандартное уравнение теплопроводности модифицируется и добавляется третий член с инволюцией. Моделирование этого процесса приводит к рассмотрению обратной задачи для одномерного уравнения дробной эволюции с инволюцией и периодическими граничными условиями по пространственной переменной. Такие уравнения также называют нелокальными уравнениями субдиффузии или нелокальными уравнениями теплопроводности. Обратная задача состоит в восстановлении (одновременно с решением) неизвестной правой части уравнения, которая зависит только от пространственной переменной. Условиями переопределения являются начальное и конечное состояния. Результаты существования и единственности решения для данной задачи получены методом разделения переменных.

Ключевые слова. Обратная задача, уравнение дробной эволюции, нелокальное уравнение субдиффузии, нелокальное уравнение теплопроводности, уравнение с инволюцией, периодические граничные условия, метод разделения переменных.

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Stability of program manifold of indirect control systems with variable coefficients

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Abstract. The absolute stability of program manifold of indirect control systems with variable coefficients and with stationary nonlinearities is considered. Conditions of the stability of indirect control systems are investigated in the neighborhood of the given program manifold. Nonlinearity satisfies to the conditions of local quadratic connection. Sufficient conditions of the absolute stability of the program manifold with respect to given vector functions are obtained by constructing Lyapunov function. A method of choosing Lyapunov matrix is specified. Also sufficient conditions of the exponential absolute stability are received.

Keywords. Program manifold, absolute stability, stationary nonlinearity, variable coefficients, Lyapunov functions, local quadratic connection.

1 Introduction

The inverse problems of the theory of ordinary differential equations have been intensively developed as applied problems [1], [2]. These are namely the problem of the analytical construction of systems of program motion, the general problem of constructing systems of differential equations, the construction of automatic control systems for a given program manifold.

At solving inverse problems of the dynamics of automatic control systems, the basic and obligatory requirement is stability of program motion in the presence of unstable actuating elements and system deviations from the given program at the initial time.

The analysis of works in this direction shows that an essential part of the literature is devoted to the study of program manifold of control systems with constant coefficients (see [3]–[11]).

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At the same time, in mathematical modelling of various physical, chemical, biological and environmental, etc. phenomena in the most cases lead to the need of research of control systems with variable coefficients. This is the movement of a point of variable mass, moving objects in which there is a change in mass and moment over time, in particular, jet thrust aircraft with variable mass (see [12]–[14]).

In the class of t times continuously-differentiable and bounded on a norm matrices Ξ we consider the program manifold $\Omega\left(t\right) \equiv \omega(t,x) = 0$, which is an integral for the system

$$\dot{x} = f\left(t,x\right) - B(t)\xi,\ \dot{\xi} = \varphi\left(\sigma\right),\ \sigma = P^T(t)\omega - Q(t)\xi,\ t \in I = \left[0,\ \infty\right), \tag{1}$$

provided $Q(t) >> 0$, where $x \in R^n$ is a state vector of the object, $f \in R^n$ is a vector-function, satisfying conditions of the existence of the solution: $x(t) = 0$, $B(t) \in \Xi^{n\times r}$, $P(t) \in \Xi^{s\times r}$ are continuous matrices, $\omega \in R^s (s \leq n)$ is a vector, $\varphi(\sigma) \in R^r$ is a vector-function of control on deviation from given program manifold, satisfying conditions of local quadratic connection:

$$\varphi(0) = 0\ \wedge\ 0 < \sigma^T\varphi(\sigma) \leq \sigma^T K\sigma\,;$$

$$K = diag\,\|\,k_1, \ldots, k_r\|,\quad K = K^T > 0. \tag{2}$$

In the space R^n we select the domain $G\left(R\right)$:

$$G\left(R\right) = \left\{(t,x) : t \in I \wedge \|\omega\left(t,x\right)\| \leq \rho < \infty\right\}. \tag{3}$$

Suppose that for all $t \geq t_0$ the following conditions are satisfied:

1) the vector-function $f(t,x)$ is continuous in all variables and satisfy Lipschitz conditions for $x \in G$;

2) the vector-function ω and its partial derivatives are continuous in some closed bounded simply connected domain $G \subset X_n$, containing the manifold $\Omega(t)$;

3) the rank of the functional matrix $rank\left\|\dfrac{\partial\omega}{\partial x}\right\| = s$ at all points of $\Omega(t)$.

Due to the fact that $\Omega(t)$ is the integral manifold for the system (1)–(2), we have

$$\dot{\omega} = \frac{\partial\omega}{\partial t} + Hf\left(t,x\right) = F(t,x,\omega), \tag{4}$$

where $H = \dfrac{\partial\omega}{\partial x}$ is Jacobi matrix and $F(t,x,\omega)$ is a certain s-dimensional Erugin vector-function, satisfying conditions $F(t,x,0) \equiv 0$ [1].

Taking into account that $\Omega(t)$ is the integral manifold for the system (1), and by choosing the Erugin function as following

$$F(t,x,\omega) = -A\omega, \tag{5}$$

where $-A \in R^{s\times s}$ is Hurwitz matrix, and differentiating the manifold $\Omega(t)$ with respect to the time t along the solutions of the system (1), we get [2]:

$$\dot{\omega} = -A(t)\omega - H(t)B(t)\xi,\ \dot{\xi} = \varphi\left(\sigma\right),\ \sigma = P^T(t)\omega - Q(t)\xi, \tag{6}$$

$$\varphi(0) = 0\ \wedge\ 0 < \sigma^T\varphi(\sigma) \leq \sigma^T K\sigma\,;$$

$$K = diag\,\|\,k_1, \ldots, k_r\|,\ K = K^T > 0. \tag{7}$$

The system (6)–(7) has only a position of equilibrium $x = \xi = 0$ if and only if, when

$$\det\left\|\begin{array}{cc} A & HB \\ -P^T & Q \end{array}\right\| \neq 0.$$

Definition 1. *A program manifold $\Omega(t)$ is called absolutely stable with respect to a vector-function ω, if it is asymptotically stable on the whole at all functions $\varphi(\sigma)$ satisfying to the conditions (7).*

Statement of the problem. To get conditions of absolute stability of the program manifold $\Omega(t)$ of the indirect control systems with variable coefficients with respect to the given vector-function ω.

2 Asymptotical stability of program manifold

First, we consider the following system with variable coefficients as a linear approximation of the system (6)–(7) with respect to the vector-function ω:

$$\dot{\omega} = -A(t)\omega,\ t \in I = [0, \infty). \tag{8}$$

If for ω we construct the Lyapunov function

$$V(t, \omega) = \omega^T L(t)\omega, \tag{9}$$

then the time derivative of V due to the system (8) is obtained in the following form

$$W(t, \omega) = \omega^T G(t)\omega,$$

where $G(t) = G^T(t)$ is a symmetric matrix of the form

$$G(t) = -\frac{dL(t)}{dt} + L(t)A(t) + A^T(t)L(t). \tag{10}$$

Let the matrix $A(t) \in \Xi^{s\times s}$ be non-degenerate, let the matrices $A(t)$ and $M(t)$ satisfy the equality

$$A^T(t)M(t) = M^T(t)A(t), \tag{11}$$

where $M(t) \in \Xi^{s\times s}$ is an arbitrary matrix. Then matrix $L(t)$ can be taken in the form

$$L(t) = M(t)A^{-1}(t). \tag{12}$$

By virtue of (12) from relations (10) we get

$$G(t) = M(t) + M^T(t) - \frac{dL(t)}{dt}A^{-1}(t) - M(t)\frac{dA^{-1}(t))}{dt}. \tag{13}$$

By the Kronecker-Capelli theorem, there always exists the matrix $M(t)$ that satisfies equation (10).

Theorem 1. *Let the Erugin function $F(t,x,\omega)$ have the form (5). Then, if the matrix $A(t)$ of the system (7) is non-degenerate and together with the matrix $M(t)$ satisfy equality (10), then whatever the given quadratic form with the matrix $G(t)$ there exists a unique quadratic form $W(t)$ with the matrix $L(t)$ and satisfies the equation*

$$-\frac{dV(t,\omega)}{dt}\Big|_{(7)} = W(t,\omega) = \omega^T G(t)\omega.$$

The following theorem is valid:

Theorem 2. *Let the Erugin function $F(t,x,\omega)$ have the form (5). Then for the asymptotic stability in the whole of the program manifold $\Omega(t)$ of a linear system with variable coefficients with respect to to the vector-function ω, it is sufficient the fulfillment of relations*

$$L(t) = M(t)A^{-1}(t) >> 0 \wedge G(t) >> 0 \quad t \in I = [0,\infty),$$

where $G(t)$ is determined by the formula (13).

3 Absolute stability of program manifold

Definition 2. *We call a function $V(t,\omega)$ definitely positive and admitting a positive higher limit as a whole with respect to the vector-function ω, if we can specify two continuous functions $V_1(\omega)$ and $V_2(\omega)$ such that for all ω the following inequalities hold*

$$V_1(\omega) \le V(t,\omega) \le V_2(\omega),$$

and the functions V_1 and V_2 are positive in the domain (3), moreover

$$\lim_{\|\omega\|\to\infty} V_!(\omega) = \infty$$

and

$$V_1(0) = V_2(0) = 0.$$

The basic theorem. *If there is a real, continuously differentiable function $V(t,\omega)$ in the domain (3) and positive-definite and allowing the highest limit in whole such that its derivative*

$$-\frac{dV}{dt}\Big|_{(6)} = W(t,\omega)$$

would be positive-definite for any function $\varphi(\sigma)$ satisfying conditions (7), then the program manifold $\Omega(t)$ is absolutely stable with respect to vector functions $\omega(t,x)$.

Theorem 3. *Let the Erugin function $F(t,x,\omega)$ have the form (5) and suppose that there exist matrices*

$$L(t) = L^T(t) > 0,\ \beta = diag\,(\beta_1, \ldots, \beta_r) > 0$$

and non-linear function $\varphi(\sigma)$ satisfies the conditions (7). Then, for the absolute stability of the program manifold $\Omega\,(t)$ with respect to the vector-function ω it is sufficient the fulfillment of the following conditions:

$$l_1(\|\omega\|^2 + \|\xi\|^2) \le V \le l_2(\|\omega\|^2 + \|\xi\|^2), \tag{14}$$

$$g_1(\|\omega\|^2 + \|\xi\|^2) \le W \le g_2(\|\omega\|^2 + \|\xi\|^2), \tag{15}$$

where l_1, l_2, g_1, g_2 are positive constants.

Proof. Let there exist matrices

$$L(t) = L^T(t) > 0,\ \beta = diag\,(\beta_1, \ldots, \beta_r) > 0,$$

then for the system (3) we can construct the Lyapunov function of the form

$$V(\omega,\ \xi) = \omega^T L(t)\omega + \int\limits_0^\sigma \varphi^T \beta d\sigma > 0. \tag{16}$$

The second term in (16) is equal to $J = \dfrac{\sigma^T h\beta\sigma}{2}$ in the case $\varphi(\sigma) = h\sigma,\ h \le K$.

For this case we have estimates:

$$l_1(t)\|z\|^2 \le V \le l_2(t)\|z\|^2, \tag{17}$$

here $l_1(t),\ l_2(t)$ are real, positive, continuous, smallest and largest roots of the characteristic equation

$$\det\|\Lambda - lE\| = 0, \Lambda = \left\| \begin{matrix} L_1 & L_2 \\ L_2^T & L_3 \end{matrix} \right\|,$$

$$L_1 = L(t) + Ph\beta P^T; \quad L_2 = Ph\beta Q; \quad L_3 = O^T h\beta Q; \quad z = \left\| \begin{matrix} \omega \\ \xi \end{matrix} \right\|.$$

A derivative on time t of this function in view of the system (6) will take the following form

$$-\dot{V} = \omega^T G_0\omega + 2\omega^T G_1\varphi + \varphi^T G_3\varphi + 2\omega^T G_2\xi + 2\varphi^T G_4\xi > 0, \tag{18}$$

where

$$G_0 = A^T L + LA - \dot{L},\ G_2 = LHB,\ G_1 = \frac{1}{2}(\beta A^T P\beta - \dot{P}\beta),$$

$$G_3 = \beta O,\ G_4 = \frac{1}{2}(\beta P^T HB + \beta\dot{Q}),\ G_5 = 0,$$

$$G = \left\| \begin{array}{ccc} G_0 & G_1 & G_2 \\ G_1^T & G_3 & G_4 \\ G_2^T & G_4^T & G_5 \end{array} \right\| >> 0. \tag{19}$$

Thus, for absolute stability of the program manifold, it is sufficient to fulfill the generalized Sylvester conditions (19).

In relation (19), in order to obtain estimates for the functions ω and ξ, we perform the following replacement $\varphi = h\sigma$, then we obtain the following inequality

$$-\dot{V} = \omega^T G_0 \omega + 2\omega^T G_1 \xi + \xi^T G_2 \xi > 0, \tag{20}$$

where

$$G_0 = A^T L + LA - \dot{L} + A^T P\beta h P^T - \dot{P}\beta h P^T + Ph\beta Q h P^T;$$

$$G_1 = LHB + \frac{1}{2}(\dot{P}\beta h Q - A^T P\beta h Q$$

$$+ Ph\beta P^T HB + Ph\beta\dot{Q} - Ph\beta Q h Q - PhQ^T \beta h Q);$$

$$G_2 = Q^T h\beta Q h Q - Q^T \beta h Q - B^T H^T P\beta h Q.$$

Based on inequality (19), the following estimates are valid

$$g_1(t)\|z\|^2 \le -\dot{V} \le g_2(t)\|z\|^2, \tag{21}$$

here $g_1(t), g_2(t)$ are real, positive, continuous, smallest and largest roots of the characteristic equation

$$\det \|\widetilde{G} - gE\| = 0, \quad \widetilde{G} = \left\| \begin{array}{cc} G_0 & G_1 \\ G_1^T & G_2 \end{array} \right\|,$$

if we assume that

$$g_1 = \inf_t g_1(t) \wedge g_2 = \sup_t g_2(t), \tag{22}$$

$$l_1 = \inf_t l_1(t) \wedge l_2 = \sup_t l_2(t). \tag{23}$$

Based on Theorem 1 and the basic theorem, we conclude: when the nonlinearity $\varphi(\sigma)$ satisfies the conditions (7), from estimates (16) and (20) it follows that conditions of Theorem 2 hold in case (21) and (22), then the program manifold $\Omega(t)$ is absolutely stable with respect to the vector-function ω. Therefore, the proof is complete.

4 Exponential absolute stability of program manifold

Taking into account the estimates (16) and (20), we obtain the inequalities

$$l_2^{-1}V_0exp\Big[-\int\limits_{t_0}^{t}\alpha_1(t)dt\Big]\leq\parallel z\parallel^2\leq l_1^{-1}V_0exp\Big[-\int\limits_{t_0}^{t}\alpha_2(t)dt\Big],\tag{24}$$

where

$$\alpha_1(t)=\frac{\big[q_2(t)\big]}{\big[l_1(t)\big]};\quad \alpha_1(t)=\frac{\big[q_2(t)\big]}{\big[l_1(t)\big]};\quad V_0=V(\omega_0,\xi_0).$$

If we assume that

$$\alpha_1=\inf_t\alpha_1(t)\wedge\alpha_2=\sup_t\alpha_2(t),$$

then from the inequality (24) we obtain simplified estimate

$$l_2^{-1}V_0exp\big[-\alpha_1(t-t_0)\big]\leq\parallel z\parallel^2\leq l_1^{-1}V_0exp\big[-\alpha_2(t-t_0)\big].\tag{25}$$

We introduce two spheres

$$\|z(t_0)\|^2=\|\omega(t_0)\|^2+\|\xi(t_0)\|^2=R^2,$$

$$\|z(t_0^\star)\|^2=\|\omega(t_0^\star)\|^2+\|\xi(t_0^\star)\|^2=\varepsilon^2,\quad R>>\varepsilon.$$

For asymptotically stable systems for all ω_0,ξ_0 on a sphere R there exists $t_0^\star$ under which

$$\|\omega(t_0^\star)\|^2+\|\xi(t_0^\star)\|^2=\varepsilon^2,\quad \|\omega(t_0^\star)\|^2+\|\xi(t_0^\star)\|^2<\varepsilon^2\quad\forall\ t>t_0^\star.$$

In view of the given inequalities (16) and (21) on the sphere R, we obtain the following relations

$$\parallel z\parallel^2\leq l_1^{-1}l_2Rexp\big[-\alpha_2(t-t_0)\big].\tag{26}$$

Corollary 1. *Let the Erugin function $F(t,x,\omega)$ have the form (5) and suppose that there exist matrices*

$$L=L^T>0,\ \beta=diag\,(\beta_1,\dots,\beta_r)>0$$

and non-linear function $\varphi(\sigma)$ satisfies conditions (4). Then, for the exponential absolute stability of the program manifold $\Omega\,(t)$ with respect to the vector-function ω on the sphere R it is sufficient the fulfillment of conditions (26).

Remark. Estimates (17) and (19) can be obtained by using properties (5) and a structure of feedback σ as following

$$0<\int\limits_0^{\sigma}\varphi^T\beta d\sigma<\frac{\beta_1k_1}{2}\|\sigma\|^2;\quad 0<\|\varphi\|^2<k_1\|\sigma\|^2;$$

$$p_1(\|\omega\|^2+\|\xi\|^2)\leq\|\sigma\|^2\leq\rho_3(\|\omega\|^2+\|\xi\|^2),\quad \rho_3=\max\{\rho_2,\ \nu_2\},\tag{27}$$

where $k_1=\min\{k_i\},\ \beta_1=\max\{\beta_i\}\ (i=1,2,\dots,r);\ \beta_i\ k_i$ are elements of the matrices β and K, ρ_1,ρ_2 and ν_1,ν_2 are determined as following:

$$\rho_1=\min_{\omega\neq 0}\frac{\omega^T PP^T\omega}{\omega^T\omega};\quad \rho_2=\max_{\omega\neq 0}\frac{\omega^T PP^T\omega}{\omega^T\omega};$$

$$\nu_1=\min_{\xi\neq 0}\frac{\xi^T RR^T\xi}{\xi^T\xi};\quad \nu_2=\max_{\xi\neq 0}\frac{\xi^T RR^T\xi}{\xi^T\xi};\quad z=\left\|\begin{array}{c}\omega\\ \xi\end{array}\right\|,$$

$$\det\|\Pi-pE\|=0,\quad \Pi=\left\|\begin{array}{cc}P_1 & P_2\\ P_2^T & P_3\end{array}\right\|,\tag{28}$$

where p_1,p_2 are real smallest and largest roots of the characteristic equation (28),

$$P_1=PP^T;\quad P_2=PQ^T;\quad P_3=Q^TQ.$$

Then, by virtue of (7) and (8) we have

$$l_1(h)(\|\omega\|^2+\|\xi\|^2)\leq V\leq l_s(h)(\|\omega\|^2+\|\xi\|^2),\tag{29}$$

where $l_1(h)=min\{(l_1+\gamma p_1),\gamma p_1\},l_s(h)=max\{(l_2+\gamma p_2),\gamma\nu_2\},\gamma=\dfrac{\beta_1 k_1}{2}$,

$$g_1(h)(\|\omega\|^2+\|\xi\|^2+\|\varphi\|^2)\leq V\leq g_s(h)(\|\omega\|^2+\|\xi\|^2+\|\varphi\|^2),\qquad (30),$$

where g_1,g_2 are real smallest and largest roots of the characteristic equation

$$\det\|G-g(h)E\|=0.$$

Taking into account estimates (27), from relations (30) we get

$$g_1(h)(\|\omega\|^2+\|\xi\|^2\leq V\leq g_s(h)\eta_2(\|\omega\|^2+\|\xi\|^2),$$

$$\eta_2=\max\{1,1+k_1\rho_3\}.$$

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Жұматов С.С. АЙНЫМАЛЫ КОЭФФИЦИЕНТТІ ТУРА ЕМЕС БАСҚАРУ ЖҮЙЕЛЕРІНІҢ БАҒДАРЛАМАЛЫҚ КӨПБЕЙНЕСІНІҢ ОРНЫҚТЫЛЫҒЫ

Стационар сызықсыздықтары бар және айнымалы коэффициентті тура емес басқару жүйелерінің бағдарламалық көпбейнесінің абсолютті орнықтылығы қарастырылады. Тура емес басқару жүйелерінің орнықтылық шарттары берілген бағдарламалық көпбейне маңайында зерттелді. Сызықсыздықтар локалды квадраттық байланыс шарттарын қанағаттандырады. Берілген вектор-функция бойынша бағдарламалық көпбейненің абсолютті орнықтылығының жеткілікті шарттары Ляпунов функциясын тұрғызу арқылы алынды. Ляпунов матрицасын таңдаудың дәйекті әдісі көрсетілді. Сонымен бірге экспоненциалды абсолютті орнықтылықтың жеткілікті шарттары алынды.

Кілттік сөздер. Бағдарламалық көпбейне, абсолютті орнықтылық, стационар сызықсыздық, айнымалы коэффициенттер, Ляпунов функциялары, локалды квадраттық байланыс.

Жуматов С.С. УСТОЙЧИВОСТЬ ПРОГРАММНОГО МНОГООБРАЗИЯ СИСТЕМ НЕПРЯМОГО УПРАВЛЕНИЯ С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ

Рассматривается абсолютная устойчивость программного многообразия систем непрямого управления с переменными коэффициентами и со стационарными нелинейностями. Условия устойчивости систем непрямого управления исследованы в окрестности заданного программного многообразия. Нелинейности удовлетворяют условиям локальной квадратичной связи. Достаточные условия абсолютной устойчивости программного многообразия относительно заданной вектор-функции получены с помощью построения функции Ляпунова. Указан конкретный метод подбора матрицы Ляпунова. Также получены достаточные условия экспоненциальной абсолютной устойчивости.

Ключевые слова. Программное многообразие, абсолютная устойчивость, стационарная нелинейность, переменные коэффициенты, функции Ляпунова, локальная квадратичная связь.

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