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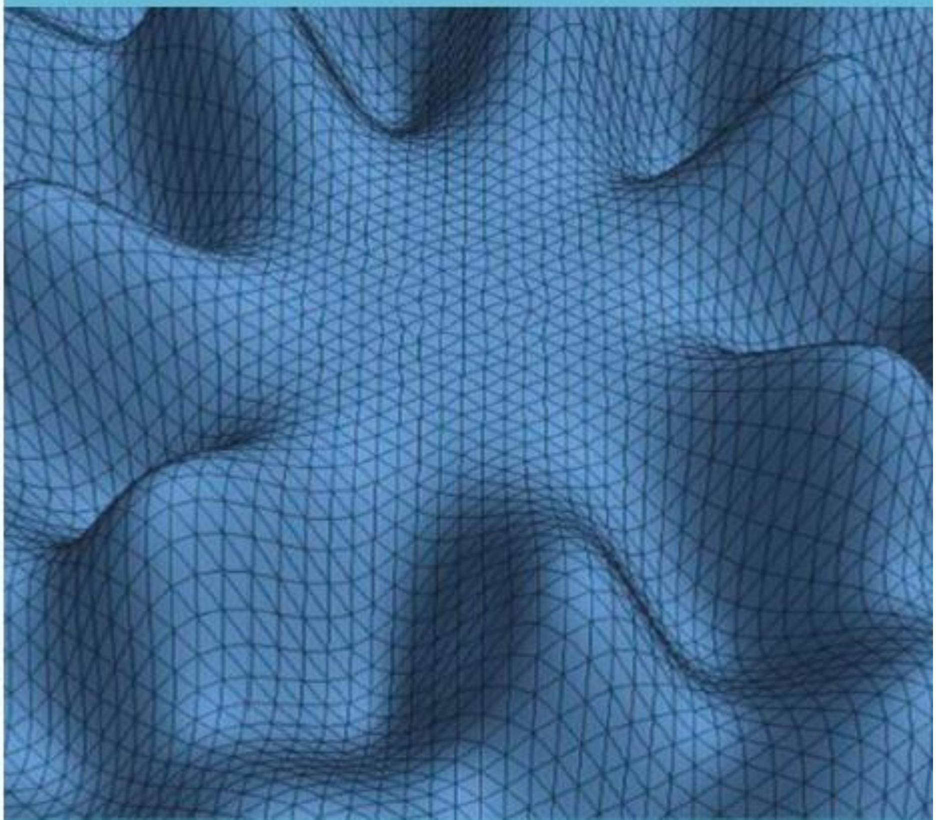

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On the conjugate diagram of one entire function

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Abstract. The analytical structure of the characteristic determinant of the eigenvalue problem for a linear differential equation on a finite interval is studied. Various representations of the specified characteristic determinant are given. The order and type of growth of the characteristic determinant are clarified. The possible types of conjugate diagram of the characteristic determinant are analyzed. Depending on the indices of nonzero minors of the boundary matrix, theorem on the localization of the eigenvalues of the investigated problem are formulated.

Keywords. Entire function, boundary matrix, characteristic determinant, eigenvalues, Kronecker symbol, degenerate and non-degenerate boundary value problems, conjugate diagram, exponential type.

1 Introduction

Let 4×8 matrix

$$A = \begin{pmatrix} a_{j1} & a_{j2} & a_{j3} & a_{j4} & a_{j5} & a_{j6} & a_{j7} & a_{j8} \\ & & & j=1,2,3,4 & & & & \end{pmatrix},$$

composed of complex numbers, be given. We call it a boundary matrix. We denote its minors by A_{kmsl}, where k, m, s, l correspond to the column numbers of the boundary matrix A. We assume that at least one of the A_{kmsl} minors is nonzero. Let the functions $q_k(\cdot)$ with $k = 0, 1, 2$ be defined on the interval [0, 1] and be continuously differentiable k times. It is also assumed that

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$$q_k(x) = (-1)^k q_k(1-x),\ x \in [0,1].$$

Then the entire function

$$f_q(\lambda) = \sum_{1\le k,m,s,l\le 4} A_{2k,2m-1,2s,2l-1} z_1^{(k)} \cdot (\lambda) \cdot z_2^{(m)}(\lambda) \cdot z_3^{(s)}(\lambda) \cdot z_4^{(l)}(\lambda) \tag{1}$$

represents the characteristic determinant of the following problem eigenvalues:

$$u^{(4)}(x) + q_2(x)u^{(2)}(x) + q_1(x)u^{(1)}(x) + q_0(x)u(x) = \lambda u(x),\ 0 < x < 1, \tag{2}$$

$$W_j(u) \equiv \sum_{p=1}^{8} a_{jp} V_p(u) = 0,\ j = 1,2,3,4. \tag{3}$$

Here

$$V_1(u) = u(1) + u(0),$$

$$V_2(u) = u(1) - u(0),$$

$$V_3(u) = u^{(1)}(1) - u^{(1)}(0),$$

$$V_4(u) = u^{(1)}(1) + u^{(1)}(0),$$

$$V_5(u) = (u^{(2)}(1) + u^{(2)}(0)) + q_2(1)(u(1) + u(0)),$$

$$V_6(u) = (u^{(2)}(1) - u^{(2)}(0)) + q_2(1)(u(1) - u(0)),$$

$$V_7(u) = (u^{(3)}(1) - u^{(3)}(0)) + q_2(1)(u^{(1)}(1) - u^{(1)}(0)) + (q_1(1) - q_2^{(1)}(1))(u(1) - u(0)),$$

$$V_8(u) = (u^{(3)}(1) + u^{(3)}(0)) + q_2(1)(u^{(1)}(1) + u^{(1)}(0)) + (q_1(1) - q_2^{(1)}(1))(u(1) + u(0)).$$

An important role in representation (1) is played by entire functions $\{z_k^{(\nu)}(\lambda), k, j = 1,2,3,4\}$. These functions were introduced in [1] and are determined according to the formulas

$$z_k^{(1)} = s_k(0,\lambda),$$

$$z_k^{(2)} = s_k^{(1)}(0,\lambda),$$

$$z_k^{(3)} = s_k^{(2)}(0,\lambda) + q_2(0)s_k(0,\lambda),$$

$$z_k^{(4)} = s_k^{(3)}(0,\lambda) + q_2(0)s_k^{(1)}(0,\lambda) - (q_1(0) - q_2^{(1)}(0))s_k(0,\lambda).$$

Moreover, $s_k(x,\lambda)$ is a solution of the homogeneous equation (2), subordinate to the Cauchy conditions at the point $x = \dfrac{1}{2}$:

$$s_k^{(\nu)}\Big(\frac{1}{2},\lambda\Big) = \sigma_{\nu+1,k},\ \nu = 0,1,2,3,$$

where $\sigma_{\nu+1,k}$ is the Kronecker symbol.

Note that representation (1) was proved in [1]. In the case of second-order differential equations, an analogue of representation (1) can be found in the monograph ([2], p. 33-50). A detailed analysis of representation (1) in the case of second-order differential equations allowed V.A. Marchenko [2] to classify the boundary conditions. V.A. Marchenko identified two classes of boundary value problems: degenerate and non-degenerate boundary value problems. It turned out that non-degenerate boundary value problems for second-order differential equations have a complete system of eigen functions and as sociated functions in the space $L_2(0,1)$. This article explores the conjugate diagram of the entire function $f_q(\lambda)$. Information on the conjugate diagram will allow calculating the type and growth order of the entire function $f_q(\lambda)$. Then the type and growth order of the entire function $f_q(\lambda)$ will allow us to carry out according to V.A. Marchenko classification of boundary conditions (3).

2 The conjugate diagram of the function $f_q(\lambda)$ for $q_k(x) \equiv 0$, $k = 0,1,2$

In this section, we study the entire function $f_q(\lambda)$, for $q_2 \equiv q_1 \equiv q_0 \equiv 0$. In this case, the functions $\{z_j^{(k)}(\lambda),\ k,j = 1,2,3,4\}$ are written as follows

$$z_j^{(k)}(\lambda) = \frac{1}{4}\sum_{p=1}^{4}(\rho\omega_p)^{k-j}\exp\Big(\frac{1}{2}\rho\omega_p\Big), \tag{4}$$

where $\omega_1 = 1$, $\omega_2 = i$, $\omega_3 = -1$, $\omega_4 = -i$, $\rho^4 = \lambda$. In work [1] the following lemma is proved.

Lemma 1. *Let the set of functions $\{z_j^{(k)}(\lambda),\ k,j = 1,2,3,4\}$ be determined according to the formulas (4). Then the product*

$$z_1^{(k)}(\lambda)\cdot z_2^{(m)}(\lambda)\cdot z_3^{(s)}(\lambda)\cdot z_4^{(l)}(\lambda)$$

has an idea

$$z_1^{(k)}(\lambda)\cdot z_2^{(m)}(\lambda)\cdot z_3^{(s)}(\lambda)\cdot z_4^{(l)}(\lambda) = \frac{1}{64}\sum_{\theta\in J}B_{kmsl}^{(\theta)}\psi^{(k+m+s+l-10)}(\theta,\lambda),$$

where $J = \{2, 1+i, \frac{3+i}{2}, \frac{3-i}{2}, 1, \frac{1+i}{2}, \frac{1-i}{2}, 0\}$,

$$\psi(x,\lambda) = \frac{1}{4}\sum_{p=1}^{4}e^{\rho\omega_{j}x}.$$

Note that the numbers $B^{(\theta)}_{kmsl}$, except for the indicated indices, depend on the numbers $\omega_1,\ \omega_2,\ \omega_3,\ \omega_4$. Here are some of the numbers $B^{(\theta)}_{kmsl}$. For example

$$B^{(1+i)}_{kmsl}=\omega_2^{s-3}\omega_3^{l-4}+\omega_3^{s-3}\omega_2^{l-4}+\omega_2^{s-3}\omega_3^{m-2}+\omega_3^{s-3}\omega_2^{m-2}+\omega_2^{s-3}\omega_3^{k-1}+\omega_3^{s-3}\omega_2^{k-1}$$
$$+\omega_2^{l-4}\omega_3^{m-2}+\omega_3^{l-4}\omega_2^{m-2}+\omega_2^{l-4}\omega_3^{k-1}+\omega_3^{l-4}\omega_2^{k-1}+\omega_3^{k-1}\omega_2^{m-2}+\omega_2^{k-1}\omega_3^{m-2},$$
$$B^{(1-i)}_{kmsl}=\omega_3^{s-3}\omega_4^{l-4}+\omega_4^{s-3}\omega_3^{l-4}+\omega_3^{s-3}\omega_4^{m-2}+\omega_4^{s-3}\omega_3^{m-2}+\omega_3^{s-3}\omega_4^{k-1}+\omega_4^{s-3}$$
$$\times\omega_3^{k-1}+\omega_3^{l-4}\omega_4^{m-2}+\omega_4^{l-4}\omega_3^{m-2}+\omega_3^{l-4}\omega_4^{k-1}+\omega_4^{l-4}\omega_3^{k-1}+\omega_4^{k-1}\omega_3^{m-2}+\omega_3^{k-1}\omega_4^{m-2}.$$

The volume of the article does not allow writing out all the numbers $B^{(\theta)}_{kmsl}$. The statement follows from representation (1).

Theorem [1]. *Let $q_2\equiv q_1\equiv q_0\equiv 0$. Then the entire function $f_q(\lambda)$ has the representation*

$$f_q(\lambda)=\sum_{p=4}^{16}\sum_{\theta\in J}c_p^{(\theta)}\psi^{(p-10)}(\theta,\lambda),$$

where $c_p^{(\theta)}=\sum_{(k,m,s)\in K_s}A_{2k-1,2m,2s-1,2(p-k-m-s)}\cdot B^{(\theta)}_{kms(p-k-m-s)}$. Here K_p is a finite set of multi-indices (k,m,s) and it changes with p.

In this paper, the coefficients are calculated explicitly $c_p^{(\theta)}$. The software package was used for this ≪MAPLE≫. It turned out that part of the coefficients $c_p^{(\theta)}$ is equal to zero regardless of the choice of the boundary matrix A. The type of indicator chart depends only on nonzero coefficients $c_p^{(\theta)}$. In the next representation of $f_q(\lambda)$ only numbers with nonzero coefficients $c_p^{(\theta)}$ are written out. Representation (13) can be clarified by calculating the coefficients $c_s^{(\theta)}$, for $\theta\in J,\ s\in\{4,\dots,16\}$. It turns out that for the values of the parameter s, the coefficients $c_s^{(\theta)}$=0 for $\theta=4,3+i,1+i,1+3i$. Nonzero coefficients $c_s^{(\theta)}$ are written below

$$c_{14}^{(2+2i)}=-8A_{5678},$$

$$c_{13}^{(2+2i)}=(-8+8i)A_{3678}+(8-8i)A_{4578},$$

$$c_{12}^{(2+2i)}=16iA_{3478}+8iA_{1678}-8iA_{2578}-8iA_{3568}+8iA_{4567},$$

$$c_{11}^{(2+2i)}=(-8-8i)A_{2378}+(8+8i)A_{1478}+(-8-8i)A_{3467}$$
$$+(8+8i)A_{3458}+(8+8i)A_{2567}+(-8-8i)A_{1568},$$

$$c_{10}^{(2+2i)}=16A_{1458}-8A_{1368}-8A_{2358}+16A_{2367}-8A_{1467}-8A_{2457}+8A_{1278}+8A_{3456},$$

$$c_9^{(2+2i)}=(-8+8i)A_{1267}+(8-8i)A_{1258}+(8-8i)A_{1456}$$

$$+(-8+8i)A_{2356}+(-8+8i)A_{1348}+(8-8i)A_{2347},$$

$$c_8^{(2+2i)}=8iA_{1346}-8iA_{2345}+8iA_{1247}-8iA_{1238}-16iA_{1256},$$

$$c_7^{(2+2i)}=(8+8i)A_{1245}+(-8-8i)A_{1236},$$

$$c_6^{(2+2i)}=-8A_{1234},$$

$$c_{13}^{(2)}=16A_{3678}+16A_{4578},$$

$$c_{12}^{(2)}=16A_{1678}+16A_{2578}-16A_{3568}-16A_{4567},$$

$$c_{11}^{(2)}=16A_{2378}+16A_{1478}+16A_{3467}+16A_{3458}-16A_{2567}-16A_{1568},$$

$$c_{10}^{(2)}=32A_{1458}-32A_{2367},$$

$$c_9^{(2)}=16A_{1267}+16A_{1258}+16A_{1456}+16A_{2356}-16A_{1348}-16A_{2347},$$

$$c_8^{(2)}=-16A_{1346}-16A_{2345}+16A_{1247}+16A_{1238},$$

$$c_7^{(2)}=16A_{1245}+16A_{1236}.$$

Therefore, the representation from the above theorem takes the form

$$\Delta_p(\lambda)=-8A_{5678}\Psi^{(4)}(2+2i)(1+O(\frac{1}{\rho}))$$

$$+16(A_{3678}+A_{4578})\Psi^{(3)}(1+O(\frac{1}{\rho}))+\rho^4(1+O(\frac{1}{\rho})) \quad (5)$$

as $\rho\to\infty$. The quantity $O(\frac{1}{\rho})$ is subject to the estimate

$$\left|O\left(\frac{1}{\rho}\right)\right|\le C\frac{1}{|\rho|+1} \quad at \quad \rho\to\infty.$$

The function $\Psi(\theta,\lambda)$ is an entire function of $\lambda=\rho^4$ and is written as

$$\Psi(\theta,\lambda)=\sum_{j=1}^{4}\exp\left(\frac{1}{2}\rho\omega_j\theta\right).$$

It follows from representation (5) that the entire function of exponential type $f_p(\lambda)$ has an conjugate diagram in the form of a square with vertices $1+i,\ 1-i,\ -1-i,\ -1+i$ and points $\pm1,\ \pm i$ marked on the sides.

According to [3] theorems on the zeros of entire functions of exponential type sufficiently large modulo zeros $f_p(\lambda)$ lie in the corners of an arbitrarily small sector with a bisector on the real (non-negative) semiaxis, if the numbers $A_{5678}\neq0$ or $|A_{3678}|+|A_{4578}|\neq0$. The above statements remain valid even for nonzero coefficients $q_2(\cdot),\ q_1(\cdot),\ q_0(\cdot)$, if $q_k \in C^{(k)}[0,1],\ k = 0,1,2$. If the above minors $A_{5678},\ A_{3678},\ A_{4578}$ are equal to zero, then you need to look at the coefficients for the exponents that determine the type of growth of the entire function $f_p(\lambda)$. Moreover, relations (5) allow us to accurately write nonzero coefficients for defining exponents. Thus, under any non-degenerate boundary conditions of the initial task, it is possible to write down the asymptotics of its eigenvalues.

3 Conclusion

In conclusion, it is appropriate to cite Sylvester's words about the amazing intellectual phenomenon that the proofs of general statements are usually simpler than the proofs of various special cases contained in them. According to Arnold, namely, they are the essence of science, even if stated, for the sake of simplicity of evidence, deductively, that is, starting from general statements.

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Кангужин Б.Е., Тасбаева Н.М., Қасбақбаева А.И., Сеитова А.А. БҮТІН ФУНКЦИЯНЫҢ ТҮЙІНДЕС ДИАГРАММАСЫ ТУРАЛЫ

Бұл жұмыста ақырлы аралықтағы сызықтық дифференциалдық теңдеу үшін меншікті мәндерді табуға арналған есептің сипаттауыш анықтауышының аналитикалық құрылымы зерттелген. Аталған сипаттауыш анықтауышының әр түрлі кейіптемелері келтірілген. Сипаттауыш анықтауышының өсу реті мен түрі нақтыланған. Сипаттауыш анықтауышының түйіндес диаграммасының мүмкін болатын түрлері талданған. Шекаралық матрицаның нөлдік емес минорларының индекстеріне байланысты бастапқы есептің меншікті мәндерін локализациялау туралы теоремалар тұжырымдалған.

Кілттік сөздер. Бүтін функция, шекаралық матрица, сипаттауыш анықтауыш, меншікті мән, Кронекер символы, айныған және айнымаған шекаралық есептер, түйіндес диаграмма, экспоненциалдық түр.

Кангужин Б.Е., Тасбаева Н.М., Касбакбаева А.И., Сеитова А.А. О СОПРЯЖЕННОЙ ДИАГРАММЕ ОДНОЙ ЦЕЛОЙ ФУНКЦИИ

В работе изучена аналитическая структура характеристического определителя задачи на собственные значения для линейного дифференциального уравнения на конечном отрезке. Приведены различные представления указанного характеристического определителя. Выяснен порядок и тип роста характеристического определителя. Проанализированы возможные виды сопряженной диаграммы характеристического определителя. В зависимости от индексов ненулевых миноров граничной матрицы сформулирована теорема о локализации собственных значений исходной задачи.

Ключевые слова. Целая функция, граничная матрица, характеристический определитель, собственное значение, символ Кронекера, вырожденные и невырожденные граничные задачи, сопряженная диаграмма, экспоненциальный тип.

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Extension operator method for solving nonstandard partial boundary value problems

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Abstract. We establish solvability criteria and construct the exact solution to some general boundary value problems for linear Fredholm integro-partial differential equations, or partial differential equations, with nonstandard integral boundary conditions. Our approach is based on a perturbation technique and the theory of extensions of operators in Banach spaces, and assumes the knowledge of the explicit solution of an ideal simpler problem involving the associated partial differential equation with simple conventional boundary conditions.

Keywords. Integro-differential equations, differential operators, integral boundary conditions, exact solutions.

1 Introduction

Integro-partial differential equations of Fredholm type appear in mathematical modeling in many disciplines of natural sciences, engineering, computer science and economics, see for example, in [1]–[6] and the references therein. Boundary value problems for these types of equations coupled with general nonlocal boundary conditions are difficult to solve analytically. Numerical methods are usually employed, whereas over the last decades there is an increasing interest in closed form solutions encouraged by the available computer algebra systems and the advances made in symbolic computations.

An approach to attack general boundary value problems is to treat multipoint and integral boundary conditions as perturbations of simpler classical boundary conditions [7]. Thus, the solution of a boundary value problem for the differential equation with nonlocal (perturbed) boundary conditions may be constructed from the solution of a corresponding ideal problem for the differential equation subject to simpler (unperturbed) boundary conditions [8]. Moreover, Fredholm integro-differential equations may be viewed as differential equations

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perturbed by one or more integral terms and therefore their solution may be obtained from the solution of the associated (unperturbed) differential equation [9]–[16].

In [17], the authors have applied this approach along with the theory of extensions of operators in Banach spaces to obtain closed form solutions for some partial boundary value problems with nonstandard perturbed boundary conditions. Here, we continue the work in [17] and derive exact solutions of two more categories of boundary value problems for integro-partial differential equations subject to some rather uncommon integral boundary conditions.

Let X, Y, Z be complex Banach spaces and X^*, Y^* the adjoint spaces of X and Y, i.e. the set of all complex-valued bounded linear functionals on X and Y, respectively. Let $A : X \to Y$ be a maximal, not necessarily closed, linear partial differential operator with domain $D(A)$ and range $R(A)$, and $\Gamma : X_A \overset{on}{\to} Z$ a bounded linear operator, where $X_A = (D(A), \|\cdot\|_{X_A})$ is a Banach space with respect to a norm $\|\cdot\|_{X_A}$. Let $\widehat{A}$ be a correct restriction of A and consider the "unperturbed" boundary value problem

$$\widehat{A}u = Au = f, \ f \in Y,$$
$$D(\widehat{A}) = \{u : u \in D(A), \ \Gamma u = 0\}. \tag{1}$$

The correct problem (1) is known to possess a unique solution for every $f \in Y$. We assume that the inverse operator $\widehat{A}^{-1}$ is known in an explicit form and the solution $u = \widehat{A}^{-1}f$ can be obtained analytically for every $f \in Y$.

In this paper, we consider the "perturbed" boundary value problem

$$Bu = Au - gF(Au) = f, \ f \in Y,$$
$$D(B) = \{u : u \in D(A), \ \Gamma u = v\Psi(Au)\}, \tag{2}$$

where $B : X \to Y$ is a linear operator, $F = \mathrm{col}(F_1, \dots, F_n)$ is a column vector of bounded linear functionals $F_i \in Y^*$, $g = (g_1, \dots, g_n)$ is a row vector of n linearly independent elements $g_i \in Y$, $\Psi = \mathrm{col}(\Psi_1, \dots, \Psi_m)$ is a column vector of bounded linear functionals $\Psi_i \in Y^*$ and $v = (v_1, \dots, v_m)$ is a vector of m elements $v_j \in Z$. We examine the solvability of problem (2) and construct its solution in closed form when the inverse operator $\widehat{A}^{-1}$ is available explicitly. Further, we contemplate the more involved "perturbed" boundary value problem

$$B_1u = A^2u - gF(A^2u) - qF(Au) = f, \ f \in Y,$$
$$D(B_1) = \{u : u \in D(A^2), \ \Gamma u = v\Psi(Au),$$
$$\Gamma(Au) = v\Psi(A^2u) + wF(Au)\}, \tag{3}$$

where now $X = Y$, $B_1 : X \to X$ is a linear operator and, in addition to above definitions, $g_i, q_i \in X$ are $2n$ linearly independent elements, $q = (q_1, \dots, q_n)$, and $w = (w_1, \dots, w_n)$ is a row vector of n elements $w_i \in Z$. We examine the existence and uniqueness of its solution and obtain it in closed form when the inverse operator $\widehat{A}^{-1}$ is known.

The paper is organized as follows. In Section 2, some preliminary results are presented. The main results are given in Sections 3, where the explicit solution formulae for problems (2), (3) are derived. An application to integro-partial differential equations is considered in Section 4. Finally, some conclusions are quoted in Section 5.

2 Preliminaries

An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a *restriction* of A_2, compactly $A_1 \subset A_2$, if $D(A_2) \supset D(A_1)$ and $A_1 u = A_2 u$, for all $u \in D(A_1)$. An operator $A : X \to Y$ is called *closed* if for every sequence u_n in $D(A)$ such that $u_n \to u_0$ in X and $Au_n \to f_0$ in Y, it follows that $u_0 \in D(A)$ and $Au_0 = f_0$. An operator A is called *maximal* if $R(A) = Y$ and $\ker A \neq \{0\}$. An operator $\widehat{A} : X \to Y$ is *correct* if $R(\widehat{A}) = Y$ and the inverse $\widehat{A}^{-1}$ exists and is continuous on Y. An operator $\widehat{A}$ is called a *correct restriction* of a maximal operator A if it is a correct operator and $\widehat{A} \subset A$.

Let $\Psi_i \in Y^*,\ i = 1, \ldots, m$, and $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_m)$. Also, let $g_j \in Y,\ j = 1, \ldots, n$, and $g = (g_1, \ldots, g_n)$. We will denote by $\Psi(g)$ the $m \times n$ matrix whose i, j-th entry $\Psi_i(g_j)$ is the value of the functional Ψ_i on the element g_j; note that

$$\Psi(gC) = \Psi(g)C, \tag{4}$$

where C is a $n \times l$ constant matrix. Further, we will designate by 0_{mn} the $m \times n$ zero matrix, by 0_m the zero $m \times m$ matrix, by I_m the identity $m \times m$ matrix, by $\mathbf{c}$ a constant vector and by $\mathbf{0}$ the zero column vector. When vectors and matrices are expressed explicitly, brackets and square brackets are used, respectively.

The boundary operator Γ in (1), (2) and (3) may be a column vector $\Gamma u = \mathrm{col}(\Gamma_1 u, \ldots, \Gamma_k u)$ of linear operators $\Gamma_i : X_A \overset{on}{\to} Z_i$, $i = 1, \ldots, k$, $(Z = Z_1 \times \cdots \times Z_k)$. In this case, the elements of the vectors v and w in (2) and (3) are column vectors; for example,

$$\begin{aligned}
\begin{pmatrix} \Gamma_1 u \\ \vdots \\ \Gamma_k u \end{pmatrix} &= \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{k1} & \cdots & v_{km} \end{bmatrix} \begin{pmatrix} \Psi_1(Au) \\ \vdots \\ \Psi_m(Au) \end{pmatrix} \\
&= \begin{pmatrix} v_{11} \\ \vdots \\ v_{k1} \end{pmatrix} \Psi_1(Au) + \cdots + \begin{pmatrix} v_{1m} \\ \vdots \\ v_{km} \end{pmatrix} \Psi_m(Au) \\
&= v_1 \Psi_1(Au) + \cdots + v_m \Psi_m(Au) \\
&= v\Psi(Au).
\end{aligned} \tag{5}$$

The following lemma is attributed to Oinarov [18] and it is recalled here together with its proof for easy of reference.

Lemma 1. *Let X, Y, Z be complex Banach spaces, $A : X \to Y$ a maximal closed linear operator, $\Gamma : X_A \overset{on}{\to} Z$ a bounded linear boundary operator, where*

$$X_A = (D(A), \|\cdot\|_{X_A}), \quad \|u\|_{X_A} = \|u\|_X + \|Au\|_Y, \quad \forall u \in D(A), \tag{6}$$

is a Banach space, and $\widehat{A}$ a correct restriction of A defined in (1). Then the restriction $\widehat{\Gamma}$ of Γ to $\ker A$, where $\ker A$ is a Banach space in the induced topology of X, is correct.

Proof. By assumption the operator Γ is bounded from X_A onto Z, i.e. there exists a constant $c > 0$ not depending on $u \in X_A$ such that

$$\|\Gamma u\|_Z \leq c\|u\|_{X_A} = c(\|u\|_X + \|Au\|_Y), \quad \forall u \in D(A). \tag{7}$$

For every $u_0 \in \ker A$, we have $\|\Gamma u_0\|_Z = \|\widehat{\Gamma} u_0\|_Z \leq c\|u_0\|_X$. Hence, the operator $\widehat{\Gamma}$ is bounded on $\ker A$. Moreover, $\widehat{\Gamma}$ is closed because $\ker A$ is closed. From the condition $\ker \Gamma = D(\widehat{A})$ and the decomposition [19],

$$D(A) = D(\widehat{A}) \oplus \ker A, \tag{8}$$

it follows that $\ker \Gamma \cap \ker A = \{0\}$ and thus the operator $\widehat{\Gamma}$ is injective. From (8) it is implied that $Z = R(\Gamma) = \Gamma D(A) = \Gamma \ker A = \widehat{\Gamma} \ker A$ and therefore $Z = R(\widehat{\Gamma})$ and the domain of $\widehat{\Gamma}^{-1}$ is the whole of Z. Since the operator $\widehat{\Gamma}^{-1}$ is closed, because $\widehat{\Gamma}$ is closed, $D(\widehat{\Gamma}^{-1}) = Z$ and Z is a Banach space, then by the Closed-Graph Theorem the operator $\widehat{\Gamma}^{-1}$ is bounded. □

We also prove the next lemma which does not require the operator A to be closed, but it suffices to be the composition $A = A_2 A_1$ of two maximal closed linear operators A_1, A_2.

Lemma 2. *Let X, Y, Z be complex Banach spaces, $A_1 : X \to X$ and $A_2 : X \to Y$ maximal closed linear operators, and $A : X \to Y$ a maximal linear operator defined by the composition $A = A_2 A_1$. Let $\Gamma : X_A \overset{on}{\to} Z$ be a continuous boundary operator with*

$$X_A = (D(A), \|\cdot\|_{X_A}), \ \|u\|_{X_A} = \|u\|_X + \|A_1 u\|_X + \|A_2 A_1 u\|_Y, \tag{9}$$

and the space

$$N = (\ker A, \|\cdot\|_N), \ \|u\|_N = \|u\|_X + \|A_1 u\|_X, \ u \in \ker A. \tag{10}$$

Let there exists a correct restriction $\widehat{A}$ of A as specified in (1), and let $\widehat{\Gamma}$ be a restriction of Γ to $\ker A$. Then:

(i) X_A and N are Banach spaces.

(ii) The operator $\widehat{\Gamma} : N \overset{on}{\to} Z$ *is correct.*

Proof.

(i) Let $\{u_n\} \subset X_A$ be a fundamental sequence, i.e. $\|u_n - u_m\|_{X_A} \to 0$, $n, m \to \infty$. Then, by (9),

$$\|u_n - u_m\|_{X_A} = \|u_n - u_m\|_X + \|A_1(u_n - u_m)\|_X + \|A_2A_1(u_n - u_m)\|_Y,\ n, m \to \infty,$$

and hence

$$\|u_n - u_m\|_X \to 0,\ \|A_1(u_n - u_m)\|_X \to 0,\ \|A_2A_1(u_n - u_m)\|_Y \to 0,\ n, m \to \infty.$$

Since X, Y are Banach spaces, there exist elements $u_0, v_0 \in X$ and $z_0 \in Y$ such that

$$u_n \to u_0,\ A_1u_n \to v_0,\ A_2A_1u_n \to z_0,\ n \to \infty.$$

By the assumption that the operators A_1, A_2 are closed, we obtain $u_0 \in D(A_1)$, $A_1u_0 = v_0$, $v_0 \in D(A_2)$ and $A_2v_0 = z_0$. Consequently, $u_0 \in D(A_2A_1) = D(A)$ and hence X_A is a Banach space.

It is easy to verify that N is a normed space. Let $\{\hat{u}_n\}$ be a fundamental sequence of N. Then $\hat{u}_n \in \ker A \subset D(A)$ and by (10),

$$\|\hat{u}_n - \hat{u}_m\|_N = \|\hat{u}_n - \hat{u}_m\|_X + \|A_1(\hat{u}_n - \hat{u}_m)\|_X \to 0,\ n, m \to \infty,$$

and thus

$$\|\hat{u}_n - \hat{u}_m\|_X \to 0,\ \|A_1(\hat{u}_n - \hat{u}_m)\|_X \to 0,\ n, m \to \infty.$$

Then there exist elements $\hat{u}_0, \hat{v}_0 \in X$ such that $\hat{u}_n \overset{X}{\to} \hat{u}_0$ and $\hat{v}_n = A_1\hat{u}_n \overset{X}{\to} \hat{v}_0$. From the closeness of A_1 follows that $\hat{u}_0 \in D(A_1)$ and $A_1\hat{u}_0 = \hat{v}_0$. From $\hat{v}_n \overset{X}{\to} \hat{v}_0$ and $0 = A_2\hat{v}_n \overset{X}{\to} 0$, $n \to \infty$, because the operator A_2 is closed, it follows that $A_1\hat{u}_0 = \hat{v}_0 \in D(A_2)$ and $A_2A_1\hat{u}_0 = 0$. So $\hat{u}_0 \in N$ and N is a Banach space.

(ii) Note that $\ker \widehat{\Gamma} = \{0\}$ and $D(\widehat{\Gamma}^{-1}) = Z$ as it has been proved already in Lemma 1. Further, since $\Gamma : X_A \overset{on}{\to} Z$ is bounded, $\widehat{\Gamma} \subset \Gamma$ and (10), for all $u \in N$ there exists a number $c > 0$ such that

$$\begin{aligned} \|\Gamma u\|_Z &= \|\widehat{\Gamma} u\|_Z \\ &\le c\|u\|_{X_A} \le c(\|u\|_X + \|A_1u\|_X + \|A_2A_1u\|_Y) \\ &= c(\|u\|_X + \|A_1u\|_X) = c\|u\|_N. \end{aligned} \tag{11}$$

So, the operator $\widehat{\Gamma} : N \overset{on}{\to} Z$ is bounded. The operator $\widehat{\Gamma}$ is closed, because $D(\widehat{\Gamma}) = \ker A$ and N is a Banach space. It follows that $\widehat{\Gamma}^{-1}$ is closed and by taking into account $D(\widehat{\Gamma}^{-1}) = Z$, it is implied that $\widehat{\Gamma}^{-1}$ is bounded. Therefore, $\widehat{\Gamma} : N \overset{on}{\to} Z$ is correct. □

3 Main results

In this section, we present the two main theorems which provide the solvability and uniqueness conditions and the solution in closed form of the general boundary value problems (2) and (3).

Theorem 1. *Let X, Y, Z be complex Banach spaces, the operators $A : X \to Y$ and $\Gamma : X_A \overset{on}{\to} Z$ as specified in Lemma 1 or Lemma 2, $\widehat{A}$ a correct restriction of A as in (1), $\widehat{\Gamma}$ a correct restriction of Γ to $\ker A$, and the operator $B : X \to Y$ defined by*

$$Bu = Au - gF(Au),$$

$$D(B) = \{u : u \in D(A),\ \Gamma u = v\Psi(Au)\}, \tag{12}$$

where $F = \mathrm{col}(F_1, \ldots, F_n)$ is a vector of linear bounded functionals $F_i \in Y^*$, $g = (g_1, \ldots, g_n)$ is a vector of n linearly independent elements $g_i \in Y$, $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_m)$ is a vector of linear bounded functionals $\Psi_i \in Y^*$ and $v = (v_1, \ldots, v_m)$ is a vector of m elements $v_j \in Z$. Then:
(i) The operator B is injective if and only if

$$\det W = \det\left[I_n - F(g)\right] \neq 0. \tag{13}$$

(ii) Moreover, under the condition (13) the operator B is correct and the unique solution to boundary value problem

$$Bu = f, \quad \forall f \in Y, \tag{14}$$

is given by the formula

$$u = B^{-1}f$$
$$= \widehat{A}^{-1}f + \left[\widehat{A}^{-1}g + \widehat{\Gamma}^{-1}v\Psi(g)\right]W^{-1}F(f) + \widehat{\Gamma}^{-1}v\Psi(f). \tag{15}$$

Proof. (i) Suppose $\det W \neq 0$. Let $u \in \ker B$, then $Bu = Au - gF(Au) = 0$ and $\Gamma u = v\Psi(Au)$. Furthermore, $F(Au - gF(Au)) = [I_n - F(g)]F(Au) = WF(Au) = \mathbf{0}$. This implies that $F(Au) = \mathbf{0}$ and as a consequence $Bu = Au = 0$ and hence $\Gamma u = 0$. Thus $u \in D(\widehat{A})$, $\widehat{A}u = Au = 0$ and $u = 0$ since $\widehat{A}$ is correct. This means $\ker B = \{0\}$ and therefore B is an injective operator. Conversely, we assume $\det W = 0$ and we will prove that the operator B is not injective. Then there exists a constant vector $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_n) \neq \mathbf{0}$ such that $W\mathbf{c} = \mathbf{0}$. Let the element $u_0 = [\widehat{A}^{-1}g + \widehat{\Gamma}^{-1}v\Psi(g)]\mathbf{c}$ and observe that $u_0 \neq 0$, because $g_1, \ldots, g_n$ is a linearly independent set and $g\mathbf{c} \neq 0$; otherwise $\mathbf{c} = \mathbf{0}$. We have

$$\Gamma u_0 - v\Psi(Au_0) = v\Psi(g)\mathbf{c} - v\Psi(g)\mathbf{c} = 0,$$

$$Bu_0 = Au_0 - gF(Au_0) = g\mathbf{c} - gF(g)\mathbf{c} = g[I_n - F(g)]\mathbf{c} = gW\mathbf{c} = g\mathbf{0} = 0. \tag{16}$$

Hence $u_0 \in \ker B \subset D(B)$, $\ker B \neq \{0\}$ and therefore B is not injective.

(ii) Let $\det W \neq 0$. Equation (14) may be written as

$$Bu = A\left(u - \widehat{\Gamma}^{-1} v \Psi(Au)\right) - gF(Au) = f, \ f \in Y,$$

$$D(B) = \{u \in D(A) : \Gamma\left(u - \widehat{\Gamma}^{-1} v \Psi(Au)\right) = 0\}, \tag{17}$$

where $\widehat{\Gamma}^{-1} v \in \ker A$. From (17) follows that $u - \widehat{\Gamma}^{-1} v \Psi(Au) \in D(\widehat{A})$ and since $\widehat{A} \subset A$, we obtain

$$Bu = \widehat{A}\left(u - \widehat{\Gamma}^{-1} v \Psi(Au)\right) - gF(Au) = f,$$

$$u - \widehat{\Gamma}^{-1} v \Psi(Au) - \widehat{A}^{-1} gF(Au) = \widehat{A}^{-1} f, \ f \in Y. \tag{18}$$

Further, by applying the vector F on both sides of (14), we get

$$F(Au - gF(Au)) = F(f),$$

$$[I_n - F(g)]F(Au) = F(f),$$

$$F(Au) = W^{-1} F(f), \tag{19}$$

for every $u \in D(B)$ and $f \in Y$. Similarly, application of the vector Ψ on both sides of (14) yields

$$\Psi(Au - gF(Au)) = \Psi(f),$$

$$\Psi(Au) = \Psi(g)F(Au) + \Psi(f). \tag{20}$$

Substituting (19) and (20) into (18), we acquire (15). Because f in (15) is arbitrary, we have $R(B) = Y$. Moreover, since the operators $\widehat{A}^{-1}$, $\widehat{\Gamma}^{-1}$ and the functionals $F_1, \ldots, F_n$, $\Psi_1, \ldots, \Psi_m$ are bounded it follows that B^{-1} is bounded too. Thus, the operator B is correct. □

The boundary value problem (3) is more cumbersome to solve. We begin by noting that

$$\widehat{A}^2 u = A^2 u,$$

$$D(\widehat{A}^2) = \{u : u \in D(A), \ \Gamma u = 0, \ \Gamma(Au) = 0\}. \tag{21}$$

Theorem 2. *Let X, Z be complex Banach spaces, the operators $A : X \to X$ and $\Gamma : X_A \overset{on}{\to} Z$ as specified in Lemma 1 or Lemma 2 with $X = Y$, $\widehat{A}$ a correct restriction of A as in (1), $\widehat{\Gamma}$ a correct restriction of Γ to $\ker A$, and the operator $B_1 : X \to X$ defined by*

$$B_1 u = A^2 u - qF(Au) - gF(A^2 u),$$

$$D(B_1) = \{u : u \in D(A^2), \ \Gamma u = v\Psi(Au),$$

$$\Gamma(Au) = v\Psi(A^2u) + wF(Au)\}, \tag{22}$$

where $F = \mathrm{col}(F_1, \dots, F_n)$ *is a vector of linear bounded functionals* $F_i \in X^*$, $q = (q_1, \dots, q_n)$ *and* $g = (g_1, \dots, g_n)$ *are vectors, where the* $2n$ *elements* $q_i, g_i \in X$ *are linearly independent,* $f \in X$, $\Psi = \mathrm{col}(\Psi_1, \dots, \Psi_m)$ *is a vector of linear bounded functionals* $\Psi_i \in X^*$, *and* $v = (v_1, \dots, v_m)$ *and* $w = (w_1, \dots, w_n)$ *are vectors of* m *and* n *elements* $v_j, w_j \in Z$, *respectively. Then:*

(i) The operator B_1 *is injective if and only if*

$$\det L \neq 0, \tag{23}$$

where

$$L = \begin{bmatrix} I_m & -\Psi(\widehat{\Gamma}^{-1}v) & -\Psi(\widehat{A}^{-1}q + \widehat{\Gamma}^{-1}w) & -\Psi(\widehat{A}^{-1}g) \\ 0_{nm} & -F(\widehat{\Gamma}^{-1}v) & I_n - F(\widehat{A}^{-1}q + \widehat{\Gamma}^{-1}w) & -F(\widehat{A}^{-1}g) \\ 0_m & I_m & -\Psi(q) & -\Psi(g) \\ 0_{nm} & 0_{nm} & -F(q) & I_n - F(g) \end{bmatrix}. \tag{24}$$

(ii) Moreover, under the condition (23) the operator B_1 *is correct and the unique solution to boundary value problem*

$$B_1u = f, \quad \forall f \in X, \tag{25}$$

is given by

$$\begin{aligned} u &= B_1^{-1}f \\ &= \widehat{A}^{-2}f + \begin{pmatrix} \widehat{\Gamma}^{-1}v & \widehat{A}^{-1}\widehat{\Gamma}^{-1}v & \widehat{A}^{-2}q + \widehat{A}^{-1}\widehat{\Gamma}^{-1}w & \widehat{A}^{-2}g \end{pmatrix} \\ &\quad \cdot L^{-1} \begin{pmatrix} \Psi(\widehat{A}^{-1}f) \\ F(\widehat{A}^{-1}f) \\ \Psi(f) \\ F(f) \end{pmatrix}. \end{aligned} \tag{26}$$

Proof. (i) Let (23) holds true. Let $u \in \ker B_1$ and hence

$$B_1u = A^2u - qF(Au) - gF(A^2u) = 0,$$

$$\Gamma u = v\Psi(Au),$$

$$\Gamma(Au) = v\Psi(A^2u) + wF(Au). \tag{27}$$

By noticing that $\widehat{\Gamma} \subset \Gamma$ and $\widehat{\Gamma}^{-1}v$, $\widehat{\Gamma}^{-1}w \in \ker A$, we can write the last two boundary equations of (27) as follows

$$\Gamma\left(u - \widehat{\Gamma}^{-1}v\Psi(Au)\right) = 0,$$

$$\Gamma\left(Au-\widehat{\Gamma}^{-1}v\Psi(A^2u)-\widehat{\Gamma}^{-1}wF(Au)\right)=0, \tag{28}$$

which means that the elements $u-\widehat{\Gamma}^{-1}v\Psi(Au)$, $Au-\widehat{\Gamma}^{-1}v\Psi(A^2u)-\widehat{\Gamma}^{-1}wF(Au)\in D(\widehat{A})$. In addition, from the first equation of (27) we may obtain

$$A^2u-qF(Au)-gF(A^2u)=0,$$

$$A\left(Au-\widehat{\Gamma}^{-1}v\Psi(A^2u)-\widehat{\Gamma}^{-1}wF(Au)\right)-qF(Au)-gF(A^2u)=0,$$

$$\widehat{A}\left(Au-\widehat{\Gamma}^{-1}v\Psi(A^2u)-\widehat{\Gamma}^{-1}wF(Au)\right)-qF(Au)-gF(A^2u)=0,$$

$$Au-\widehat{\Gamma}^{-1}v\Psi(A^2u)-(\widehat{A}^{-1}q+\widehat{\Gamma}^{-1}w)F(Au)-\widehat{A}^{-1}gF(A^2u)=0, \tag{29}$$

and since $Au=A\left(u-\widehat{\Gamma}^{-1}v\Psi(Au)\right)=\widehat{A}\left(u-\widehat{\Gamma}^{-1}v\Psi(Au)\right)$, we get

$$u=\widehat{\Gamma}^{-1}v\Psi(Au)+\widehat{A}^{-1}\widehat{\Gamma}^{-1}v\Psi(A^2u)+(\widehat{A}^{-2}q+\widehat{A}^{-1}\widehat{\Gamma}^{-1}w)F(Au)$$
$$+\widehat{A}^{-2}gF(A^2u), \tag{30}$$

or conveniently in matrix form

$$u=\begin{pmatrix}\widehat{\Gamma}^{-1}v & \widehat{A}^{-1}\widehat{\Gamma}^{-1}v & \widehat{A}^{-2}q+\widehat{A}^{-1}\widehat{\Gamma}^{-1}w & \widehat{A}^{-2}g\end{pmatrix}\begin{pmatrix}\Psi(Au)\\ \Psi(A^2u)\\ F(Au)\\ F(A^2u)\end{pmatrix}. \tag{31}$$

Acting by the vectors Ψ and F on both sides of (29) and the first equation of (27), we acquire

$$\Psi(Au)-\Psi(\widehat{\Gamma}^{-1}v)\Psi(A^2u)-\Psi(\widehat{A}^{-1}q+\widehat{\Gamma}^{-1}w)F(Au)-\Psi(\widehat{A}^{-1}g)F(A^2u)=0,$$

$$F(Au)-F(\widehat{\Gamma}^{-1}v)\Psi(A^2u)-F(\widehat{A}^{-1}q+\widehat{\Gamma}^{-1}w)F(Au)-F(\widehat{A}^{-1}g)F(A^2u)=0,$$

$$\Psi(A^2u)-\Psi(q)F(Au)-\Psi(g)F(A^2u)=0,$$

$$F(A^2u)-F(q)F(Au)-F(g)F(A^2u)=0,$$

or in matrix form

$$L\begin{pmatrix}\Psi(Au)\\ \Psi(A^2u)\\ F(Au)\\ F(A^2u)\end{pmatrix}=\mathbf{0}, \tag{32}$$

where the matrix L is specified in (24). From (32) and the hypothesis it is implied that $\operatorname{col}\left(\Psi(Au),\,\Psi(A^2u),\,F(Au),\,F(A^2u)\right)=\mathbf{0}$, which upon substitution into (31) yields $u=0$.

Thus, $\ker B_1 = \{0\}$ and therefore B_1 is injective.

Conversely. We assume $\det L = 0$. Observe that

$$\det L = \det \begin{bmatrix} -F(\widehat{\Gamma}^{-1}v) & I_n - F(\widehat{A}^{-1}q + \widehat{\Gamma}^{-1}w) & -F(\widehat{A}^{-1}g) \\ I_m & -\Psi(q) & -\Psi(g) \\ 0_{nm} & -F(q) & I_n - F(g) \end{bmatrix}$$

$$= \pm \det \begin{bmatrix} I_n - F(\widehat{A}^{-1}q + \widehat{\Gamma}^{-1}w) - F(\widehat{\Gamma}^{-1}v)\Psi(q) & -F(\widehat{A}^{-1}g) - F(\widehat{\Gamma}^{-1}v)\Psi(g) \\ -F(q) & I_n - F(g) \end{bmatrix}$$

$$= \pm \det L_2 = 0, \tag{33}$$

by multiplying from the left the second line by $F(\widehat{\Gamma}^{-1}v)$ and adding to the first line of the matrix. Equation (33) suggests that there exists a vector $\mathbf{c} = \mathrm{col}(\mathbf{c}_1, \mathbf{c}_2) \neq \mathbf{0}$, where $\mathbf{c}_1 = col(c_{11}, \ldots, c_{1n})$ and $\mathbf{c}_2 = \mathrm{col}(c_{21}, \ldots, c_{2n})$, such that $L_2\mathbf{c} = \mathbf{0}$. Consider the element

$$u_0 = \widehat{A}^{-2}(q\mathbf{c}_1 + g\mathbf{c}_2) + \widehat{A}^{-1}\widehat{\Gamma}^{-1}\big[\,(w + v\Psi(q))\,\mathbf{c}_1 + v\Psi(g)\mathbf{c}_2\big]$$
$$+\widehat{\Gamma}^{-1}v\big[\Psi(\widehat{A}^{-1}q) + \Psi\left(\widehat{\Gamma}^{-1}[w + v\Psi(q)]\right)\big]\mathbf{c}_1$$
$$+\widehat{\Gamma}^{-1}v\big[\Psi(\widehat{A}^{-1}g) + \Psi(\widehat{\Gamma}^{-1}v)\Psi(g)\big]\mathbf{c}_2, \tag{34}$$

and notice that $u_0 \neq 0$; otherwise $q\mathbf{c}_1 + g\mathbf{c}_2 = 0$ which implies $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{0}$ since the vectors q, g are linearly independent. Notice that

$$\Gamma u_0 = v\big[\Psi(\widehat{A}^{-1}q) + \Psi\left(\widehat{\Gamma}^{-1}[w + v\Psi(q)]\right)\big]\mathbf{c}_1$$
$$+v\big[\Psi(\widehat{A}^{-1}g) + \Psi(\widehat{\Gamma}^{-1}v)\Psi(g)\big]\mathbf{c}_2,$$
$$\Gamma(Au_0) = [w + v\Psi(q)]\mathbf{c}_1 + v\Psi(g)\mathbf{c}_2,$$
$$Au_0 = \widehat{A}^{-1}(q\mathbf{c}_1 + g\mathbf{c}_2) + \widehat{\Gamma}^{-1}[w + v\Psi(q)]\mathbf{c}_1 + \widehat{\Gamma}^{-1}v\Psi(g)\mathbf{c}_2,$$
$$A^2u_0 = q\mathbf{c}_1 + g\mathbf{c}_2,$$
$$\Psi(Au_0) = \Psi(\widehat{A}^{-1}q)\mathbf{c}_1 + \Psi(\widehat{A}^{-1}g)\mathbf{c}_2 + \Psi\left(\widehat{\Gamma}^{-1}[w + v\Psi(q)]\right)\mathbf{c}_1$$
$$+\Psi(\widehat{\Gamma}^{-1}v)\Psi(g)\mathbf{c}_2,$$
$$\Psi(A^2u_0) = \Psi(q)\mathbf{c}_1 + \Psi(g)\mathbf{c}_2,$$

$$F(Au_0) = F(\widehat{A}^{-1}q)\mathbf{c}_1 + F(\widehat{A}^{-1}g)\mathbf{c}_2 + F\left(\widehat{\Gamma}^{-1}[w + v\Psi(q)]\right)\mathbf{c}_1$$
$$+F(\widehat{\Gamma}^{-1}v)\Psi(g)\mathbf{c}_2,$$
$$F(A^2u_0) = F(q)\mathbf{c}_1 + F(g)\mathbf{c}_2. \tag{35}$$

By using (35), we have

$$\Gamma u_0 - v\Psi(Au_0) = 0,$$
$$\Gamma(Au_0) - v\Psi(A^2u_0) - wF(Au_0) = (w, \mathbf{0})L_2\mathbf{c} = 0, \tag{36}$$

which means that $u_0 \in D(B_1)$. Further,

$$B_1u_0 = A^2u_0 - qF(Au_0) - gF(A^2u_0)$$
$$= q\mathbf{c}_1 + g\mathbf{c}_2 - q\Big[F(\widehat{A}^{-1}q)\mathbf{c}_1 + F(\widehat{A}^{-1}g)\mathbf{c}_2 + F\left(\widehat{\Gamma}^{-1}[w + v\Psi(q)]\right)\mathbf{c}_1$$
$$+F(\widehat{\Gamma}^{-1}v)\Psi(g)\mathbf{c}_2\Big] - g\left[F(q)\mathbf{c}_1 + F(g)\mathbf{c}_2\right]$$
$$= \begin{pmatrix} q & g \end{pmatrix} L_2\mathbf{c} = 0, \tag{37}$$

and hence $u_0 \in \ker B_1$ which implies $\ker B_1 \neq \{0\}$ and thus the operator B_1 is not injective.

(ii) Let $\det L \neq 0$ and consider the boundary value problem (25), i.e.

$$B_1u = A^2u - qF(Au) - gF(A^2u) = f,$$
$$\Gamma u = v\Psi(Au),$$
$$\Gamma(Au) = v\Psi(A^2u) + wF(Au), \tag{38}$$

for any $f \in X$. By taking into account (28) and repeating the same steps in (29), we get

$$A^2u - qF(Au) - gF(A^2u) = f,$$
$$Au - \widehat{\Gamma}^{-1}v\Psi(A^2u) - (\widehat{A}^{-1}q + \widehat{\Gamma}^{-1}w)F(Au) - \widehat{A}^{-1}gF(A^2u) = \widehat{A}^{-1}f, \tag{39}$$

and

$$u = \widehat{A}^{-2}f$$
$$+\begin{pmatrix} \widehat{\Gamma}^{-1}v & \widehat{A}^{-1}\widehat{\Gamma}^{-1}v & \widehat{A}^{-2}q + \widehat{A}^{-1}\widehat{\Gamma}^{-1}w & \widehat{A}^{-2}g \end{pmatrix} \begin{pmatrix} \Psi(Au) \\ \Psi(A^2u) \\ F(Au) \\ F(A^2u) \end{pmatrix}. \tag{40}$$

Acting now by the vectors Ψ and F on both sides of (39) and the first equation of (38), we get

$$L\begin{pmatrix}\Psi(Au)\\ \Psi(A^2u)\\ F(Au)\\ F(A^2u)\end{pmatrix}=\begin{pmatrix}\Psi(\widehat{A}^{-1}f)\\ F(\widehat{A}^{-1}f)\\ \Psi(f)\\ F(f)\end{pmatrix}, \tag{41}$$

where the matrix L is given in (24). Since $\det L \neq 0$ the system (41) has a unique solution for any $f \in X$. By inverting the system and substituting into (40), we obtain (26) which is the unique solution to the problem (38). Moreover, the operator B_1 is injective and onto, i.e. $R(B_1) = X$, while the operator B_1^{-1} is bounded since the operators $\widehat{A}^{-2}$, $\widehat{A}^{-1}$, $\widehat{\Gamma}^{-1}$ as well as the elements of the vectors F and Ψ are bounded. Thus, the operator B_1 is correct. □

4 Examples

To show the implementation and the effectiveness of the method unfolded in the previous section, we find the solution of a boundary value problem for a partial integro-differential equation with two integral boundary conditions.

Let the Fredholm integro-partial differential equation

$$u_{xy}-2xy\int_0^1\int_0^1 x^2u_{xy}(x,y)dxdy=6xy(5x-6),\quad (x,y)\in\Omega, \tag{42}$$

subject to two boundary conditions

$$u_x(x,0)=4x\int_0^1\int_0^1 u_{xy}(x,y)dxdy,$$

$$u(0,y)=\frac{2}{5}(y-1)\int_0^1\int_0^1 y^2u_{xy}(x,y)dxdy\}, \tag{43}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\}$, $\overline{\Omega} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$, and $C(\overline{\Omega})$ denotes the space of all continuous functions $u = u(x, y)$ defined on $\overline{\Omega}$.

Comparing problem (42), (43) with (12) and (14), it seems natural to take the operator $A : C(\overline{\Omega}) \to C(\overline{\Omega})$, the vectors $F = (F_1) \in C(\overline{\Omega})^*$ and $g = (g_1) \in C(\overline{\Omega})$, and the function $f \in C(\overline{\Omega})$ defined as follows

$$Au=u_{xy},\quad D(A)=\{u\in C(\overline{\Omega}):\, u_x,\, u_{xy}\in C(\overline{\Omega})\},$$

$$F(Au)=\left(\int_0^1\int_0^1 x^2u_{xy}(x,y)dxdy\right),$$

$$
\begin{aligned}
g &= (2xy),\\
f &= 6xy(5x-6).
\end{aligned}
\tag{44}
$$

In addition, the boundary operator $\Gamma : X_A \overset{on}{\to} Z$, the vector $\Psi = \mathrm{col}(\Psi_1, \Psi_2) \in \left[C(\overline{\Omega})^*\right]^2$ and the vector $v \in Z$ are specified as

$$
\Gamma u = \begin{pmatrix} u_x(x,0) \\ u(0,y) \end{pmatrix},
$$

$$
\Psi(Au) = \begin{pmatrix} \Psi_1(Au) \\ \Psi_2(Au) \end{pmatrix} = \begin{pmatrix} \int_0^1\int_0^1 u_{xy}(x,y)dxdy \\ \int_0^1\int_0^1 y^2 u_{xy}(x,y)dxdy \end{pmatrix},
$$

$$
v = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \left(\begin{pmatrix} 4x \\ 0 \end{pmatrix} \; \begin{pmatrix} 0 \\ \frac{2}{5}(y-1) \end{pmatrix} \right), \tag{45}
$$

where the boundary space

$$
Z = \left\{ z = \begin{pmatrix} z_1(x) \\ z_2(y) \end{pmatrix} : \; z_1(x) \in C[0,1], \quad z_2(y) \in C^1[0,1] \right\},
$$

$$
\text{with,} \quad \|z\|_Z = \|z_1(x)\|_C + \|z_2(y)\|_{C^1} \tag{46}
$$

is a Banach space, and

$$
\begin{aligned}
X_A &= (D(A), \|\cdot\|_{X_A}), \quad \|u\|_{X_A} = \|u\|_{C(\overline{\Omega})} + \|u_x\|_{C(\overline{\Omega})} + \|u_{xy}\|_{C(\overline{\Omega})},\\
N &= (\ker A, \|\cdot\|_N), \quad \|u\|_N = \|u\|_{C(\overline{\Omega})} + \|u_x\|_{C(\overline{\Omega})},
\end{aligned}
\tag{47}
$$

are also Banach spaces by Lemma 2, since $Au = A_2A_1u$, where $A_1u = u_x$, $D(A_1) = \{u \in C(\overline{\Omega}) : u_x \in C(\overline{\Omega})\}$ and $A_2u = u_y$, $D(A_2) = \{u \in C(\overline{\Omega}) : u_y \in C(\overline{\Omega})\}$ are maximal closed operators. Moreover, by the same lemma it is concluded that the operator $\widehat{\Gamma} \subset \Gamma$ is correct. To find the inverse operator $\widehat{\Gamma}^{-1}$, we notice that $\widehat{\Gamma}^{-1}z = u(x,y) \in \ker A$, $\forall z \in Z$, i.e. $z_1(x) = u_x(x,0)$, $z_2(y) = u(0,y)$ and $Au = u_{xy} = 0$, which after double integration yields

$$
\widehat{\Gamma}^{-1}z = \int_0^x z_1(s)ds + z_2(y), \quad \forall z \in Z. \tag{48}
$$

Also, the correct operator $\widehat{A}$ is defined by

$$
\widehat{A}u = u_{xy}, \quad D(\widehat{A}) = \left\{ u \in D(A) : \begin{pmatrix} u_x(x,0) \\ u(0,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \tag{49}
$$

and its inverse is given by

$$\widehat{A}^{-1}f = \int_0^x \int_0^y f(t,s)dsdt, \quad \forall f \in C(\overline{\Omega}). \tag{50}$$

Finally, it is easy to show that the functionals F, Ψ_1, Ψ_2 are bounded. Thus, the operator $B : C(\overline{\Omega}) \to C(\overline{\Omega})$ is defined by

$$Bu = Au - gF(Au) = u_{xy} - 2xy \int_0^1 \int_0^1 x^2 u_{xy}(x,y)dxdy,$$

$$D(B) = \{u \in D(A) : \Gamma u = v\Psi(Au)\}, \tag{51}$$

and the given problem is formulated as

$$Bu(x,y) = f(x,y). \tag{52}$$

We now apply Theorem 1. To examine its solvability, we compute

$$F(g) = \left[\int_0^1 \int_0^1 x^2(2xy)dxdy \right] = \left[\ \tfrac{1}{4} \ \right],$$

$$W = \left[\ 1 - F(g) \ \right] = \left[\ \tfrac{3}{4} \ \right]. \tag{53}$$

As a consequence $\det W = \frac{3}{4} \neq 0$ and therefore the given problem possesses a unique solution for every $f \in C(\overline{\Omega})$. To find the solution, we perform the following calculations

$$\widehat{A}^{-1}f = \int_0^x \int_0^y f(t,s)dsdt = x^2y^2(5x-9),$$

$$\widehat{A}^{-1}g = \left(\int_0^x \int_0^y g(t,s)dsdt \right) = \left(\frac{1}{2}x^2y^2 \right),$$

$$F(f) = \left(\int_0^1 \int_0^1 x^2 f(x,y)dxdy \right) = \left(-\frac{3}{2} \right),$$

$$\Psi(f) = \begin{pmatrix} \Psi_1(f) \\ \Psi_2(f) \end{pmatrix} = \begin{pmatrix} \int_0^1 \int_0^1 f(x,y)dxdy \\ \int_0^1 \int_0^1 y^2 f(x,y)dxdy \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix},$$

$$\Psi(g) = \begin{bmatrix} \Psi_1(g) \\ \Psi_2(g) \end{bmatrix} = \begin{bmatrix} \int\limits_0^1 \int\limits_0^1 g(x,y)dxdy \\ \int\limits_0^1 \int\limits_0^1 y^2 g(x,y)dxdy \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix},$$

$$\widehat{\Gamma}^{-1}v = \begin{pmatrix} \widehat{\Gamma}^{-1}v_1 & \widehat{\Gamma}^{-1}v_2 \end{pmatrix} = \begin{pmatrix} 2x^2 & \frac{2}{5}(y-1) \end{pmatrix}, \tag{54}$$

and then by substituting the above into the formula (15), we obtain

$$u(x,y) = 5x^3y^2 - 10x^2(y^2+1) - y + 1. \tag{55}$$

5 Conclusions

By employing a perturbation technique and the theory of extensions of operators in Banach spaces we have developed a method for examining the solvability and constructing the unique solution in closed form of a kind of boundary value problems incorporating a linear Fredholm integro-partial differential equation and nonlocal multipoint and integral boundary conditions of special type. The method has been proven to be effective, easy to use and simple to program to a computer algebra system.

The method can be adjusted for solving general boundary value problems for partial differential equations with nonstandard boundary conditions.

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Евтимиос Провидас, Иван Нестерович Парасидис СТАНДАРТТЫ ЕМЕС ДЕРБЕС ТУЫНДЫЛЫ ШЕТТІК ЕСЕПТЕРДІ ШЕШУГЕ АРНАЛҒАН КЕҢЕЙТУ ОПЕРАТОРЫ ӘДІСІ

Біз стандартты емес интегралдық шекаралық шарттары бар және дербес туындылы сызықты Фредгольм интегралдық-дифференциалдық теңдеулері үшін кейбір жалпы шеттік есептердің шешілімділік критерийлерін тағайындаймыз және дәл шешімін тұрғызамыз. Біздің тәсіліміз ауытқулар әдісі мен Банах кеңістіктеріндегі операторлардың кеңейтулер теориясына негізделген әрі қарапайым кәдуілгі шекаралық шарттары бар дербес туындылы сәйкес теңдеуді қамтитын мүлтіксіз барынша қарапайым есептің айқын шешімін білуді жорамалдайды.

Кілттік сөздер. Дербес туындылы интегралдық-дифференциалдық теңдеулер, дифференциалдық операторлар, интегралдық шекаралық шарттар, дәл шешімдер.

Евтимиос Провидас, Иван Нестерович Парасидис МЕТОД ОПЕРАТОРА РАСШИРЕНИЯ ДЛЯ РЕШЕНИЯ НЕСТАНДАРТНЫХ ЧАСТИЧНЫХ КРАЕВЫХ ЗАДАЧ

Мы устанавливаем критерии разрешимости и строим точное решение некоторых общих краевых задач для линейных интегро-дифференциальных уравнений Фредгольма с частными производными и нестандартными интегральными граничными условиями. Наш подход основан на методе возмущений и теории расширений операторов в Банаховых пространствах и предполагает знание явного решения идеальной более простой задачи, включающей соответствующее уравнение в частных производных с простыми обычными граничными условиями.

Ключевые слова. Интегро-дифференциальные уравнения с частными производными, дифференциальные операторы, интегральные граничные условия, точные решения.

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A randomly determined unpredictable function

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Abstract. Recently, we have introduced unpredictable oscillations, which are in the basis of Poincaré chaos. For theoretical analysis as well as for applications, it is necessary to provide constructive examples of unpredictable functions. We have already provided such functions utilizing orbits of the logistic map, and in the present paper we suggest another way of construction of the functions by applying the Bernoulli random process. A simulation for a randomly determined unpredictable function is provided.

Keywords. Unpredictable functions, Unpredictable sequences, Bernoulli process, Poincaré chaos, Symbolic dynamics.

1 Introduction and preliminaries

The theory of oscillations extremely rely on functions, which can be either tabulated or formalized. The ones in the second category are based first of all on the functions which are trigonometric, polynomials, hyperbolic trigonometric and others. All of them have been tabulated in computer memories. Next ones are functions, which can be presented as finite or infinite sums of the former ones. They are evaluated by developing software programs and very helpful in applications. Other are oscillations produced as solutions of differential equations. There exists, even, the large class in the qualitative theory of differential equations – oscillatory differential equations. The solutions are approved as oscillations by special type of criteria for the existence. In this study, we focus on functions which are shaped through qualitative conditions of definitions. They make the core of the research area in the theory of dynamical systems, issued by H. Poincaré, G. Birkhoff, and others [1], [2]. These are periodic, quasi-periodic, almost periodic oscillations, recurrent and Poisson stable orbits [1]–[4]. A special type of Poisson stable orbit called an unpredictable trajectory, which leads

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to Poincaré chaos in the quasi-minimal set, was introduced in the paper [5]. Moreover, the papers [6]–[9] were concerned with unpredictable solutions of various types of quasi-linear differential equations. In the present paper, we introduce a new way for unpredictable functions construction benefiting from the dynamics associated with the discrete distribution [10]. We consider the process with a finite number of possible outcomes to generate an unpredictable sequence. The sequence is then used to construct a continuous unpredictable function. Thus, unpredictable oscillations appeared as solutions of linear or quasi-linear differential equations, i.e., as outputs of the systems, provided that there is an unpredictable input. The natural question how it is possible to choose the inputs being unpredictable arises. For this reason in the previous papers [6]–[9], we introduced unpredictable functions built by applying orbits of the logistic map, which were verified to be unpredictable sequences. One can confirm that in this way we utilize several other discrete equations with dynamics topologically equivalent to the symbolic dynamics [11], [12]. This is why, they are in some sense the same as those functions, which have been already determined in our research. For that reason, the task of construction of new unpredictable oscillations is undertaken in the present paper. We utilize the two principal issues for solving the problem. The first one is that the set of all orbits of the symbolic dynamics coincides with all possible sequences of the symbols. Moreover, realizations of the Bernoulli random process altogether are the set of sequences. Consequently, constructing an orbit of a random process, we obtain an orbit of the symbolic dynamics and simulate a part of the unpredictable sequence. Thus, we obtain that a single iteration of the Bernoulli shift is the same as a trial for the Bernoulli process.

The next definitions are concerned with unpredictable sequences and functions.

Definition 1.1 [8]. *A bounded sequence $\{\nu_k\}$, $k \in \mathbb{Z}$, in $\mathbb{R}^p$ is called unpredictable if there exist a positive number ε_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$, $n \in \mathbb{N}$, of positive integers both of which diverge to infinity such that $\|\nu_{k+\zeta_n} - \nu_k\| \to 0$ as $n \to \infty$ for each k in bounded intervals of integers and $\|\nu_{\zeta_n+\eta_n} - \nu_{\eta_n}\| \geq \varepsilon_0$ for each $n \in \mathbb{N}$.*

Definition 1.2 [6]. *A uniformly continuous and bounded function $h : \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ε_0, σ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $h(t+t_n) \to h(t)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|h(t+t_n)-h(t)\| \geq \varepsilon_0$ for each $t \in [u_n - \sigma, u_n + \sigma]$ and $n \in \mathbb{N}$.*

Consider the space Σ_m of bi-infinite sequences $\ldots i_{-2}i_{-1}.i_0i_1i_2\ldots$ on finite number of complex numbers $a_1, \ldots, a_m$, with the metric

$$d(I, J) = \sum_{k=-\infty}^{\infty} \frac{|i_k - j_k|}{2^{|k|}}, \tag{1}$$

where $I = (\ldots i_{-2}i_{-1}.i_0i_1i_2\ldots), J = (\ldots j_{-2}j_{-1}.j_0j_1j_2\ldots)$, and $|.|$ is the absolute value. Introduce the Bernoulli shift $\varphi : \Sigma_m \to \Sigma_m$ such that

$$\varphi((\ldots i_{-2}i_{-1}.i_0i_1i_2\ldots)) = (\ldots i_{-2}i_{-1}i_0.i_1i_2i_3\ldots). \tag{2}$$

The map φ is continuous and the metric space Σ_m is compact [12].

Let us, now, build an unpredictable point for the dynamics (Σ_m, d, φ). Without loss of generality, we consider a particular case of the space when $m = 2$, $a_1 = 0$, $a_2 = 1$. We need a collection of finite sequences i_k^r, $r \in \mathbb{N}$, $k = 1, 2, \ldots, 2^r$, consisting of 0's and 1's. Let us use the notations $i_1^1 = (0)$ and $i_2^1 = (1)$ for the sequences of length 1. For each natural number r, we recursively define $i_{2k-1}^{r+1} = (i_k^r 0)$ and $i_{2k}^{r+1} = (i_k^r 1)$, $k = 1, 2, \ldots, 2^r$, where i_{2k-1}^{r+1} and i_{2k}^{r+1} are obtained by respectively inserting 0 and 1 to the end of the sequence i_k^r of length r. For instance, $i_1^2 = (i_1^1 0) = (00)$, $i_2^2 = (i_1^1 1) = (01)$, $i_3^2 = (i_2^1 0) = (10)$, and $i_4^2 = (i_2^1 1) = (11)$ are the sequences of length 2. Now consider the following sequence $i^* = (\ldots i_8^3 i_6^3 i_4^3 i_2^3 i_4^2 i_2^2 . i_1^1 i_1^2 i_3^2 i_1^3 i_3^3 i_5^3 i_7^3 \ldots)$. In [5] it was proved that i^* is an unpredictable point of the dynamics.

Because the trajectory which initiates at i^* is dense in the quasi-minimal set Σ_m, the dynamics is Poincaré chaotic according to Theorem 3.1 presented in paper [5]. Moreover, there is an uncountable set of unpredictable points in the set. From this discussion it implies that any numerical simulation of a discrete finite distribution is an approximation of an unpredictable sequence. Indeed, the metric peculiarity implies that if one considers the point i^* in Σ_m as bi-infinite sequence, then it is easily seen that it is an unpredictable sequence in the sense of Definition 1.1. This is in the base of the construction of an unpredictable function in the next section.

2 Main result

Let us fix a finite string $i_k, \ldots, i_p, 1 \leq k < p$, on the set of complex numbers $a_1, \ldots, a_m$. It can be accepted as an arc of a sequence from Σ_m. Since of the last section discussion, the string can be approximated with arbitrary precision by a shift of the sequence i^*. More precisely, the peculiarity of the metric implies that as the result of the shifting we have coincidence of elements in arcs, and the term approximation relates only to the length of the coincidence. This possibility to approximate by the shifts is the main advantage of the Poincaré chaos against other types of chaos. Taking into account that there are limits for the approximation validity in numerical simulations by computers, we can admit that simulation of the string is simulation of the unpredictable sequence itself. This is why, we accept that finite realizations of the Bernoulli process, which are obtained randomly present the unpredictable sequence, since, at first, they are not periodic even on a sufficiently large interval of discrete time, and, secondly, since of the above explanation the simulation is an approximation of the sequence with arbitrary precision. The arbitrariness guarantees that in applications we can get the simulations as the unpredictable sequence with the atributes listed in the definition. Moreover, we must not be confused with the approximations in the basis of the definition. This is true for all types of functions, which are determined through infinitely long algorithms such as series, for instance.

Fix an unpredictable sequence i^*, which is defined on the two real numbers a and b. One can find that the unpredictability constant ϵ_0 can be taken equal to $|a-b|$. Define the function $\chi(t) : \mathbb{R} \to \mathbb{R}$ through the equation

$$\chi(t) = \int_{-\infty}^{t} e^{-(t-s)}\pi(s)ds, \tag{3}$$

where $\pi(t) : \mathbb{R} \to \mathbb{R}$ is the piecewise constant function satisfying $\pi(t) = i_k^*$ for $t \in [k, k+1)$, $k \in \mathbb{Z}$. One can confirm that $\sup\limits_{t\in\mathbb{R}} |\chi(t)| \le M_\chi$, where $M_\chi = \max\{|a|, |b|\}$.

We will show that the function $\chi(t)$ defined by (3) is unpredictable. Consider a fixed compact interval $[\alpha, \beta]$ and a positive number ε. We assume without loss of generality that α and β are integers. Let us fix a positive number ξ and an integer $\gamma < \alpha$ which satisfy the inequalities $Me^{-2(\alpha-\gamma)} < \varepsilon/4$ and $\xi(1 - e^{-2(\beta-\gamma)}) < \varepsilon$. Suppose that n is a sufficiently large natural number satisfying $|\pi(t + \zeta_n) - \pi(t)| < \xi$ for every t in $[\gamma, \beta]$. Accordingly, we have for $t \in [\alpha, \beta]$ that

$$|\chi(t + \zeta_n) - \chi(t)| \le \int_{-\infty}^{\gamma} e^{-2(t-s)} \left|\pi(s + \zeta_n) - \pi(s)\right| ds$$

$$+ \int_{\gamma}^{\beta} e^{-2(t-s)} \left|\pi(s + \zeta_n) - \pi(s)\right| ds \le \int_{-\infty}^{\gamma} e^{-2(t-s)} 2ds + \int_{\gamma}^{\beta} e^{-2(t-s)}\xi ds$$

$$\le 2Me^{-2(\alpha-\gamma)} + \frac{\xi}{2}[1 - e^{-2(\beta-\gamma)}] < \varepsilon.$$

Thus, $|\chi(t + \zeta_n) - \chi(t)| \to 0$ as $n \to \infty$ uniformly on the interval $[\alpha, \beta]$.

Let us fix a number n and consider two alternative cases: (i) $|\chi(\eta_n + \zeta_n) - \chi(\eta_n)| < \frac{\epsilon_0}{8}$ and (ii) $|\chi(\eta_n + \zeta_n) - \chi(\eta_n)| \ge \frac{\epsilon_0}{8}$.

(i) There exists a positive number $\kappa < 1$ such that $e^{-2\kappa} = \frac{2}{3}$. Using the relation

$$\chi(t + \zeta_n) - \chi(t) = \chi(\eta_n + \zeta_n) - \chi(\eta_n) + \int_{\eta_n}^{t} e^{-2(t-s)}(\pi(s + \zeta_n) - \pi(s))ds \tag{4}$$

we obtain that

$$|\chi(t + \zeta_n) - \chi(t)| \ge \left|\int_{\eta_n}^{t} e^{-2(t-s)}|\pi(s + \zeta_n) - \pi(s))|ds - |\pi(\eta_n + \zeta_n) - \pi(\eta_n)|\right.$$

$$\ge \int_{\eta_n}^{t} e^{-2(t-s)}\epsilon_0 ds - \frac{\epsilon_0}{8} \ge \frac{\epsilon_0}{2}(1 - e^{-2\kappa}) - \frac{\epsilon_0}{8} = \frac{\epsilon_0}{24}$$

for $t \in [\eta_n + \kappa, \eta_n + 1)$.

(ii) There exists a positive number $\kappa < 1$ such that $1 - e^{-2\kappa} = \frac{\epsilon_0}{12}$. From the relation (4) we get

$$|\chi(t + \zeta_n) - \chi(t)| \geq |\chi(\eta_n + \zeta_n) - \chi(\eta_n)| - |\int_{\eta_n}^{t} e^{-2(t-s)}(\pi(s + \zeta_n) - \pi(s))ds$$

$$\geq \frac{\epsilon_0}{8} - \int_{\eta_n}^{t} e^{-2(t-s)} 2ds \geq \frac{\epsilon_0}{8} - [1 - e^{-2\kappa}] = \frac{\epsilon_0}{24}$$

for $t \in [\eta_n, \eta_n + \kappa)$.

Thus, $\chi(t)$ is an unpredictable function.

It is easy to see that $\chi(t)$ is a solution of the differential equation

$$x' = -x + h(t) \tag{5}$$

with

$$\chi(0) = \int_{-\infty}^{0} e^s h(s)ds, \tag{6}$$

but we do not know the value $\chi(0)$ precisely, since it cannot be evaluated by the improper integral (6). Nevertheless, we utilize that $\chi(t)$ is an exponentially stable solution of equation (5). Therefore, any solution $\varphi(t)$ of (5) approximates $\chi(t)$. The approximation is better for larger t such that $\|\chi(t) - \varphi(t)\| \leq \|\chi(0) - \varphi(0)\| e^{-t}$, $t \geq 0$. For that reason we take $\varphi(0) = 0.5$, so that $\|\chi(t) - \varphi(t)\| \leq e^{-50} < 10^{-17}$ for $t \in [50, 100]$. It is less than Matlab precision between 50 and 100. Hence, the part of the time series of $\varphi(t)$ for $50 \leq t \leq 100$ can be accepted as the graph of the function $\chi(t)$.

In Figure 1 we depict the unpredictable function $\chi(t)$ defined by equation (3). For the simulation, we use the function $\pi(t) = i_k$, $t \in [\mu(k-1), \mu k)$, $\mu = 0.1$, $k \in \mathbb{N}$. The sequence i_k is generated randomly such that $i_k = 0, 1$ for each $k = 1, 2, \ldots$.

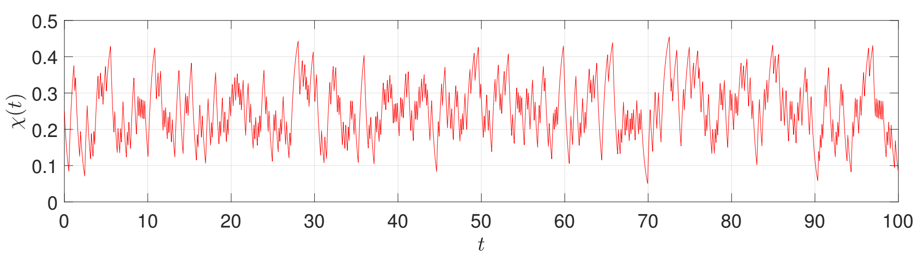

Figure 1 – Time series of the unpredictable function $\chi(t)$

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Ахмет М., Фен М.О., Аледжайли Е.М. СТОХАСТИКАЛЫҚ ТҮРДЕ АНЫҚТАЛҒАН БОЛЖАНБАҒАН ФУНКЦИЯ

Пуанкаре хаосы негізінде жатқан болжанбаған тербелістерді біз жақында енгізген болатынбыз. Теориялық талдаумен қатар, қолданыс тұрғысынан алғанда, болжанбаған функциялардың құрылымдық мысалдарын ұсыну қажет болады. Бұрын біз болжанбаған функцияларды логистикалық бейнелеу орбиталарын пайдалана отырып құрған болсақ, осы мақаламызда аталған функцияларды кездейсоқ Бернулли процесін қолдана отырып құрудың басқа әдісін ұсынамыз. Кездейсоқ анықталған болжанбаған функцияға арналған моделдеу ұсынылады.

Кілттік сөздер. Болжанбаған функция, болжанбаған тізбек, Бернулли процесі, Пуанкаре хаосы, символдық динамика.

Ахмет М., Фен М.О., Аледжайли Е.М. СТОХАСТИЧЕСКИ ОПРЕДЕЛЕННАЯ НЕПРЕДСКАЗУЕМАЯ ФУНКЦИЯ

Непредсказуемые колебания, которые лежат в основе хаоса Пуанкаре, были введены нами недавно. Как для теоретического анализа, так и для приложений необходимо предоставлять конструктивные примеры непредсказуемых функций. Ранее мы строили непредсказуемые функции, используя орбиты логистического отображения, и в настоящей статье предлагаем другой способ построения функций с применением случайного процесса Бернулли. Предложено моделирование для случайно определенной непредсказуемой функции.

Ключевые слова. Непредсказуемые функция, непредсказуемая последовательность, процесс Бернулли, хаос Пуанкаре, символическая динамика.

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Well-posedness of Dirichlet and Poincare problems in multi-dimensional domain for degenerate hyperbolic equations

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Abstract. It is well-known that the Dirichlet problem is ill-posed for general hyperbolic equations. Earlier work mostly studied this problem using the functional analysis methods, which is inconvenient for applications. This paper establishes a multi-dimensional domain, where the Dirichlet and Poincare problems for the degenerate hyperbolic equations are well-posed.

Keywords. Well-posedness, multi-dimensional domain, degenerate equations, spherical functions.

1 Introduction

It has been shown (see [1]) that one of the fundamental problems of mathematical physics, that is, the study of the behavior of an oscillating string, is incorrect on a plane in the case when the boundary conditions are given on the entire boundary of the region. As noted in [2], [3], the Dirichlet problem is ill-posed not only for the wave equation but also for general hyperbolic equations. In [4] it has been shown that a solution to the Dirichlet problem exists in rectangular domains. Subsequently, this problem was investigated using the functional analysis methods [5], which is however inconvenient for the use in applied work.

A more complete bibliography of works devoted to this subject can be found in the monographs [3], [6].

In [7]–[10], the Dirichlet and Poincare problems for degenerate multidimensional hyperbolic equations were analyzed. These works show that the well-posedness of these problems crucially depends on the height of the cylindrical region under study.

In this paper, we establish a multi-dimensional region inside a characteristic conoid in which the Dirichlet and Poincare problems for the Gellerstedt equation are uniquely solvable.

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2 Statement of the problem and the main result

Let D be a finite region of the Euclidean space E_{m+1} of points $(x_1, ..., x_m, t)$, bounded at $t > 0$ with the conic surface $K : t = \varphi(r)$, $\varphi(0) = \varphi(1) = 0$, $\varphi(r) \in C^1([0,1]) \cap C^2((0,1))$, $|\varphi'(r)| < 1$, and the plane $t = 0$, where $r = |x|$ is the length of the vector $x = (x_1, ..., x_m)$. Let us denote with S the set $\{t = 0, 0 < |x| < 1\}$ of the points in E_m.

In the domain D, let us analyze the multi-dimensional Gellerstedt equation

$$t^p \Delta_x u - u_{tt} = 0, \tag{1}$$

where $p = const > 0$, Δ_x is the Laplace operator for variables $x_1, ..., x_m$, $m \geq 2$.

The multi-dimensional versions of the Dirichlet and Poincare problems for the equation (1) are the following problems.

Problem 1. In the domain D, find the solution to the equation (1) from the class $C(\bar{D}) \cap C^1(D \cup S) \cap C^2(D)$, satisfying the boundary conditions

$$u|_S = \tau(x), \ \ u|_K = \sigma(x), \tag{2}$$

or

$$u_t|_S = \nu(x), \ \ u|_K = \sigma(x). \tag{3}$$

Hereafter, it is convenient to switch from the Cartesian coordinates $x_1, ..., x_m$, t to spherical ones $r, \theta_1, ..., \theta_{m-1}, t$, $r \geq 0$, $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi$, $i = 2, 3, ..., m-1$.

Let $\{Y_{n,m}^k(\theta)\}$ be a system of linearly independent spherical functions of order n, $1 \leq k \leq k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2), \theta = (\theta_1, ..., \theta_{m-1})$, $W_2^l(S)$, $l = 0, 1, ...$, being the Sobolev spaces.

The following lemma holds [11].

Lemma 1. *Let $f(r,\theta) \in W_2^l(S)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{4}$$

as well as the series obtained from it by differentiation of order $p \leq l - m + 1$, converge absolutely and uniformly in S.

Lemma 2. *For $f(r,\theta) \in W_2^l(S)$ to hold, it is necessary and sufficient that the coefficients of the series (4) satisfy the inequalities*

$$|f_0^1(r)| \leq c_1, \ \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} n^{2l} |f_n^k(r)|^2 \leq c_2, \ \ c_1, \, c_2 = const.$$

Let us denote by $\bar{\tau}_n^k(r),\ \bar{\nu}_n^k r,\ \bar{\sigma}_n^k(r)$ the coefficients of the series (4), respectively, of the functions $\tau(r,\theta),\ \nu(r,\theta),\ \sigma(r,\theta)$.

Let also $\tau(r,\theta) = r^4\tau^*(r,\theta)$, $\nu(r,\theta) = r^4\nu^*(r,\theta)$, $\sigma(r,\theta) = r^4\sigma^*(r,\theta), \tau^*(r,\theta), \nu^*(r,\theta)$, $\sigma^*(r,\theta) \in W_2^l(S)$, $l > \frac{3m}{2} + 4$.

Then, the following theorem holds.

Theorem. *Problem 1 is uniquely solvable.*

Note that [12] proves this theorem for the multidimensional wave equation.

3 Proof of the Theorem

In the spherical coordinates, the equation (1) has the form [11]

$$t^p(u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u) - u_{tt} = 0, \tag{5}$$

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j \sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\frac{\partial}{\partial\theta_j}\right),\ g_1 = 1,\ g_j = (\sin\theta_1 ... \sin\theta_{j-1})^2,\ j > 1.$$

Since the solution to Problem 1 that we are looking for belongs to the class $C(\bar{D})\cap C^2(D)$, it can be sought in the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{u}_n^k(r,t)Y_{n,m}^k(\theta), \tag{6}$$

where $\bar{u}_n^k(r,t)$ are functions to be determined.

Substituting (6) into (5), and using the orthogonality of spherical functions $Y_{n,m}^k(\theta)$ [11], we obtain

$$t^p(\bar{u}_{nrr}^k + \frac{m-1}{r}\bar{u}_{nr}^k - \frac{\lambda_n}{r^2}\bar{u}_n^k) - \bar{u}_{ntt}^k = 0,\ \lambda_n = n(n+m-2),\ k = \overline{1,k_n},\ n = 0,1,..., \tag{7}$$

with the boundary conditions (2) and (3), taking into account Lemma 1, take the forms, respectively,

$$\bar{u}_n^k(r,0) = \bar{\tau}_n^k(r),\ \bar{u}_n^k(r,\varphi(r)) = \bar{\sigma}_n^k(r),\ k = \overline{1,\ k_n},\ n = 0,1,...\ , \tag{8}$$

$$\bar{u}_{nt}^k(r,0) = \bar{\nu}_n^k(r),\ \bar{u}_n^k(r,\varphi(r)) = \bar{\sigma}_n^k(r),\ k = \overline{1,\ k_n},\ n = 0,1,... . \tag{9}$$

Doing the substitutions $\bar{u}_n^k(r,t) = r^{\frac{(1-m)}{2}}u_n^k(r,t)$ and putting $r = r,\ x_0 = \dfrac{2}{2+p}t^{\frac{(2+p)}{2}}$, the problems (7), (8) and (7), (9) reduce to the following problems

$$L_\alpha \upsilon_{\alpha,n}^k \equiv \upsilon_{\alpha,nrr}^k - \upsilon_{\alpha,nx_0x_0}^k - \frac{\alpha}{x_0}\upsilon_{\alpha,nx_0}^k + \frac{\bar{\lambda}_n}{r^2}\upsilon_{\alpha,n}^k = 0, \tag{10_α}$$

$$\upsilon^k_{\alpha,n}(r,0)=\tau^k_n(r),\ \upsilon^k_{\alpha,n}(r,\varphi_1(r))=\sigma^k_n(r),\ k=\overline{1,\,k_n},\ n=0,1,\dots, \tag{11}$$

$$\lim_{x_0\to 0} x_0^\alpha \frac{\partial}{\partial x_0}\upsilon^k_{\alpha,n}=\nu^k_n(r),\upsilon^k_{\alpha,n}(r,\varphi_1(r))=\sigma^k_n(r),\ k=\overline{1,\,k_n},\ n=0,1,\dots, \tag{12}$$

where

$$0<\alpha=\frac{p}{2+p}<1,\ \bar\lambda_n=\frac{((m-1)(3-m)-4\lambda_n)}{4},\ \upsilon^k_{\alpha,n}(r,x_0)=u^k_n\left[r,(\frac{2+p}{2}x_0)^{\frac{2}{2+p}}\right],$$

$$\varphi_1(r)=\frac{2}{2+p}[\varphi(r)]^{\frac{2+p}{2}},\tau^k_n(r)=r^{\frac{(m-1)}{2}}\bar\tau^k_n(r),\ \nu^k_n(r)=r^{\frac{(m-1)}{2}}\bar\nu^k_n(r),\ \ \sigma^k_n(r)=r^{\frac{(m-1)}{2}}\bar\sigma^k_n(r).$$

Along with the equation (10_α), let us consider the equation

$$L_0\upsilon^k_{0,n}\equiv\upsilon^k_{0,nrr}-\upsilon^k_{0,nx_0x_0}+\frac{\overline{\lambda}_n}{r^2}\upsilon^k_{0,n}=0. \tag{10_0}$$

As has been shown in [13] (see also [14]), there exists the following functional connection between the solutions to the Cauchy problem for the equations (10_α) and (10_0).

Proposition 1. *If $\upsilon^{k,1}_{0,n}(r,x_0)$ is a solution to the Cauchy problem for the equation (10_0) that satisfies the conditions*

$$\upsilon^{k,1}_{0,n}(r,0)=\tau^k_n(r),\ \frac{\partial}{\partial x_0}\upsilon^{k,1}_{0,n}(r,0)=0, \tag{13}$$

then the function

$$\upsilon^{k,1}_{\alpha,n}(r,x_0)=\gamma_\alpha\int\limits_0^1\upsilon^{k,1}_{0,n}(r,\xi x_0)(1-\xi^2)^{\frac{\alpha}{2}-1}d\xi\equiv 2^{-1}\gamma_\alpha\Gamma\left(\frac{\alpha}{2}\right)x_0^{1-\alpha}D^{-\frac{\alpha}{2}}_{0x_0^2}\left[\frac{\upsilon^{k,1}_{0,n}(r,x_0)}{x_0^2}\right] \tag{14}$$

for $\alpha>0$ is a solution of the equation (10_α) with the conditions (13).

Proposition 2. *If $\upsilon^{k,1}_{0,n}(r,x_0)$ is a solution to the Cauchy problem for the equation (10_0) that satisfies the conditions*

$$\upsilon^{k,1}_{0,n}(r,0)=\frac{\nu^k_n(r)}{(1-\alpha)(3-\alpha)...(2q+1-\alpha)},\ \frac{\partial}{\partial x_0}\upsilon^{k,1}_{0,n}(r,0)=0, \tag{15}$$

then for $0<\alpha<1$ the function

$$\upsilon^{k,2}_{\alpha,n}(r,x_0)=\gamma_{2-k+2q}\left(\frac{1}{x_0}\frac{\partial}{\partial x_0}\right)^q\left[x_0^{1-\alpha+2q}\int_0^1\upsilon^{k,1}_{0,n}(r,\xi x_0)(1-\xi^2)^{q-\frac{\alpha}{2}}d\xi\right]\equiv$$

$$\equiv\gamma_{2-k+2q}2^{q-1}\Gamma\left(q-\frac{\alpha}{2}+1\right)D^{\frac{\alpha}{2}-1}_{0x_0^2}\left[\frac{\upsilon^{k,1}_{0,n}(r,x_0)}{x_0}\right] \tag{16}$$

is a solution of the equation (10_α) *with the initial conditions*

$$\upsilon^{k,2}_{\alpha,n}(r,0)=0,\ \lim_{x_0\to 0} x_0^\alpha \frac{\partial}{\partial x_0}\upsilon^{k,2}_{\alpha,n}=\nu^k_n(r), \tag{17}$$

where $\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)\gamma_\alpha = 2\Gamma\left(\frac{\alpha+1}{2}\right)$, $\Gamma(z)$ *is the gamma-function,* D^α_{0t} *is the Riemann-Liouville operator* [15], *whereas* $q\geq 0$ *is the smallest integer satisfying the inequality* $2-\alpha+2q\geq m-1$.

We will seek for the solution of the problem (10_α), (11) in the form

$$\upsilon^{k}_{\alpha,n}(r,x_0)=\upsilon^{k,1}_{\alpha,n}(r,x_0)+\upsilon^{k,2}_{\alpha,n}(r,x_0), \tag{18}$$

where $\upsilon^{k,1}_{\alpha,n}(r,x_0)$ is a solution to the Cauchy problem (10_α), (13), while $\upsilon^{k,2}_{\alpha,n}(r,x_0)$ is the solution to the boundary value problem for the equation (10_α) with the conditions

$$\upsilon^{k,2}_{\alpha,n}(r,0)=0,\ \upsilon^{k,2}_{\alpha,n}(r,\varphi_1(r))=\sigma^k_n(r)-\upsilon^{k,1}_{\alpha,n}(r,\varphi_1(r)). \tag{19}$$

Taking into account the expressions (14), (16), as well as the invertibility of the operator D^α_{0t} [15] the problems (10_α), (13) and (10_α), (19), respectively, reduce to the Cauchy problem (10_0), (13), which has a unique solution [13], and to the problem for (10_0) with the conditions

$$\frac{\partial}{\partial x_0}\upsilon^{k,1}_{0,n}(r,0)=0,\ \upsilon^{k,1}_{0,n}(r,\varphi_1(r))=\sigma^k_{1n}(r), \tag{20}$$

where $\sigma^k_{1n}(r)$ is a function expressed through $\tau^k_n(r)$, $\sigma^k_n(r)$.

It has been shown in [12] that the problem (10_0), (20) is uniquely solvable.

Next, using Propositions 1 and 2, the unique solvability of the problems (10_α), (13) and (10_α), (19) are established.

Now we can solve the problem (10_α), (12) in the form (18), where $\upsilon^{k,2}_{\alpha,n}(r,x_0)$ is a solution to the problem Cauchy (10_α), (17), and $\upsilon^{k,1}_{\alpha,n}(r,x_0)$ is a solution to the problem for (10_α) with the conditions

$$\frac{\partial}{\partial x_0}\upsilon^{k,1}_{\alpha,n}(r,0)=0,\ \upsilon^{k,1}_{\alpha,n}(r,\varphi_1(r))=\sigma^k_n(r)-\upsilon^{k,2}_{\alpha,n}(r,\varphi_1(r)). \tag{21}$$

Using the expressions (16), (14), the problems (10_α), (17) and (10_α), (21), respectively reduce to the Cauchy problem (10_0), (15) and the problem (10_0), (20), where $\sigma^k_{1n}(r)$ is a function now expressed through $\nu^k_n(r)$, $\sigma^k_n(r)$.

Thus, the problem (10_α), (12), also has a unique solution.

Therefore, the solutions of the problem (1), (2) are in the form

$$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n} r^{\frac{(1-m)}{2}} u^k_n(r,t)Y^k_{n,m}(\theta), \tag{22}$$

where $u_n^k(r,t)$ are found from (10_α), (11).

In a similar way, we find the solution to the problem (1), (3) in the form (22), where $u_n^k(r,t)$ are determined from (10_α), (12).

Given the restrictions on the given functions $\tau(r,\theta)$, $\nu(r,\theta)$, $\sigma(r,\theta)$, the lemmas and the formulas ([16])

$$\frac{d^m}{dz^m}P_\mu(z) = \frac{\Gamma(\mu+m+1)}{2^m\Gamma(\mu-m+1)}F\Big(1+m+\mu,\, m-\mu,\, m+1;\, \frac{1-z}{2}\Big),$$

$$\frac{\Gamma(z+\alpha)}{z+\beta} = z^{\alpha-\beta}[1+\frac{1}{2z}(\alpha-\beta(\alpha-\beta-1)+0(z^{-2}))],$$

as well as the estimates ([11])

$$|k_n| \le c_1 n^{m-2},\ \Big|\frac{\partial^q}{\partial\theta_j^q}Y_{n,m}^k(\theta)\Big| \le c_2 n^{\frac{m}{2}-1+q},\ j=\overline{1,m-1},\ q=0,1,\dots,$$

where $F(a,b,c,z)$ are hypergeometric function, α,β are arbitrary real numbers, as in [13], [12], we prove that the resulting solution (22) belongs to the class $C(\bar{D})\cap C^1(D\cup S)\cap C^2(D)$.

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Алдашев С.А. КӨП ӨЛШЕМДІ ОБЛЫСТА АЗЫНҒАН ГИПЕРБОЛАЛЫҚ ТЕҢДЕУГЕ ДИРИХЛЕ ЖӘНЕ ПУАНКАРЕ ЕСЕПТЕРІНІҢ ҚИСЫНДЫЛЫҒЫ

Дирихле есебінің жалпы гиперболалық теңдеулер үшін қисынды болмайтыны баршаға мәлім. Ертеректегі жұмыстарда бұл мәселе, негізінен, функционалдық талдау әдістерін пайдалана отырып зерттелген болатын, ол қолданбалар үшін ыңғайсыз болып табылады. Осы мақала азынған гиперболалық теңдеулер үшін Дирихле мен Пуанкаре есептері қисынды болатын көпөлшемді облысты айқындайды.

Кілттік сөздер. Қисындылық, көп өлшемді облыс, азынған теңдеулер, сфералық функциялар.

Алдашев С.А. КОРРЕКТНОСТЬ ЗАДАЧ ДИРИХЛЕ И ПУАНКАРЕ В МНОГОМЕРНОЙ ОБЛАСТИ ДЛЯ ВЫРОЖДАЮЩИХСЯ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ

Хорошо известно, что задача Дирихле некорректна для общих гиперболических уравнений. В ранних работах, в основном, изучалась эта проблема с использованием методов функционального анализа, что неудобно для приложений. Эта статья устанавливает многомерную область, в которой задачи Дирихле и Пуанкаре для вырожденных гиперболических уравнений являются корректными.

Ключевые слова. Корректность, многомерная область, вырождающиеся уравнения, сферические функции.

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Using full limit order book for price jump prediction

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Abstract. Institutional investors, especially high frequency traders, employ the order information contained in the Limit Order Book (LOB). The main purpose of the paper is to investigate how full information about the LOB can help in predicting the price jump. Normally, a full LOB contains total volumes of orders for hundreds of prices. Using the full information runs into the curse of dimensionality which manifests itself in multicollinearity, insignificant coefficients, inflated estimate variances and high computation time. Due to these problems, order volumes for prices that are distant from ask and bid prices are usually not used in prediction procedures. For this reason we call such information a silent crowd. Here we propose a summary measure of the silent crowd and quantify its influence on trade jump prediction. We use a realistically simulated LOB as a vehicle for experiments and logistic regression as the prediction tool. The full code in Matlab includes 18 blocks.

Keywords. Simulation, trade jump prediction, high frequency trading, logistic regression, limit order book.

1 Introduction

The advent of information technologies made possible the transition from quote-driven markets to order-driven trading platforms. On many stock exchanges, including NYSE, NASDAQ, and the London Stock Exchange, trade orders are submitted and executed electronically [1]. Outstanding orders are recorded in what is called a Limit Order Book (LOB). For a fee, clients can have access to either partial or full information contained in the LOB. High speed communications, fast computers and computer algorithms enabled high frequency trading, when orders are submitted every millisecond. Analysing the LOB and making predictions regarding possible market moves in real time is essential for participants of this market.

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One direction of research focuses on mathematical models of the LOB [2]–[6]. They provide kind of a common denominator for financial phenomena but are too judgmental in the sense that they typically impose restrictions which are hard to validate in practice [7], [8].

On the other hand, machine learning methods do not impose any a priori conditions and attempt to reveal the regularities that are in the data [9]–[12]. In particular, statistical methods are used to predict quantities that can be used profitably. The paper [13] presents a non-parametric model for trade sign inference. [14] uses logistic regression to predict occurrence of price jumps. [15], [16] employ support vector machines to capture the dynamics of price movements. [17] suggest a model that describes the evolution of the distribution of limit orders and whose estimates can be used in a regression. [18] analyze the contribution to price discovery of market and limit orders by high-frequency traders (HFTs) and non-HFTs. See the last paper for a valuable review and latest references.

The above references use real-world data. We work with a simulated LOB. The two approaches have different focuses.

The main value of real-world data is that it contains traces of investors' decisions, which are influenced by the shape of the LOB, among other things. The challenge is to infer about investors decisions and use that inference to successfully predict future price movements. This is complicated by many realities: different investors react differently to the market signals contained in the LOB, there are events outside the LOB influencing investors moves and the very invention of successful prediction mechanisms affects investors behaviour.

A simulated LOB should incorporate and exhibit the stylized facts of the real LOB. Different types of orders are posted in accordance with distributional patterns observed in practice, but other than that they are random and independent, at least in our implementation. There are no built-in behavioral assumptions. The simulated LOB is impartial, so to speak. It serves better the purpose of revealing relative importance of quantities contained in the LOB, as opposed to inferring about investors motivations. A real LOB is a snapshot of what has happened, while a simulated LOB can be produced as many times as needed and allows one to fine-tune model parameters to achieve the desired patterns.

In Section 2 we describe the standard features of limit order books. In Section 3 we detail the simulations. Section 4 presents the main results. Section 5 contains conclusions. The Matlab code is available on request.

2 Order types and LOB structure

In order-driven markets investors can submit three order types: limit orders, cancel orders and market orders. The minimum allowed price increment is called a tick. For simulation purposes the tick can be taken to be 1 without loss of generality.

A sell limit order is an order to sell a certain number of shares at a certain price (called ask) or higher. A buy limit order is an order to buy a certain number of shares at a certain price (called bid) or lower. If there is no offsetting order at the same price, a limit order is recorded in the LOB. Limit orders are executed against offsetting incoming orders in the order they (limit orders) were recorded. Limit orders have an expiration date, unless the investor specifies that the order is good until canceled. Order expiration dates are not seen in the LOB investors have access to. For modeling purposes all limit orders are considered as orders with no expiration date.

An investor can cancel his/her limit order (or its remaining part) at any time. In fact, most limit orders are canceled before their execution.

It is useful to imagine the LOB as consisting of two parts, with a vertical price axis. The upper part contains all sell orders, and the lower one contains all buy orders (more precisely, total volumes against each tick). The lowest sell price is called the best ask and the highest buy price is called the best bid. Because of opposite order matching the best ask is always higher than the best bid. The midprice is defined by $midprice = (best\ ask + best\ bid)/2$. The difference $best\ ask - best\ bid$ is called a spread. The prices and total volumes at the best ask and bid are called first level quotes, the prices and total volumes one tick away from the best ask and bid are called second level quotes and so on.

A market sell order is an order to sell a certain number of shares at the best available price, that is at the best bid. Similarly, a market buy order is an order to buy a certain number of shares at the best available price, that is at the best ask. When a market sell order arrives, the total volume at the best bid may be smaller than the market order size. In this case the market order consumes all of the volume at the best bid, the best bid moves down and the remaining part of the market order is executed against the limit orders at the new best bid. Some exchanges use a different rule: if, say, a sell market order size is larger than the outstanding volume at the best bid, the remaining part of the market order stays in the LOB as a sell limit order. The difference between the first case, when the market order may be executed at several prices, and the second one, when it may be partially executed and the remainder stays as a limit order at the best bid, is that in the first case the best bid moves down (and the spread increases), while in the second case it is the best ask that moves down. In the first case the downward move of the midprice is determined by the relative size of the market order and liquidity at the bid side. In the second case this downward move depends on the spread, and the midprice right after execution of the market order will be lower than the best bid right before the execution. The midprice is more stable under the first arrangement, which we adopt in our simulations. Stability of market prices is one of desirable features.

Market orders are executed immediately, so in case of a real LOB, one can know about their arrival and size only from a change in total volumes of limit orders at the best ask and bid. HFT's often place orders just to cancel them a moment later. There also can be errors in the way the LOB is recorded. This kind of problems do not arise with a simulated LOB. Experiments on a real stock exchange are costly and likely to disrupt its operations; in case of a change in rules governing an exchange, large and technologically advanced players will win at the expense of small investors.

All the information above the ask price characterizes the supply, whereas all the information below the bid price characterizes the demand side.

3 Simulation description

The task of modeling the LOB is complex because the impact of an order on the book depends on the state of the book. Therefore one cannot sum the incoming orders over a period of time and post the sum to the book. The orders have to be generated and posted immediately one by one. This requires a lot of calculation, only a small part of which can be made faster using parallel computing. We have not been able to use the CUDA (parallel computing language from NVIDIA^{TM}) because it can handle only specific types of code.

Application of logit requires measuring depths at equally spaced moments, and their number should be large enough. With short time intervals (on the order of several milliseconds) the LOB is too poor. Increasing the lengths of time intervals increases the complexity of calculations.

Following the empirical pattern [7], the distribution of orders is defined in such a way that the spread of limit orders is very large, $\pm 50\%$ of the midprice or more. On both sides of the midprice the distribution declines as a power law, up to 100 ticks from the midprice, and then falls to zero. Orders arrive independently at exponential rates.

Cancel order sizes are given as a fraction of the order depth.

The Matlab code consists of 18 programs. The first character in the program name indicates its level. The lowest-level programs start with A, the next-level programs start with B and so on. The level of a program is determined by the references contained in it. For example, the program C_AllOrdersTimesAndPrices.m may refer to levels A and B but not higher.

The function A_InDistr creates the initial distribution of orders.

The function A_OrderTimes generates a sequence of order placement times up a to given moment.

The function A_Revert just makes some code more convenient to read.

A_NormConstant realizes an empirically observed pattern in the distribution of orders from [7].

B_OrderTimesFixedPrice generates lists of limit, cancel and market order times (for all price ticks from 1 to $MaxPrice$).

B_AskAndBid finds the best ask (the lowest ask price at which the order size is not zero) and the best bid (the highest bid price at which the order size is not zero).

The function B_FindCum creates cumulative sums starting from the lower end of B_T. This is the most important part of the method. The silent crowd should be summarized in such a way that the prices close to the midprice should have larger weights. The weights should not be so heavy as to dampen the tail of the silent crowd.

B_LODensity generates sizes of limit orders in the range $(midprice-dist, midprice+dist)$, currently under the condition $MaxPrice = 4 * dist$.

C_AllOrdersTimesAndPrices puts into one $MaxPrice \times 3$ matrix M:

- all order times from lists of limit, cancel and market orders, unsorted (first column of M),
- order types (1 for Limit, 2 for Cancel, 3 for Market) (second column of M),
- and corresponding prices, numbered 1 through $MaxPrice$ (third column of M).

This is necessary to create a line of orders that later will be posted to the LOB.

Next there are three functions that post three types of orders: C_PostCancelOrder, C_PostLimitOrder and C_PostMarketOrder.

E_Inference_A_B collects statistical characteristics of the LOB. It is important that after about 50 orders the simulated LOB stabilizes and its two-humped shape corresponds to what is observed in practice.

Next we need to see how informative are the prices close to the midprice, compared to the informativeness of the silent crowd.

F_band_A_B finds bands of order sizes of width *band* (*band* up from ask in A_T and *band* down from bid in B_T).

F_weight_A_B prepares weights for averaging order sizes.

Finally, comparison is made between the contribution of the prices that are close to the midprice (in the band) and the contribution of the silent crowd.

3 Simulation results

The density of incoming limit orders is generated according to what is observed in practice. 200 ticks up and down from the initial midprice the density tapers off. After that, we set it to zero, see Figure 1.

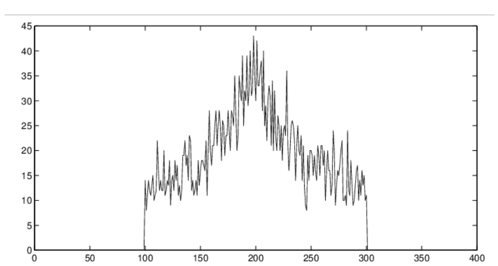

Figure 1 – Density of limit orders

As it was mentioned above, after about 50 orders the simulated LOB stabilizes. The midprice falls from the one defined in the initial distribution and afterwards is pretty stable (Figure 2).

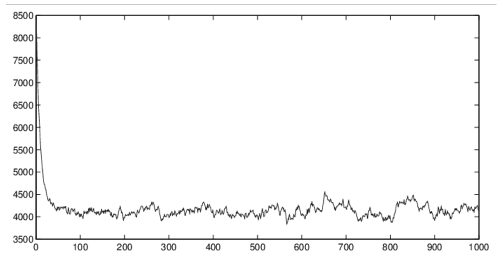

Figure 2 – Stabilization of the midprice

The standard deviation of the midprice also stabilizes (Figure 3).

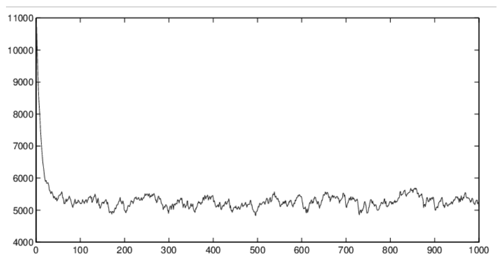

Figure 3 – Stabilization of the standard deviation of the midprice

Its two-humped shape corresponds to what is observed in practice, see Figure 4. This is a sign that relative order sizes have been chosen correctly (orders do not accumulate to infinity and are not consumed entirely by incoming buy orders).

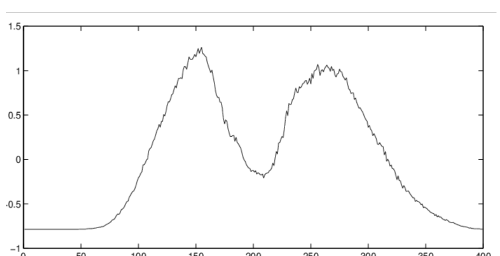

Figure 4 – Two-humped distribution of order sizes

Another sign that the LOB is being simulated correctly is that the order lists in the LOB behave pretty irregularly. See in Figure 5 the behavior of the first five ask sizes.

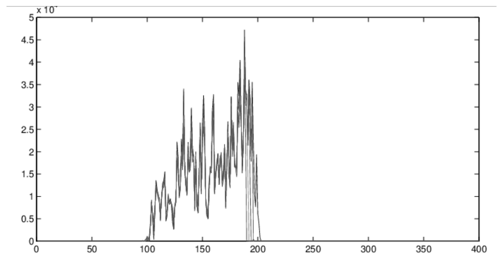

Figure 5 – Ask sizes at the first 5 prices

We use the logit model to predict the price jump. This is done with two sets of predictors: one includes only prices close to the midprice and the other additionally includes the index of the silent crowd. Specifically, let a_{it}, b_{it} denote the ask and bid sizes at time t, where $i = 1, 2, ...$ is the quote level. The price jump $j_t = \text{sgn}(midprice_{t+1} - midprice_t)$ is regressed on $a_{it}, b_{it},\ i = 1, ..., I$, in the first regression and on $a_{it}, b_{it},\ i = 1, ..., I, index_{It}$ in the second regression. Here $index_{It} = \sum_{i=I+1}^{MaxPrice} w_i(a_{it} + b_{it})$ is a weighted sum of the representatives of the silent crowd. We change $I = 1, 2, ..., 24$ to see how the two regressions compare.

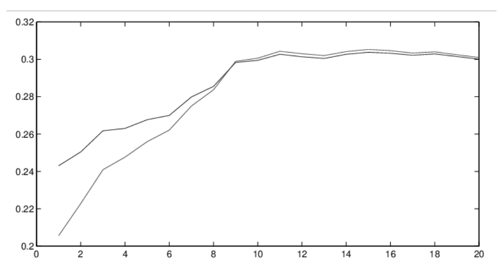

Figure 6 – R squared for two sets f predictors

From Figure 6 it is clear that the silent crowd significantly improves prediction if the number of prices included is low (less than or equal to five). Then its contribution falls and becomes negligible after the number of prices included exceeds eight.

5 Conclusions

We have been able to reproduce the stylized facts of the LOB. Those include the hump-shaped density distribution of order sizes. Using a simulated LOB allows one to achieve desirable distributional properties while preserving unpredictability and to test various forecasting techniques in different scenarios. In our simulations, the midprice stabilizes, which is not a feature observed in practice. It can be easily avoided by introducing a random-walk-like disturbances. However, to obtain distributions of order sizes one would have to detrend the resulting midprice series using moving averages. Because of the lagging nature of moving averages, this would introduce an additional error in estimation. However, we believe that reversion to the mean would at least partially mitigate this problem and the final result would not be very different from ours.

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Мыңбаев Қ. БАҒА ӨСУІН БОЛЖАУ ҮШІН ШЕКТЕУЛІ ТАПСЫРЫСТАРДЫҢ ТОЛЫҚ КІТАБЫН ПАЙДАЛАНУ

Институционалды инвесторлар, әсіресе жоғары жиілікті трейдерлер, Шектеулі Тапсырыстар Кітабындағы (LOB) тапсырыстар туралы ақпаратты пайдаланады. Мақаланың негізгі мақсаты – инвесторларға қызықты болатын әралуан оқиғаларды болжауға LOB туралы толық ақпараттың қалай көмектесе алатындығын зерттеу. Әдетте, LOB жүздеген бағалар бойынша тапсырыстың жалпы көлемдерін камтиды. Толық ақпаратты пайдалану өлшемділік қарғысына кездеседі, ол көпколлинеарлықтан, коэффициенттердің маңыздылығы төмен болуынан, бағалаулардың дисперсияларының шамадан тыс өсуінен және есептеулер уақытының ұзақтығынан байқалады. Осы мәселелерге байланысты, bid пен ask бағаларынан алшақтап кеткен бағалар бойынша тапсырыстар көлемі әдетте болжау рәсімдерінде қолданылмайды. Осы себепті, біз осындай ақпаратты үнсіз тобыр деп атаймыз. Мұнда біз үнсіз тобырдың жиынтық өлшемін ұсынамыз және оның сауда секірісін болжауға ықпалын сан жағынан бағалаймыз. Біз шынайы моделденген LOB-ты тәжірибелерге арналған құрал ретінде, ал логистикалық регрессияны болжау құралы ретінде пайдаланамыз. Matlab-тағы толық код 18 блоктан тұрады.

Кілттік сөздер. Моделдеу, сауда секірісін болжау, жоғары жиілікті сауда, логистикалық регрессия, шектеулі тапсырыстар кітабы.

Мынбаев К. ИСПОЛЬЗОВАНИЕ ПОЛНОЙ КНИГИ ПРЕДЕЛЬНЫХ ЗАКАЗОВ ДЛЯ ПРОГНОЗИРОВАНИЯ СКАЧКА ЦЕН

Институциональные инвесторы, особенно высокочастотные трейдеры, используют информацию о заказах, содержащуюся в Книге Лимитных Заказов (LOB). Основная цель статьи - изучить, как полная информация о LOB может помочь в прогнозировании различных событий, представляющих интерес для инвесторов. Обычно LOB содержит общие объемы заказов по сотням цен. Использование полной информации наталкивается на проклятие размерности, которое проявляется в мультиколлинеарности, низкой значимости коэффициентов, завышенных дисперсиях оценки и большом времени вычислений. Из-за этих проблем объемы заказов по ценам, далеким от цен bid и ask, обычно не используются в процедурах прогнозирования. По этой причине мы называем такую информацию молчаливой толпой. Здесь мы предлагаем сводную меру молчаливой толпы и количественно оцениваем ее влияние на прогнозирование торгового скачка. Мы используем реалистично смоделированную LOB в качестве средства для экспериментов и логистическую регрессию в качестве инструмента прогнозирования. Полный код в Matlab включает 18 блоков.

Ключевые слова. Моделирование, прогнозирование торгового скачка, высокочастотная торговля, логистическая регрессия, книга лимитных ордеров.

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Survivor sets in subshifts of finite type

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Abstract. In this article, we consider symbolic dynamics (X, T) with holes H and corresponding interval maps. Depending on location and size of the hole, the survivor set given by $\Omega_H(T) = \{x \in X : T^n(x) \notin H,$ for every $n \geq 0.\}$ maybe finite or infinite. Our goal is to find the sufficient condition for the survivor sets $\Omega_H(T)$ of general open subshifts of finite type to be uncountable and also to have positive entropy and Fractal dimension.

Keywords. Dynamical systems, fractal dimension, interval maps, survivor sets, open systems, irregular sets.

1 Introduction

The entire universe is full of changes. In fact, there is almost nothing that is stable and constant. The changes happen subject to various rules and physical laws. Applied mathematics tries to simplify real life phenomena and obtain a mathematical model that helps to understand the original system to certain extent. Dynamical systems theory deals with systems that vary in time subject to a given rule. In mathematical terms, let X be a set and let T be a self map $T : X \to X$, then (X, T) is called a (closed) *dynamical system.* For a given point $x \in X$, its *orbit* or *trajectory* is given by

$$O_T(x) := \{T^n(x) : n = 0, 1, 2, ...\},$$

where T^n is n-fold composition of T with itself, namely $T \circ T \circ \cdots \circ T$. Dynamical systems theory tries to understand orbits of points of X and their properties. For a given $x \in X$, some of the research questions of interest are

1. Is x (eventually) periodic, that is, is $O_T(x)$ finite?

2. Is $O_T(x)$ bounded or unbounded?

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3. Is $O_T(x)$ infinite, if so is it countable or uncountable?
4. Is $O_T(x)$ dense in X?
5. Is $O_T(x)$ uniformly distributed?
6. What is the fractal dimension of the closure of $O_T(x)$?

However, it is often difficult to understand the orbit of a single point, except in certain specific cases such as when (X, T) is minimal [1] or uniquely ergodic [2]. We note here that this difficulty is not due to the research gap in the literature, instead it is related to the presence of chaos in the system. In any case, to overcome this difficulty we usually study sets of orbits with certain characteristics and understanding topological aspects of sets is much easier.

While the closed dynamical systems have much studied, the open dynamical systems have many research directions awaiting to be explored. In general terms, let (X, T) be a dynamical system as before and H a subset of X, called a *hole*. (X, T, H) is called an *open dynamical system*. An orbit of x exists as far as its iteration under the map T does not fall into H, if it visits the H, then the point is said to have escaped, and the dynamics stops. Open dynamical systems are analogous to systems with terminal nodes in stochastic processes. A *survivor set* Ω_H is the set of all points whose orbits under T miss H. In other words,

$$\Omega_H(T) := \{x \in X : T^n(x) \notin H, \text{ for every } n \geq 0.\}. \tag{1}$$

Clearly, for any $x \in \Omega_H(T)$ we have $T(x) \in \Omega_H(T)$. Thus, $T : \Omega_H(T) \to \Omega_H(T)$ is well-defined and $(\Omega_H(T), T)$ gives rise to a new dynamical system for each given hole H. The questions listed above are still valid for the open dynamical systems and this list can be extended. In particular, for open dynamical systems, one of important research questions is if to investigate the 'size' of the survivor sets. Here, 'size' could be related to some topological notions or measure theoretic notions. To be more specific, for a given $H \subset X$, some of the further research questions are

1. Is $\Omega_H(T)$ finite?
2. Is $\Omega_H(T)$ infinite, if so is it countable or uncountable?
3. What is the fractal dimension of $\Omega_H(T)$?
4. What is the topological entropy of $\Omega_H(T)$?

Another interesting question is about the speed of orbits escape to the hole. In this paper, our primary objective is to address the above four questions for subshifts of finite type and corresponding k-transformations of unit interval. Before we state our findings, we introduce the related notions and review the literature on the related work.

1.1 k-transformations and symbolic space

In the rest of the paper, we fix a positive integer $k \geq 2$. Let $X = [0,1)$ and k-transformation $T_k : X \to X$ given by $T(x) = kx \mod 1$. More specifically,

$$T_k(x) = \begin{cases} kx, \text{ if } kx < 1, \\ kx - 1, \text{ if } 1 \leq kx < 2, \\ \dots \\ kx - (k-1), \text{ if } k-1 \leq kx < k. \end{cases}$$

Let $H = (a, b)$ be an open interval in X. The case of $k = 2$ was studied by Glendinning and Sidorov in [3], [4]. They prove,

Theorem. *The Hausdorff dimension of the survivor set $\Omega_H(T_k)$ is positive and in particular it is uncountable if $b-a < 1-2a_*$, where $a_* \approx 0.41245$ is the Thue-Morse constant. Moreover, if the hole H contains the midpoint 0.5, then $\Omega_H(T_k)$ has positive Hausdorff dimension if and only if $b < \chi(a)$, where $\chi(\cdot)$ is given in [4, Theorem 2.3].*

The above theorem is recently generalized to k-transformations for arbitrary $k \geq 2$ by Agarwal in [5].

The main idea of the proofs in the above mentioned results are to transfer the problem to symbolic space, namely the full shift on k letters.

We now define the shift space and more generally subshifts of finite type. To this end, let Λ_k denote the alphabet of k symbols, namely, $\Lambda = \{1, 2, \dots, k\}$ and

$$\Sigma_k := \{1, 2, \dots, k\}^{\mathbb{N}} = \{\omega = a_0 a_1 a_2 \cdots \mid a_i \in \{1, 2, \dots, k\}, i = 0, 1, 2, \dots\}$$

be the space of infinite words from the alphabet Λ_k equipped with the product topology. Let $A = (A_{ij})$ be a $k \times k$ transition matrix with entries consisting of 0's and 1's. We define Σ_A to be the subspace of $\{1, 2, \dots, k\}^{\mathbb{N}}$ given by

$$\Sigma_A := \{w = a_0 a_1 a_2 \cdots \in \{1, 2, \dots, k\}^{\mathbb{N}} \mid A_{a_i a_{i+1}} = 1, \forall i = 0, 1, 2, \dots\}.$$

If $A_{ij} = 0$, it means that the phrase ij is forbidden. *A subshift of finite type* (Σ_A, σ_k) is a dynamical system with a shift map $\sigma_k : \Sigma_A \to \Sigma_A$ given by

$$\sigma_k(a_0 a_1 a_2 \dots) = a_1 a_2 a_3 \dots \ .$$

We define a metric d_k on Σ_A given by

$$d_k(\omega_1, \omega_2) = k^{-t(\omega_1, \omega_2)} \text{ where } t(\omega_1, \omega_2) = \max\{n \geq 0 : x_i = y_i, 0 \leq i < n\},$$

where $\omega_1 = x_0 x_1 x_2 \dots$ and $\omega_2 = y_0 y_1 y_2 \dots$.

For A consisting of only 1's we get the full shift space Σ_k, and otherwise Σ_A is a proper subspace. Clearly, for any hole in $\{1, 2, \dots, k\}^{\mathbb{N}}$ if a point escapes to the hole under the full shift, then it obviously escapes under the shift of finite type for any transition matrix A.

To relate the k-transofrmation on interval to the symbolic space we define $\pi_k : \Sigma_k \to [0, 1)$ by

$$\pi_k(a_0 a_1 a_2 \dots) = \sum_{n=0}^{\infty} \frac{a_n}{k^{n+1}}.$$

We note that for any $\omega = a_0 a_1 a_2 \dots$ we have

$$T_k \circ \pi_k(\omega) = k \sum_{n=0}^{\infty} \frac{a_n}{k^{n+1}} \mod 1 = \sum_{n=0}^{\infty} \frac{a_{n+1}}{k^{n+1}} = \pi_k \circ \sigma_k(\omega),$$

so that the diagram (Figure 1) commutes.

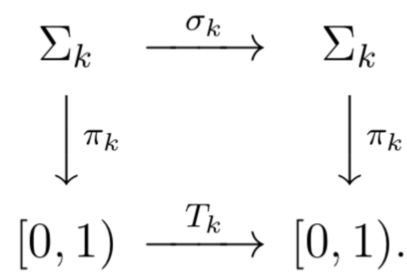

Figure 1 – Commuting diagram

1.2 Statement of results

As it is seen from the literature, the previous work related to the survivor set was mainly to understand the cardinally and fractal dimension of survivor set given the open dynamical system. We would like to extend these results with a slight twist. Mainly, we ask the following question.

Main question: For a given interval $H = (a, b)$, which intervals maps with hole H induce the survivor set Ω_H with positive fractal dimension?

We state our result in symbolic space and leave it to the interested reader to rephrase the theorem in terms of interval k-transformation using the above mentioned commuting diagram.

Theorem 1. *Let $k \geq 2$ be an integer, A a $k \times k$ transition matrix, and (Σ_A, σ_k) the induced subshift of finite type. Assume that there exist two distinct symbols $i, j \in \Lambda_k$ such that $A_{ii} = A_{jj} = A_{ij} = A_{ji} = 1$. For any $\ell \geq 0$ we define the subset of Σ_A*

$$S_\ell(i, j) := \{a_1 a_2 \dots \in \{i, j\}^{\mathbb{N}} : a_m = i, \implies a_{m+n} = j, n = 1, 2, \dots, \ell\}. \tag{2}$$

If the hole H in Σ_A is disjoint from S_ℓ for some ℓ, then the survivor set satisfies

- $\Omega_H(\sigma_k)$ *is uncountable,*
- $\Omega_H(\sigma_k)$ *has topological entropy at least* $\frac{\log 2}{\ell+1}$,
- $\Omega_H(\sigma_k)$ *has Hausdorff dimension at least* $\frac{\log 2}{\ell \log k}$.

2 Proof of main results

To prove Theorem 1, we show that the sets $S_\ell(i,j)$ have the desired properties. We recall that an infinite work is in $S_\ell(i,j)$ if between two symbols 'i', there should be symbols 'j' repeated at least ℓ times, see Figure 2.

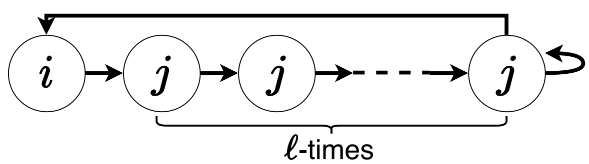

Figure 2 – Letter transitions in $S_\ell(i,j)$ illustrated

Proof of Theorem 1. From the assumption on transition matrix we see that the subshift of finite type $\{i,j\}^{\mathbb{N}}$ is embedded in Σ_A. Let ℓ be a positive integer such that $S_\ell(i,j)$ is disjoint from H, as $S_\ell(i,j)$ is σ_k-invariant and it is contained in $\{i,j\}^{\mathbb{N}}$ we conclude that $S_\ell(i,j) \subset \Sigma_H(\sigma_k)$. Thus, it suffices to show the desired properties for the dynamical system $(S_\ell(i,j), \sigma_k)$. The next three lemmas complete the proof. □

Lemma 1. *Let i, j be two distinct non-negative integers and ℓ a positive integer. Then, the set $S_\ell(a,b)$ is uncountable.*

Proof. The proof of the lemma is analogous to Cantor's diagonal argument from the set theory. For the sake of completeness we will reproduce the proof here.

Assume by contradiction that for some i, j, ℓ, we have $S_\ell(i,j)$ (at most) countable. Let x and y be two finite words of length $\ell+1$ and ℓ, respectively, given by

$$y = \underbrace{jjj\dots j}_{\ell-\text{times}} \text{ and } x = yi. \tag{3}$$

Let us take a set $U(x,y) \subset S_\ell(i,j)$ given by $U(x,y) = \{x,y\}^{\mathbb{N}}$. Then, $U(x,y)$ must be (at most) countable too, say $U(x,y) = \{\omega_1, \omega_2, \dots\}$. We now consider, x and y as letters. For any i,j by ω_i^j we denote the jth letter of ω_i and define a new word $\omega \in U(x,y)$ as follows, for any $j = 1,2,3\dots$, the jth letter is given by

$$\omega^j = \begin{cases} x & \text{if } w_j^j = y, \\ y & \text{if } w_j^j = x. \end{cases}$$

Since $\omega_j^j \neq w^j$ for any j we conclude that $\omega \neq \omega_j$, that is, $\omega \notin U(x,y)$ a contradiction. Thus, $U(x,y)$ is uncountable and so is $S_\ell(i,j)$. This concludes the proof of the lemma. □

As noted before the sets $S_\ell(a,b)$ are shift-invariant, that is, $\omega \in S_\ell(a,b)$ implies $\sigma_k(\omega) \in S_\ell(a,b)$. Hence, it makes sense to study the topological entropy of the set, which is another way to measure the complexity of a set. A set of positive entropy is necessarily uncountable, but the converse is false. Namely, the uncountability of sets $S_\ell(a,b)$ is not sufficient to deduce positive topological entropy. The next lemma exactly does this.

Lemma 2. *Let i,j be two distinct non-negative integers and ℓ a positive integer. The set $S_\ell(i,j)$ has positive topological entropy. In fact*

$$h(S_\ell(i,j),\sigma_k) \geq \frac{\log 2}{\ell+1}.$$

Proof. We note that the topological entropy can be computed using the number of periodic orbits. More specifically, let $N(n)$ denote the number of periodic orbits in $S_\ell(a,b)$ with period n. Then, the topological entropy $h(\sigma_k, S_\ell(a,b))$ satisfies

$$h(\sigma_k, S_\ell(a,b)) = \lim_{n\to\infty} \frac{\log N(n)}{n}. \tag{4}$$

To finish the proof, we estimate $N(n)$. As in (3) with slight modification we let x and y be two finite words of length $\ell+1$, given by

$$y = \underbrace{jjj\dots j}_{\ell+1-\text{times}} \text{ and } x = \underbrace{jjj\dots j}_{\ell-\text{times}} i.$$

Analogously, set $U(x,y) = \{x,y\}^{\mathbb{N}}$ which is a subset of $S_\ell(x,y)$. We will use the periodic orbits of $U(x,y)$ to estimate $N(n)$ from below. To this end, we note that there are at least two periodic orbits of length $\ell+1$, namely x^∞ and y^∞, so $N(\ell+1) \geq 2$. Inductively, we see that $N((\ell+1)n) \geq 2^n$. Let m be any large integer, we may find a positive integer n such that $(\ell+1)n \leq m < (\ell+1)(n+1)$. For any periodic orbit z^∞ in $U(x,y)$ of length $(\ell+1)n$ we may associate a periodic orbit ω of $U(x,y) \subset S_\ell(x,y)$ of length m letting

$$\omega = (z \quad \underbrace{jjj\dots j}_{(m-(\ell+1)n)-\text{times}} \quad)^\infty.$$

Thus, $N(m) \geq 2^n$ for $(\ell+1)n \leq m < (\ell+1)(n+1)$. Using the formula (4) we arrive at

$$h(\sigma_k, S_\ell(a,b)) = \lim_{m\to\infty} \frac{\log N(m)}{m} \geq \lim_{n\to\infty} \frac{\log 2^n}{(\ell+1)(n+1)} = \frac{\log 2}{\ell+1}.$$

This finishes the proof. □

Lemma 3. *Let i, j be two distinct non-negative integers and ℓ a positive integer. The set $S_\ell(i,j)$ has positive Hasudroff dimension. In fact*

$$\dim_H(S_\ell(i,j)) \geq \frac{\log 2}{\ell \log k}.$$

Proof. To prove the lemma, we make use of Mass Distribution Principle, see e.g. [6, § 4]. As in (3) with slight modification we let x and y be two finite words of length $\ell+1$, given by

$$y = \underbrace{jjj\ldots j}_{\ell+1-\text{times}} \text{ and } x = \underbrace{jjj\ldots j}_{\ell-\text{times}} i.$$

Analogously, set $U(x,y) = \{x,y\}^{\mathbb{N}}$ which is a subset of $S_\ell(x,y)$. Thus, it suffices to estimate the Hausdorff dimension of $U(x,y)$. To this end, we inductively define a probability measure μ on Σ_A supported on $U(x,y)$ as follows:

For any finite word z of length ℓ we call $C(z) = \{a_0a_1\dots \in \Sigma_k \mid a_0a_1\dots a_{\ell-1} = z\}$ *a cylinder set.*

We let $\mu(\Sigma_k) = 1$ and set $E_0 = \Sigma_k$. There are $k^{\ell+1}$ finite words of length $\ell+1$.

In the first step, we may split Σ_k into $k^{\ell+1}$ cylinder sets of equal length and pick two of them, namely, $E_1 = \{C(x), C(y)\}$ and let $\mu(C(x)) = \mu(C(y)) = 1/2$.

Next, in step 2, we split each $C(x)$ and $C(y)$ into $k^{\ell+1}$ cylinder sets of equal length and pick the four $E_2 = \{C(xx), C(xy), C(yx), C(yy)\}$ and set $\mu(C(xx)) = \mu(C(xy)) = \mu(C(yx)) = \mu(C(yy)) = 1/4$.

Inductively, in step n we further split each previously obtained 2^{n-1} cylinder sets into $k^{\ell+1}$ smaller cylinder sets. We note that exactly 2^n of these cylinder sets are defined using x, y and we place them into E_n and assign measure 2^{-n} to each.

This inductively defines a probability measure supported in $U(x,y)$.

We notice that $d(x,y) = k^{-\ell}$ and inductively one can show that for any two distinct cylinder sets $C, C' \in E_n$ one has $d(C,C') \geq k^{-\ell n}$. Now, let U be any subset of Σ_A with diameter $\delta > 0$. We may find non-negative integer n such that $k^{-(n+1)\ell} \leq \delta < k^{-n\ell}$, that is, clearly, on E_n there exists at most one cylinder set C that intersects with U. Hence, the measure of U satisfies $\mu(U) \leq \mu(C) = 2^{-n}$. That is,

$$\mu(U) \leq 2^{-n} = (k^{-n\ell})^{\log 2/\ell \log k} \leq (k^\ell \delta)^{\log 2/\ell \log k} \ll \delta^{\log 2/\ell \log k}.$$

Hence, it follows from the Mass Distribution Principle that the Hausdorff dimension of the support $U(x, y)$ satisfies

$$\dim_H(U(x, y)) \geq \frac{\log 2}{\ell \log k}.$$

Hence, $\dim_H(S_\ell(x, y)) \geq \dim_H(U(x, y)) \geq \log 2/\ell \log k$ which finishes the proof. □

3 Conclusion

We studied open dynamical systems in subshifts of finite type and obtained sufficient condition when the survivor set $\Omega_H(\sigma_k)$ is uncountable and estimated from below the Hausdorff dimension and topological entropy. We make no claim on the sharpness of our estimates. Indeed, it is an interesting question to find exact Hausdorff dimension and topological entropy of survivor sets. Another interesting question is to investigate the necessary condition for the survivor sets in the subshifts of finite type to be uncountable.

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Кадыров Ш., Айту Н. ЖЫЛЖЫМАЛАРДЫҢ АҚЫРЛЫ ТИПТЕРІНДЕГІ ТҮСПЕЙ ҚАЛҒАН ЭЛЕМЕНТТЕР ЖИЫНДАРЫ

Бұл мақалада біз берілген H аралықтары бар символдық динамиканы және сәйкесінше олардың $T : [0,1) \to [0,1)$ аралықтық бейнелеулерін қарастырамыз. Аралықтың орны мен өлшеміне байланысты берілген жиынның $\Omega_H(T) = \{x \in X : T^n(x) \notin H, n \geq 0.\}$ түспей қалған элементтер жиындары ақырлы немесе ақырсыз болуы мүмкін. Біздің мақсатымыз – жалпы ашық жылжымалардың ақырлы типіндегі $\Omega(H)$ түспей қалған элементтер жиындары есептелмейтін болуының, сонымен қатар оң энтропия мен фракталдық өлшемді иеленуінің жеткілікті шартын табу.

Кілттік сөздер. Динамикалык жүйелер, фракталдық өлшемдер, аралықтық функциялар, тірі қалған элементтер жиындары, ашық жүйелер, регуляр емес жиындар.

Кадыров Ш., Айту Н. МНОЖЕСТВА НЕПОПАДАЮЩИХ ЭЛЕМЕНТОВ В ПОДВИГАХ КОНЕЧНОГО ТИПА

В этой статье мы рассмотрим символическую динамику с H интервалами и соответствующие $T : [0,1) \to [0,1)$ интервальные отображения. В зависимости от местоположения и размера интервала, множества непопадающих элементов $\Omega_H(T) = \{x \in X : T^n(x) \notin H, n \geq 0.\}$ может быть конечным или бесконечным. Наша цель – найти достаточное условие того, чтобы множества непопадающих элементов $\Omega(H)$ в общих открытых подвигах конечного типа были неисчислимыми, а также имели положительную энтропию и фрактальную размерность.

Ключевые слова. Динамические системы, фрактальная размерность, интервальные функции, множества выживших элементов, открытые системы, нерегулярные множества.

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Some functional inequalities for convex functions via fractional integrals with non-singular kernel

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Abstract. The aim of this paper is to establish Hermite-Hadamard, Hermite-Hadamard-Fejér and Liu-Ngo-Huy type inequalities for fractional integral operators with Mittag-Leffler non-singular kernel.

Keywords. Hermite–Hadamard inequality, Hermite-Hadamard-Fejér inequality, Liu-Ngo-Huy inequality, new fractional integral operator, integral inequalities.

1 Introduction

The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature (see, e.g., [1], [2]). These inequalities state that if [3], [4] $u: I \to \mathbb{R}$ is a convex function on the interval $I \subset \mathbb{R}$ and $a, b \in I$ with $b > a$, then

$$u\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b u(x)dx \leq \frac{u(a)+u(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if u is concave. We note that Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $u : [a, b] \to \mathbb{R}$.

The most well-known inequalities related to the integral mean of a convex function u are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

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In [5], Fejér established the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Let $u : [a, b] \to \mathbb{R}$ be convex function. Then the inequality

$$u\left(\frac{a+b}{2}\right)\int_a^b v(x)dx \le \int_a^b u(x)v(x)dx \le \frac{u(a)+u(b)}{2}\int_a^b v(x)dx \tag{2}$$

holds; here $v : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$.

In [6], Liu, Ngo and Huy established the following results:

Let u and v be two positive continuous functions on $[a, b]$ and $u \le v$ on $[a, b]$. If $\frac{u(x)}{v(x)}$ is decreasing and u is increasing on $[a, b]$, then for any convex function ϕ; $\phi(0) = 0$. Then the inequality

$$\frac{\int_a^b u(s)ds}{\int_a^b v(s)ds} \ge \frac{\int_a^b \phi(u(s))ds}{\int_a^b \phi(v(s))ds} \tag{3}$$

holds.

Many generalizations and extensions of the Hermite-Hadamard, Hermite-Hadamard-Fejér, Liu-Ngo-Huy type inequalities were obtained for various classes of functions using fractional integrals; see [7]–[19] and references therein.

Definition 1. The function $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$u(\mu x + (1-\mu)y) \le \mu u(x) + (1-\mu)u(y)$$

for all $x, y \in [a, b]$ and $\mu \in [0, 1]$. We say that u is concave if $(-u)$ is convex.

In the following, we will give some necessary definitions and mathematical preliminaries of new fractional integral which are used further in this paper.

Definition 2. Let $f \in L_1(a, b)$. The fractional integrals $\mathcal{I}_a^\alpha$ and $\mathcal{I}_b^\alpha$ of order $\alpha \in (0, 1)$ are defined by

$$\mathcal{I}_a^\alpha u(x) = \frac{1}{\alpha}\int_a^x E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(x-s)^\alpha\right)u(s)ds,\ x > a,$$

and

$$\mathcal{I}_b^\alpha u(x) = \frac{1}{\alpha}\int_x^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(s-x)^\alpha\right)u(s)ds,\ x < b,$$

respectively. Here $E_{\alpha,1}(z)$ is a Mittag-Leffler function is defined as (see e.g. [20])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0. \tag{4}$$

If $\alpha = 1$, then

$$\lim_{\alpha \to 1} \mathcal{I}_a^{\alpha} u(x) = \int_a^x u(s)ds, \ \lim_{\alpha \to 1} \mathcal{I}_b^{\alpha} u(x) = \int_x^b u(s)ds.$$

Therefore the operators $\mathcal{I}_a^{\alpha}$ and $\mathcal{I}_b^{\alpha}$ are called fractional integrals of order α.

The aim of this paper is to establish some functional inequalities for the above new fractional integral operators with exponential kernel. We henceforth denote $\mathcal{A} = \frac{1-\alpha}{\alpha}(b - a)^{\alpha}$.

2 Hermite-Hadamard type inequality

Theorem 1. *Let $u : [a,b] \to \mathbb{R}$ be an integrable function on $[a,b]$, i.e. $u \in L_1(a,b)$. If u is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$u\left(\frac{a+b}{2}\right) \le \frac{\alpha}{2(b-a)E_{\alpha,2}(-\mathcal{A})}\left[\mathcal{I}_a^{\alpha} u(b) + \mathcal{I}_b^{\alpha} u(a)\right] \le \frac{u(a)+u(b)}{2}. \tag{5}$$

Proof. Since u is a convex function on $[a,b]$, we get for x and y from $[a,b]$ with $\mu = \frac{1}{2}$

$$u\left(\frac{x+y}{2}\right) \le \frac{u(x)+u(y)}{2}, \tag{6}$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2u\left(\frac{a+b}{2}\right) \le u(ta + (1-t)b) + u((1-t)a + tb). \tag{7}$$

Multiplying both sides of (7) by $E_{\alpha,1}(-\mathcal{A}t^{\alpha})$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$2E_{\alpha,2}(-\mathcal{A})\, u\left(\frac{a+b}{2}\right)$$

$$\le \int_0^1 E_{\alpha,1}(-\mathcal{A}t^{\alpha})\left[u(ta + (1-t)b) + u((1-t)a + tb)\right] dt$$

$$= \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u(ta+(1-t)b)dt$$

$$+ \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u((1-t)a+tb)dt$$

$$= \frac{1}{b-a}\int_a^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(b-s)^{\alpha}\right) u(s)ds$$

$$+\frac{1}{b-a}\int_a^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(s-a)^{\alpha}\right) u(s)ds$$

$$= \frac{\alpha}{b-a}\left[\mathcal{I}_a^{\alpha}u(b)+\mathcal{I}_b^{\alpha}u(a)\right].$$

As a result, we obtain

$$2E_{\alpha,2}\left(-\mathcal{A}\right) u\left(\frac{a+b}{2}\right) \leq \frac{\alpha}{b-a}\left[\mathcal{I}_a^{\alpha}u(b)+\mathcal{I}_b^{\alpha}u(a)\right].$$

The first inequality of (5) is proved.

For the proof of the second inequality in (5) we first note that if u is a convex function, then, for $t \in [0,1]$, it yields

$$u(ta+(1-t)b) \leq tu(a)+(1-t)u(b)$$

and

$$u((1-t)a+tb) \leq (1-t)u(a)+tu(b).$$

By adding these inequalities we get

$$u(ta+(1-t)b)+u((1-t)a+tb) \leq u(a)+u(b). \tag{8}$$

Then multiplying both sides of (8) by $E_{\alpha,1}\left(-\mathcal{A}t\right)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$E_{\alpha,2}\left(-\mathcal{A}\right)[u(a)+u(b)] \geq \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u(ta+(1-t)b)dt$$

$$+ \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u((1-t)a+tb)dt,$$

i.e.

$$\frac{\alpha}{b-a}\left[\mathcal{I}_a^\alpha u(b)+\mathcal{I}_b^\alpha u(a)\right] \le E_{\alpha,2}\left(-\mathcal{A}\right)\left[u(a)+u(b)\right],$$

and the second inequality in (5) is proved. The proof of Theorem 1 is completed. □

Corollary 1. *Let $u:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $u\in L_1(a,b)$. If u is a concave function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$u\left(\frac{a+b}{2}\right) \ge \frac{\alpha}{2(b-a)E_{\alpha,2}\left(-\mathcal{A}\right)}\left[\mathcal{I}_a^\alpha u(b)+\mathcal{I}_b^\alpha u(a)\right] \ge \frac{u(a)+u(b)}{2}.$$

Remark 1. *For $\alpha\to 1$, we get*

$$\lim_{\alpha\to 1}\frac{\alpha}{2(b-a)E_{\alpha,2}\left(-\mathcal{A}\right)} = \frac{1}{2(b-a)}.$$

Then the under assumptations of Theorem 1 with $\alpha=1$, we have Hermite-Hadamard inequality of (1).

3 Hermite-Hadamard-Fejér type inequality

Theorem 2. *Let $u:[a,b]\to\mathbb{R}$ be convex and integrable function with $a<b$. If $v:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\frac{a+b}{2}$, i.e. $v(a+b-x)=v(x)$, then the following inequalities hold*

$$u\left(\frac{a+b}{2}\right)\left[\mathcal{I}_a^\alpha v(b)+\mathcal{I}_b^\alpha v(a)\right]$$

$$\le \left[\mathcal{I}_a^\alpha\left(uv\right)(b)+\mathcal{I}_b^\alpha\left(uv\right)(a)\right] \le \frac{u(a)+u(b)}{2}\left[\mathcal{I}_a^\alpha v(b)+\mathcal{I}_b^\alpha v(a)\right]. \tag{9}$$

Proof. Since u is a convex function on $[a,b]$, we have for all $t\in[0;1]$ the inequality (7). Multiplying both sides of (7) by

$$E_{\alpha,1}\left(-\mathcal{A}t^\alpha\right)v\left((1-t)a+tb\right), \tag{10}$$

then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$2u\left(\frac{a+b}{2}\right)\int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^\alpha\right)v\left((1-t)a+tb\right)dt$$

$$\le \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^\alpha\right)u\left(ta+(1-t)b\right)v\left((1-t)a+tb\right)dt$$

$$+\int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u\left((1-t)a+tb\right) v\left((1-t)a+tb\right) dt$$

$$=\frac{1}{b-a}\int_a^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(s-a)^{\alpha}\right) u\left(a+b-s\right) v(s) ds$$

$$+\frac{1}{b-a}\int_a^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(s-a)^{\alpha}\right) u(s)v(s) ds$$

$$=\frac{1}{b-a}\int_a^b E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(b-s)^{\alpha}\right) u(s)v\left(a+b-s\right) ds$$

$$+\frac{\alpha}{b-a}\mathcal{I}_b^{\alpha}\left[u(a)v(a)\right]=\frac{\alpha}{b-a}\left[\mathcal{I}_a^{\alpha}\left[u(a)v(a)\right]+\mathcal{I}_b^{\alpha}\left[u(a)v(a)\right]\right],$$

i.e.

$$2u\left(\frac{a+b}{2}\right)\int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) v\left((1-t)a+tb\right) dt$$

$$\leq \frac{\alpha}{b-a}\left[\mathcal{I}_a^{\alpha}\left[u(a)v(a)\right]+\mathcal{I}_b^{\alpha}\left[u(a)v(a)\right]\right].$$

Since v is symmetric with respect to $\frac{a+b}{2}$, then the following equalities hold

$$\mathcal{I}_a^{\alpha}v(b)=\mathcal{I}_b^{\alpha}v(a)=\frac{1}{2}\left[\mathcal{I}_a^{\alpha}v(b)+\mathcal{I}_b^{\alpha}v(a)\right].$$

Therefore, we have

$$u\left(\frac{a+b}{2}\right)\left[\mathcal{I}_a^{\alpha}v(b)+\mathcal{I}_b^{\alpha}v(a)\right]\leq \mathcal{I}_a^{\alpha}\left[v\left(b\right)u(b)\right]+\mathcal{I}_b^{\alpha}\left[v\left(a\right)u(a)\right]$$

and the first inequality of Theorem 2 is proved.

For the proof of the second inequality in (9) we first note that if u is a convex function, then, for all $t\in[0;1]$, it yields the inequality (8). Then multiplying both sides of (7) by (10) and integrating the resulting inequality with respect to t over $[0;1]$, we get

$$\int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u\left(ta+(1-t)b\right) v\left((1-t)a+tb\right) dt$$

$$+\int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) u\left((1-t)a+tb\right) v\left((1-t)a+tb\right) dt$$

$$\leq [u(a)+u(b)] \int_0^1 E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right) v\left((1-t)a+tb\right) dt.$$

As a result, we obtain

$$\mathcal{I}_a^{\alpha}\left[v\left(b\right)u(b)\right]+\mathcal{I}_b^{\alpha}\left[v\left(a\right)u(a)\right] \leq \frac{u(a)+u(b)}{2}\left[\mathcal{I}_a^{\alpha}v(b)+\mathcal{I}_b^{\alpha}v(a)\right].$$

Theorem 2 is proved. □

Corollary 2. *Let $u : [a, b] \to \mathbb{R}$ be concave and integrable function with $a < b$. If $v : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, i.e. $v(a+b-x) = v(x)$, then the following inequalities hold*

$$u\left(\frac{a+b}{2}\right)\left[\mathcal{I}_a^{\alpha}v(b)+\mathcal{I}_b^{\alpha}v(a)\right] \geq \left[\mathcal{I}_a^{\alpha}\left(uv\right)(b)+\mathcal{I}_b^{\alpha}\left(uv\right)(a)\right]$$

$$\geq \frac{u(a)+u(b)}{2}\left[\mathcal{I}_a^{\alpha}v(b)+\mathcal{I}_b^{\alpha}v(a)\right].$$

Remark 2. *Under assumptations of Theorem 2 with $\alpha = 1$, we have Hermite-Hadamard-Fejér inequality of* (2)*.*

4 Liu-Ngo-Huy inequality

Theorem 3. *Let u and v be two positive continuous functions on $[a, b]$ and $u \leq v$ on $[a, b]$. If $\dfrac{u(x)}{v(x)}$ is decreasing and u is increasing on $[a, b]$, then for any convex function ϕ; $\phi(0) = 0$, the inequality*

$$\frac{\mathcal{I}_a^{\alpha}u(b)}{\mathcal{I}_a^{\alpha}v(b)} \geq \frac{\mathcal{I}_a^{\alpha}\left(\phi(u(b))\right)}{\mathcal{I}_a^{\alpha}\left(\phi(v(b))\right)} \tag{11}$$

is valid.

Proof. The function ϕ is convex and $\phi(0) = 0$. Then the function $\dfrac{\phi(x)}{x}$ is increasing. Since u is increasing, then $\dfrac{\phi(u)}{u}$ is also increasing. This and the fact that $\dfrac{u(x)}{v(x)}$ is decreasing yield

$$\frac{\phi(u(s))}{u(s)}\frac{u(t)}{v(t)}+\frac{\phi(u(t))}{u(t)}\frac{u(s)}{v(s)}-\frac{\phi(u(t))}{u(t)}\frac{u(t)}{v(t)}-\frac{\phi(u(s))}{u(s)}\frac{u(s)}{v(s)} \geq 0 \tag{12}$$

for all $s, t \in [a, b]$.

Hence, we can write

$$\frac{\phi(u(s))}{u(s)}u(t)v(s) + \frac{\phi(u(t))}{u(t)}u(s)v(t)$$
$$-\frac{\phi(u(t))}{u(t)}u(t)v(s) - \frac{\phi(u(s))}{u(s)}u(s)v(t) \geq 0. \tag{13}$$

Now, multiplying both sides of (12) by $\frac{1}{\alpha}E_{\alpha,1}\left(-\frac{1-\alpha}{\alpha}(x-s)^{\alpha}\right)$, then integrating the resulting inequality with respect to s over $[a, b]$, we get

$$u(t)\mathcal{I}_a^{\alpha}\left(\frac{\phi(u(b))}{u(b)}v(b)\right) + \frac{\phi(u(t))}{u(t)}v(t)\mathcal{I}_a^{\alpha}u(b)$$
$$-\frac{\phi(u(t))}{u(t)}u(t)\mathcal{I}_a^{\alpha}v(b) - v(t)\mathcal{I}_a^{\alpha}\left(\frac{\phi(u(b))}{u(b)}u(b)\right) \geq 0. \tag{14}$$

With the same argument as before, we obtain

$$\mathcal{I}_a^{\alpha}u(b)\mathcal{I}_a^{\alpha}\left(\frac{\phi(u(b))}{u(b)}v(b)\right) \geq \mathcal{I}_a^{\alpha}v(b)\mathcal{I}_a^{\alpha}\left(\phi(u(b))\right). \tag{15}$$

Since $u \leq v$ on $[a, b]$, then using the fact that the function $\frac{\phi(x)}{x}$ is increasing, we can write

$$\frac{\phi(u(s))}{u(s)} \leq \frac{\phi(v(s))}{v(s)},\ s \in [a, b]. \tag{16}$$

By virtue of (16), from (15) we obtain (11). □

Remark 3. *Under assumptations of Theorem 3 with $\alpha = 1$, we have Liu-Ngo-Huy inequality of* (3).

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Данияр Дүкенбай, Берікбол Т. Төребек БӨЛШЕК РЕТТІ БЕЙСИНГУЛЯРЛЫ ӨЗЕГІ БАР ИНТЕГРАЛДЫ ҚАМТИТЫН ДӨҢЕС ФУНКЦИЯЛАР ҮШІН КЕЙБІР ФУНКЦИОНАЛДЫҚ ТЕҢСІЗДІКТЕР

Мақаланың негізгі мақсаты – бөлшек ретті Миттаг-Леффлер бейсингулярлы өзегі бар интегралдық операторлар үшін Эрмит-Адамар, Эрмит-Адамар-Фейер және Лю-Нго-Хай тектес теңсіздіктерді алу.

Кілттік сөздер. Эрмит-Адамар теңсіздігі, Эрмит-Адамар-Фейер теңсіздігі, Лю-Нго-Хай теңсіздігі, жаңа бөлшек ретті интегралдық оператор, интегралдық теңсіздіктер.

Данияр Дукенбай, Берикбол Т. Торебек НЕКОТОРЫЕ ФУНКЦИОНАЛЬНЫЕ НЕРАВЕНСТВА ДЛЯ ВЫПУКЛЫХ ФУНКЦИЙ, СОДЕРЖАЩИЕ ДРОБНЫЕ ИНТЕГРАЛЫ С НЕСИНГУЛЯРНЫМИ ЯДРАМИ

Цель данной статьи – установить неравенства типа Эрмита-Адамара, Эрмита-Адамара-Фейера и Лю-Нго-Хая для дробных интегральных операторов с несингулярным ядром Миттаг-Леффлера.

Ключевые слова. Неравенство Эрмита-Адамара, неравенство Эрмита-Адамара-Фейера, неравенство Лю-Нго-Хая, новый дробный интегральный оператор, интегральные неравенства.

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On the completeness of root vectors of regular boundary value problems for one-dimensional differential operators

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Abstract. In this paper, we constructed a Volterra invariant subspace of regular boundary value problems for one-dimensional differential operators. A Volterra criterion of the extension of a first-order operator was proved in [1]. We using the Keldysh's theorem, proved that there exist correct restrictions such that their eigenvectors are incomplete in $L_2(0,1)$, and also there exist correct restrictions such that their eigenvectors are complete in $L_2(0,1)$. The description of Volterra invariant subspaces is given.

Keywords. Volterra invariant subspace, one-dimensional differential operator, extension, correct restriction.

Let us consider the differential equation

$$Lu\left(x\right) = u'\left(x\right) + q\left(x\right)u\left(x\right) = f\left(x\right), \quad x \in (0,1). \tag{1}$$

We will find a general solution of equation (1) in $W_2^1\left(0,1\right)$, which continuously depends on $f\left(x\right) \in L_2\left(0,1\right)$. It is well-known that the general solution of the homogeneous equation (1)

$$Lu_0\left(x\right) = u_0'\left(x\right) + q\left(x\right)u_0\left(x\right) = 0, \quad x \in (0,1), \tag{2}$$

can be represented as

$$u_0\left(x\right) = c \cdot e^{-\int\limits_0^x q(\xi)d\xi},$$

where $c = const$, $q(x) \in C[0,1]$.

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Cauchy problem: Find a solution to equation (1) satisfying the homogeneous initial condition

$$u(0) = 0. \tag{3}$$

By calculation, we verify that the unique solution of Cauchy problem (1), (3) is given by the formula

$$u_K(x) = L_K^{-1} f(x) = \int_0^x e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi.$$

Therefore, the general solution of equation (1) is representable as

$$u(x) = L_K^{-1} f(x) + u_0(x),$$

i.e.

$$u(x) = L_K^{-1} f(x) + u_0(x) = \int_0^x e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + c \cdot e^{-\int_0^x q(\xi)d\xi}.$$

Since the solution of equation (1) is continuously dependent on $f(x) \in L_2(0,1)$, then the constant c should continuously depend on $f(x) \in L_2(0,1)$, that is c should be a linear continuous functional of f, i.e. $c = c(f)$. According to the Riesz theorem on the representation of a linear functional in the Hilbert space $L_2(0,1)$ there exists a unique element $v(x) \in L_2(0,1)$ such that

$$c(f) = \int_0^1 f(x)\, v(x)\, dx.$$

Thus, the general solution of equation (1) is continuously dependent on $f(x) \in L_2(0,1)$ and can be represented as

$$u(x) = L_K^{-1} f(x) + u_0(x)$$

$$= \int_0^x e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + \int_0^1 f(\xi)\, v(\xi)\, d\xi \cdot e^{-\int_0^x q(\xi)d\xi}$$

$$= \int_0^1 \theta(x-\xi)\, e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + \int_0^1 f(\xi)\, v(\xi)\, d\xi \cdot e^{-\int_0^x q(\xi)d\xi},$$

where $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We denote by L_0 the closure in $L_2(0,1)$ of the differential operator (1) on a subset of the function $u \in \overset{0}{W}{}_2^1(0,1)$, and by L_0^+ the closure in $L_2(0,1)$ of the differential operator given by

$$L^+ v = -\frac{d}{dx} v(x) + \bar{q}(x) v(x) = g(x),$$

on a subset $v \in \overset{0}{W}{}_2^1(0,1)$. By L_0^* and $\left(L_0^+\right)^*$ denote the adjoint operators to L_0 and L_0^+, respectively. An operator L is called a correct restriction of maximal operator L_0^+ if there exists a bounded inverse operator L^{-1}, on all of $L_2(0,1)$ and $L \subset L_0^+$. We say that the operator L is a boundary extension of L_0, if L is simultaneously the correct restriction of the maximal operator L_0^+ and the correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset L_0^+$.

The theory of correct restrictions of the maximal operator and correct extensions of the minimal operator for the case of ordinary differential operators was first developed by M. Otelbaev [2], [3], and it was further developed in the works of his disciples.

Otelbaev's theorem [2]. *Let L_0^+ be a maximal linear operator in $L_2(0,1)$, L_K any known correct restriction of the operator L_0^+ and K an arbitrary linear bounded operator in $L_2(0,1)$, satisfying the following condition $R(K) \subset Ker(L_0^+)$. Then the operator L^{-1}, defined by the formula $L^{-1} f = L_K^{-1} f + K f$ describes the inverse to all possible correct restrictions of L maximal operator L_0^+, i.e. $L \subset L_0^+$.*

The closest topics to what in this article can be found from [1], [5]–[11]. In the work [1], correct restrictions and some of their spectral properties for an ordinary differential operator of the first order were considered. As for ordinary differential operators, in [5]–[10], the spectral properties of correct restrictions and extensions were considered for the partial differential operators. In [11], the criterion of Volterra property of well-posed boundary value problems for the Sturm-Liouville operator was proved. This criterion is the condition of symmetry of the least coefficient of the equation.

In this paper, we study the question of Volterra invariant subspaces for the correct restrictions of an ordinary differential operator of the first order.

The correct restriction of the operator $\left(L_0^+\right)^*$ is given by the integral form:

$$L^{-1} f = \int_0^1 \theta(x-\xi)\, e^{-\int_\xi^x q(\xi_1) d\xi_1} \cdot f(\xi)\, d\xi + u_0(x) \cdot \int_0^1 f(\xi)\, v(\xi)\, d\xi, \tag{4}$$

where $u_0(x) = e^{-\int_0^x q(\xi) d\xi}$. We will find an adjoint operator to the correct restriction

$$\left(L^{-1} f(x), g(x)\right) = \int_0^1 f(x) \cdot \left(L^{-1}\right)^* g(x)\, dx$$

$$= \int_0^1 f(x)\left[\int_0^1 \theta(x-\xi)\, e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot g(\xi)\, d\xi + \int_0^1 g(\xi)\, v(\xi)\, d\xi \cdot e^{-\int_0^x q(\xi_1)d\xi_1}\right] dx$$

$$= \int_0^1 f(x) \int_0^1 \theta(x-\xi)\, e^{-\int_\xi^x q(\xi_1)d\xi_1} \cdot g(\xi)\, d\xi dx + \int_0^1 f(x) \int_0^1 g(\xi)\, v(\xi)\, d\xi \cdot u_0(x)\, dx,$$

i.e.

$$\left(L^{-1}\right)^* g(x) = \int_0^1 \theta(\xi - x)\, e^{\int_\xi^x \bar{q}(\xi_1)d\xi_1} \cdot g(\xi)\, d\xi + v_0(x) \cdot \int_0^1 g(\xi)\, u_0(\xi)\, d\xi. \qquad (5)$$

In work [12], Kalmenov T.Sh., Otelbaev M. proved that a correct restriction is a regular boundary extension if and only if $L^* \subset L_0^*$, that is, $\left(L^{-1}\right)^* g$ is a solution of the equation $L^* v = g(x)$. From the representation (5) we find this only if the function $v(x) \equiv v_0(x)$ is a solution of the adjoint homogeneous equation

$$\left(-\frac{d}{dx} + \bar{q}(x)\right) v_0(x) = 0.$$

This proves that:

Theorem 1. *For arbitrary $v(x) \in L_2(0,1)$ formula (4) defines the correct restriction of $\left(L_0^+\right)^*$, and for $v \equiv v_0(x) \in \ker(L_0)^*$ the representation (4) gives elements of regular boundary extension.*

It should be noted that the arbitrary function $v(x) \in L_2(0,1)$ in formula (4) determines an arbitrary correct restriction of $\left(L_0^+\right)^*$ and a regular boundary extension in the case of $v \equiv v_0(x) \in \ker(L_0)^*$.

To determine a boundary condition of regular boundary extension, we assume that $v_0(\xi) \in \ker\left(L_0^+\right)^*$, i.e. $L_0^+ v_0(\xi) = -\frac{d}{d\xi} v_0(\xi) + q(\xi)\, v_0(\xi) = 0$. The general solution of the homogeneous equation can be represented as

$$v_0(x) = c_0 \cdot e^{\int_0^x q(\xi_1)d\xi_1},$$

where $c_0 = const$. Substituting in equality (4) $f(\xi) = \frac{d}{d\xi} u(\xi) + q(\xi)\, u(\xi)$ for $x = 0$, we notice that

$$u(0) = \int_0^1 \left(\frac{d}{d\xi} + q(\xi)\right) u(\xi)\, v_0(\xi)\, d\xi$$

$$= u(\xi)\, v_0(\xi)|_0^1 - \int_0^1 u(\xi)\, \frac{d}{d\xi} v_0(\xi)\, d\xi + \int_0^1 u(\xi)\, q(\xi)\, v_0(\xi)\, d\xi$$

$$= u(\xi)\, v_0(\xi)|_0^1 + \int\limits_0^1 u(\xi)\left(-\frac{d}{d\xi} + q(\xi)\right) v_0(\xi)\, d\xi$$

$$= u(1)\, v_0(1) - u(0)\, v_0(0) - \int\limits_0^1 u(\xi)\cdot 0\ d\xi$$

$$= u(1)\, v_0(1) - u(0)\, v_0(0).$$

Since $v_0(0) = c_0$, $v_0(1) = c_0 \cdot e^{\int\limits_0^1 q(\xi_1)d\xi_1}$, then the general regular boundary condition can be rewritten as

$$(1 + c_0)\cdot u(0) = c_0 \cdot e^{\int\limits_0^1 q(\xi_1)d\xi_1} \cdot u(1), \tag{6}$$

where $c_0 = const$.

Theorem 2. *The regular boundary condition for equation* (1) *is given by formula* (6)*, where* c_0 *is an arbitrary constant.*

Now we will solve the spectral problem with the regular boundary condition (6)

$$Lu = \frac{d}{dx}u(x) + q(x)\, u = \lambda u(x). \tag{7}$$

Remark. *For* $c_0 = 0$ *or* $1 + c_0 = 0$*, we obtain Cauchy boundary conditions* $u(0) = 0$ *or* $u(1) = 0$*. The Cauchy problem is Volterra, i.e. does not have a spectrum. The general solution of equation* (7) *is a function*

$$u(x) = c \cdot e^{-\int\limits_0^x q(\xi)d\xi + \lambda x}. \tag{8}$$

Then satisfying $u(x)$ boundary condition (6) and since

$$u(1) = c \cdot e^{-\int\limits_0^1 q(\xi)d\xi + \lambda},$$

assuming that $c_0 \cdot (1 + c_0) \neq 0$, we will get

$$u(1) = \frac{1}{\beta}\cdot e^{-\int\limits_0^1 q(\xi)d\xi} \cdot c = c \cdot e^{-\int\limits_0^1 q(\xi)d\xi + \lambda},$$

$$e^{\lambda} = \frac{1}{\beta},$$

$$\lambda_n = -\ln\beta + 2\pi ni,\ n \in Z, \tag{9}$$

where $\beta = \frac{c_0}{1+c_0} \neq 0$. The eigenfunction corresponding to the eigenvalues of $\lambda_n = -\ln\beta + 2\pi ni$ is

$$u_n(x) = e^{-\int\limits_0^x q(\xi)d\xi + (-\ln\beta + 2\pi ni)x}. \tag{10}$$

Obviously, the set of $\{u_n(x)\}$ forms a Riesz basis in $L_2(0,1)$ which proves the following:

Theorem 3. *The eigenvalue of the regular boundary-value problem* (7), (6) *is given by formula* (9), *and the eigenfunction defined by formula* (10) *form the Riesz basis in* $L_2(0,1)$.

Now we are considering the spectral questions of correct restrictions. In this case, assume that the function $v(x)$ determining the correct restriction by formula (4) does not belong to $v(x) \notin \ker L_0^*$.

Let $v(x) \in W_2^1(0,1)$ and $v(x) \equiv 0$ when $x \in (\alpha, 1)$, where $0 < \alpha < 1$. Then we transform formula (4) to the form

$$L^{-1}f = \int\limits_0^1 \theta(x-\xi)\, e^{-\int\limits_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + u_0(x) \cdot \int\limits_0^\alpha f(\xi)\, v(\xi)\, d\xi. \tag{11}$$

As $f(\xi) = \frac{d}{d\xi}u(\xi) + q(\xi)\, u(\xi)$, integrating by parts (11) we get

$$u(x) = L^{-1}f = \int\limits_0^x e^{-\int\limits_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + u_0(x) \cdot \int\limits_0^\alpha \left(\frac{d}{d\xi} + q(\xi)\right) u(\xi)\, v(\xi)\, d\xi$$

$$= \int\limits_0^x e^{-\int\limits_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\, d\xi + u_0(x) \left[u(\alpha)\, v(\alpha) - u(0)\, v(0) - \int\limits_0^\alpha u(\xi) \left(\frac{d}{d\xi} - q(\xi)\right) v(\xi)\, d\xi\right].$$

When $x = 0$, from (23) it follows

$$u(0) = u(\alpha)\, v(\alpha) - u(0)\, v(0) + \int\limits_0^\alpha u(\xi) \left(-\frac{d}{d\xi} + q(\xi)\right) v(\xi) d\xi.$$

Consider the following spectral problem

$$Lu = \left(\frac{d}{dx} + q(x)\right) u = \lambda u, \tag{12}$$

$$u(0) = u(\alpha)v(\alpha) - u(0)v(0) + \int\limits_0^\alpha u(\xi) \left(-\frac{d}{d\xi} + q(\xi)\right) v(\xi)\, d\xi. \tag{13}$$

Substantiate the general solution of equation (12) in the form of (8) in equation (13)

$$c = c \cdot e^{-\int\limits_0^\alpha q(\xi)d\xi + \lambda\alpha} \cdot v(\alpha) - v(0) + c \cdot \int\limits_0^\alpha e^{-\int\limits_0^\xi q(\xi_1)d\xi_1 + \lambda\xi} \left(-\frac{d}{d\xi} + q(\xi)\right) v(\xi)\, d\xi.$$

Without loss of generality we assume that $v(\alpha) = 1$ and $v(0) = 0$, then to determine λ we will solve the next transcendental equation

$$1 = e^{-\int\limits_0^\alpha q(\xi)d\xi + \lambda\alpha} + \int\limits_0^\alpha e^{-\int\limits_0^\xi q(\xi_1)d\xi_1 + \lambda\xi} \cdot \left(-\frac{d}{d\xi} + q(\xi)\right) v(\xi)\, d\xi. \tag{14}$$

Integrating by parts, we get

$$\int\limits_0^\alpha e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1 + \lambda\xi} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi)\, d\xi = \int\limits_0^\alpha \frac{e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1}}{\lambda} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi) \tfrac{d}{d\xi} e^{\lambda\xi} d\xi$$
$$= \frac{e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1 + \lambda\xi}}{\lambda} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi)\Bigg|_0^\alpha - \tfrac{1}{\lambda} \cdot \int\limits_0^\alpha e^{\lambda\xi} \tfrac{d}{d\xi} \left[e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi) \right] d\xi$$
$$= \tfrac{1}{\lambda} \left[e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1 + \lambda\xi} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi)\Bigg|_0^\alpha - \int\limits_0^\alpha e^{\lambda\xi} \tfrac{d}{d\xi} \left[e^{-\int\limits_0^\xi q(\xi)_1 d\xi_1} \left(-\tfrac{d}{d\xi} + q(\xi)\right) v(\xi) \right] d\xi \right].$$

Therefore, for large λ the asymptotic behavior of the spectrum is determined by the main term of (14):

$$1 = e^{-\int\limits_0^\alpha q(\xi)d\xi + \lambda\alpha}.$$

From the above it follows that

$$e^{\lambda\alpha} = e^{\int\limits_0^\alpha q(\xi)d\xi},$$

hence

$$\lambda\alpha = \int\limits_0^\alpha q(\xi)\, d\xi + 2\pi n i,\ \ n = 1, 2, 3, ...,$$

i.e.

$$\lambda_n = \frac{\int\limits_0^\alpha q(\xi)\, d\xi}{\alpha} + \frac{2\pi n i}{\alpha} = \bar{q} + \frac{2\pi n i}{\alpha}, \quad 0 < \alpha < 1,$$

where $\bar{q} = \frac{\int\limits_0^\alpha q(\xi)d\xi}{\alpha}$. The corresponding eigenfunctions are given as

$$u_n(x) = e^{-\int\limits_0^x q(\xi)d\xi + \left(\bar{q} + \frac{2\pi n i}{\alpha}\right)x}.$$

Since $\alpha < 1$ the eigenfunctions $u_n(x)$ are incomplete in $L_2(0,1)$, and complete in $L_2(0,\alpha)$, similarly, it can be established that the eigenfunctions of the initial problem are represented in the form

$$\bar{u}_n(x) = \left(u_n(x) + \frac{1}{\lambda_n} u_n^{\perp}\right).$$

Hence we can see that the system $u_n(x)$ is incomplete in $L_2(0,1)$, but it is complete and a basis in the space $L_2(0,\alpha)$ according to the criteria of N. Bari.

Let

$$h(x) = \begin{cases} 0, & x \in (\alpha, 1), \\ \bar{h}(x) \neq 0, & x \in (0,\alpha) \end{cases}.$$

Then, due to the incompleteness of the eigenfunctions $\bar{u}_n(x)$ in $L_2(\alpha,1)$ it follows that there exists $h(x)$ such that

$$(h(x), \bar{u}_n(x))_{L_2(0,1)} \equiv 0.$$

Then the function $h(x)$ is orthogonal to all root vectors $\{\bar{u}_n(x)\}$. Therefore, adjoint operator to the operator L^{-1} according to Keldysh M.V. theorem is a Volterra invariant operator on elements of arbitrary $h(x) \perp \bar{u}$. Denote by L_λ a root subspace of the correct restriction of L, and by $L_\lambda^{\perp}$ an orthogonal complement to L_λ. M.V. Keldysh showed that the adjoint operator L^* to $L_\lambda^{\perp}$ is a Volterra operator. Using the Keldysh's theorem, we can prove:

Theorem 4. *There is a correct restriction of L such that its root vectors are incomplete in $L_2(0,1)$ and the adjoint operator to L is an invariant Volterra operator on $L_\lambda^{\perp}$.*

Now, consider the case $v(x) \equiv 0$, $x \in (\alpha, 1)$. In this case, as a fixed correct restriction we take the solution of the Cauchy problem

$$Lu(x) = u'(x) + q(x)u(x) = f(x), \quad x \in (\alpha, 1), \tag{15}$$

$$u(1) = 0. \tag{16}$$

By the direct calculation, we find that the solution to problem (15)–(16) can be represented as

$$u_{K^*} = L_{K^*}^{-1} f = \int\limits_\alpha^x e^{-\int\limits_\xi^x q(\xi_1)d\xi_1} \cdot f(\xi)\,d\xi.$$

Therefore, as above, the solution continuously dependent on $f(\xi) \in L_2(0,1)$ can be represented as:

$$u(x) = L_{K^*}^{-1} f(x) + c \cdot u_0(x) = \int\limits_{\alpha}^{x} e^{-\int\limits_{\xi}^{x} q(\xi_1) d\xi_1} f(\xi)\, d\xi + u_0(x) \cdot \int\limits_{\alpha}^{1} f(x)\, v(x)\, dx, \qquad (17)$$

where $u_0(x)$ is an arbitrary solution of the homogeneous equation

$$Lu_0(x) = u_0^{'}(x) + q(x)\, u_0(x) = 0,$$

i.e.

$$u_0(x) = c \cdot e^{-\int\limits_{\alpha}^{x} q(\xi) d\xi}, \quad c = const,$$

and $v(x) \in L_2(0,1)$ is an arbitrary function.

Now, we will find the regular boundary condition (17) for this, substituting in the equality (17) $x = 1$ and $Lu = u'(x) + q(x)\, u(x)$, we get

$$\begin{aligned} u(1) &= u_0(1) \int\limits_{\alpha}^{1} \left(u'(\xi) + q(\xi)\, u(\xi)\right) v(\xi)\, d\xi \\ &= u_0(1) \left[v(1)\, u(1) - v(\alpha)\, u(\alpha)\right] + u_0(1) \left[\int\limits_{\alpha}^{1} u(\xi) \left(\frac{d}{d\xi} - q(\xi)\, u(\xi)\right) v(\xi)\, d\xi\right]. \end{aligned} \qquad (18)$$

Since $u_0(1) = 1$, assume that $u(1) \cdot (1 - v(1)) = k$, then the last equality of (18) takes the following form

$$k = -v(\alpha)\, u(\alpha) + \left[\int\limits_{\alpha}^{1} u(\xi) \left(\frac{d}{d\xi} - q(\xi)\, u(\xi)\right) v(\xi)\, d\xi\right].$$

Thus, we will study the spectral problem

$$Lu = u' + q(x)\, u = \lambda u, \qquad (19)$$

$$k = -u(\alpha)\, v(\alpha) + \int\limits_{\alpha}^{1} u(\xi) \left(\frac{d}{d\xi} - q(\xi)\right) v(\xi)\, d\xi. \qquad (20)$$

Substituting the solution to equation (19)

$$u(x) = e^{-\int\limits_{1}^{x} q(\xi) d\xi + \lambda(x-1)},$$

from the equality (20) we have

$$k=-e^{-\int_1^{\alpha} q(\xi)d\xi+\lambda(\alpha-1)}v(\alpha)+\int_{\alpha}^{1} e^{-\int_1^{\alpha} q(\xi)d\xi}\left(\frac{d}{d\xi}-q(\xi)\right)v(\xi)\frac{1}{\lambda}\frac{d}{d\xi}e^{\lambda(\xi-1)}d\xi$$
$$=-e^{-\int_1^{\alpha} q(\xi)d\xi+\lambda(\alpha-1)}v(\alpha)-\frac{1}{\lambda}\left[e^{-\int_1^{\alpha} q(\xi)d\xi+\lambda(\xi-1)}\left(-\frac{d}{d\xi}+q(\xi)\right)v(\xi)\right]\Bigg|_{\alpha}^{1} \tag{21}$$
$$-\frac{1}{\lambda}\int_{\alpha}^{1} e^{\lambda(\xi-1)}\frac{d}{d\xi}e^{-\int_1^{\xi} q(\xi_1)d\xi_1}\left(\frac{d}{d\xi}-q(\xi)\right)v(\xi)\,d\xi.$$

We assume that $v(x)\in W_2^2(0,1)$, from (21) we have

$$k=e^{-\int_1^{\alpha} q(\xi)d\xi+\lambda(\alpha-1)}\left(1-\frac{k_1}{\lambda}\right),$$

where $k_1=k_1(v)$ is a bounded number, what is more $\frac{k_1}{\lambda}\neq 1$. Hence

$$\frac{k}{\left(1-\frac{k_1}{\lambda}\right)}e^{\int_1^{\alpha} q(\xi)d\xi}=e^{\lambda_n(\alpha-1)+2\pi ni},\ n=1,2,3,\dots . \tag{22}$$

We take the logarithm of both sides

$$\lambda_n(\alpha-1)+2\pi ni=\ln k_2-\ln\left(1-\frac{k_1}{\lambda_n}\right).$$

Since for large λ_n

$$\ln\left(1-\frac{k_1}{\lambda_n}\right)=-\frac{k_1}{\lambda_n}+O\left(\frac{1}{\lambda_n^2}\right),$$

then we rewrite equality (22) in the form

$$\lambda_n(\alpha-1)=\left(2\pi ni+O\left(\frac{1}{\lambda_n^2}\right)\right)\left(1+\frac{k_1}{\lambda_n}\right)^{-1}.$$

Therefore, as $n\to\infty$ we have

$$\lambda_n=\frac{2\pi ni}{(\alpha-1)}\left(1+O\left(\frac{(\alpha-1)}{2\pi ni}\right)^2\right). \tag{23}$$

The eigenfunction corresponding to the eigenvalue is represented as

$$u_n=e^{-\int_1^{x} q(\xi)d\xi+\frac{2\pi ni}{(\alpha-1)}\left(1+O\left(\frac{(\alpha-1)}{2\pi ni}\right)^2\right)(x-1)}.$$

Therefore, for $0<\alpha<1$ the system of eigenfunctions given by formulas (23) is incomplete in $L_2(0,1)$. Thus proved:

Theorem 5. *The eigenvectors of the correct restriction generated by the function $v(x) \in L_2(0,1)$ are incomplete in $L_2(0,1)$.*

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Кәлменов Т.Ш., Қахарман Н. БІР ӨЛШЕМДІ ДИФФЕРЕНЦИАЛДЫҚ ОПЕРАТОРЛАР ҮШІН РЕГУЛЯРЛЫ ШЕТТІК ЕСЕПТЕРДІҢ ТҮБІРЛІК ВЕКТОРЛАРЫНЫҢ ТОЛЫҚТЫҒЫ ТУРАЛЫ

Бұл жұмыста біз бір өлшемді дифференциалдық операторлар үшін регулярлы шеттік есептердің вольтерралық инвариантты ішкікеңістігін құрдық. Бірінші ретті оператордың сығылуларының вольтерралығының критерийі [1] жұмысында дәлелденген. Біз, Келдыштың теоремасын пайдалана отырып, $L_2(0,1)$-де меншікті векторлары толық емес болатын қисынды сығылу бар екенін, сондай-ақ $L_2(0,1)$-де меншікті векторлары толық болатын қисынды сығылулар да бар екенін дәлелдедік. Вольтерралық инвариантты ішкікеңістіктердің сипаттамасы берілді.

Кілттік сөздер. Вольтерралық инвариантты ішкікеңістіктер, қисынды сығылулар, кеңейтулер, бір өлшемді дифференциалдық операторлар.

Кальменов Т.Ш., Кахарман Н. О ПОЛНОТЕ КОРНЕВЫХ ВЕКТОРОВ РЕГУЛЯРНЫХ КРАЕВЫХ ЗАДАЧ ДЛЯ ОДНОМЕРНЫХ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ

В этой работе мы построили вольтерровое инвариантное подпространство регулярных краевых задач для одномерных дифференциальных операторов. В [1] был доказан критерий вольтерровости сужений оператора первого порядка. Используя теорему Келдыша мы доказали, что существуют корректные сужения максимального оператора, собственные вектора которых являются неполными в $L_2(0,1)$; в то же время также существуют корректные сужения максимального оператора, собственные вектора которых полны в $L_2(0,1)$. Дано описание вольтерровых инвариантных подпространств.

Ключевые слова. Вольтерровое инвариантное подпространство, корректные сужения, расширения, одномерные дифференциальные операторы.

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On optimization problem of chemical reaction kinetics

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Abstract. In this article, one deterministic model of the kinetics of chemical reactions is considered. A mathematical model of this chemical kinetics has been constructed. For the given experimental data of reaction speeds, i.e. parameters (coefficients) of the model, the exact optimal parameters (coefficients) of the model were found using the least-squares methods. For one concentration (blood concentration) and for the reaction rate of this concentration an optimal parameter, that is, the reaction rate of this concentration was found. The task is practical and is of great importance in chemical kinetics and in medicine.

Keywords. Chemical kinetics, ODEs system, optimization, least-squares method, reaction, concentration, rate.

1 Introduction

Chemical kinetics can be defined as a science that studies the flow rates of chemical reactions and the factors affecting them. The task of the mathematical theory of the kinetics of a chemical reaction is the description of the change in the concentrations of reacting substances with time, moreover, the concentration is defined as the number of molecules in a certain constant volume. In the classical deterministic theory of concentration, they are described using real continuous functions of time, and the reaction mechanism is modeled by a system of differential (or integral) equations [1].

In a probabilistic theory, the basic random variables are the concentrations of the reacting substances at time t, and the task is to determine the distributions of these concentrations. Probabilistic models were considered in the works: [2]–[4].

Montroll and Schuler [5], developed a general theory of the kinetics of chemical reactions based on the theory of first-time distribution. Some stochastic models were considered in [6].

Other deterministic models of chemical kinetics are solved by S.I. Kabanikhin and his scientists and PhD students [7], [8].

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Some probabilistic models of chemical kinetics are the simple model of an autocatalytic reaction, the unimolecular reaction, the bimolecular reaction and the law of effective masses, and the sequence of monomolecular reactions of the form

$$\mathbf{A} \xrightarrow{l_1} \mathbf{B} \xrightarrow{l_2} \mathbf{C}$$
$$\mathbf{A} \xleftarrow[l_3]{} \mathbf{B} \xleftarrow[l_4]{} \mathbf{C},$$

where l_1 and l_2 are constants characterizing the rate of the direct reaction, l_3 and l_4 are constants characterizing the rate of backlash.

In this article, we considered a deterministic model of the kinetics of chemical reactions [10]. We consider one inverse problem of pharmacokinetics and pharmacodynamics (**PK** & **PD**): finding unknown optimal coefficients (parameters) of the process of the dynamics (reactions) of chemical kinetics. According to practical experiments and observations, the processes of chemical kinetics in pharmacokinetics and pharmacodynamics proceed according to the following scheme: there are 4 chambers **A, B, C, D** and the kinetics of the reaction system are described in Schemes: 1.

$$\mathbf{A} \xrightarrow{k_1} \mathbf{B} \xrightarrow{k_2} \mathbf{C}$$

and 2.

$$\mathbf{B} \xrightarrow{k_3} \mathbf{D}$$
$$\mathbf{B} \xleftarrow[k_4]{} \mathbf{D}$$

(**PK** & **PD**) model of these schemes has the form

$$\mathbf{A}[y_1] \xrightarrow{k_1} \mathbf{B}[y_2] \xrightarrow{k_2} \mathbf{C}[y_3] \tag{1}$$

and

$$\begin{aligned} &\mathbf{B}[y_2] \xrightarrow{k_3} \mathbf{D}[y_4] \\ &\mathbf{B}[y_2] \xleftarrow[k_4]{} \mathbf{D}[y_4] \end{aligned} \tag{2}$$

Comment to scheme (2). The concentration in chamber **B** is equal to y_2 and this concentration flows into chamber **D** with speed k_3, that is, is described by the scheme

$$\mathbf{B}[y_2] \xrightarrow{k_3} \mathbf{D}[y_4].$$

The concentration in chamber **D** is equal to y_4 and this concentration flows back to chamber **B** at a speed k_4, that is, is described by the scheme

$$\mathbf{B}[y_2] \xleftarrow[k_4]{} \mathbf{D}[y_4].$$

A model of schemes (1) and (2) was constructed in [10]

$$\frac{y_1(t)}{dt} = -k_1 y_1(t), \tag{3}$$

$$\frac{y_2(t)}{dt} = k_1 y_1(t) - \big(k_2 y_2(t) + k_3 y_2(t)\big) + k_4 y_4(t), \tag{4}$$

$$\frac{y_3(t)}{dt} = k_2 y_2(t), \tag{5}$$

$$\frac{y_4(t)}{dt} = k_3 y_2(t) - k_4 y_4(t), \tag{6}$$

$$y_1(0) = y_0, \tag{7}$$

$$y_2(0) = 0, \tag{8}$$

$$y_3(0) = 0, \tag{9}$$

$$y_4(0) = 0, \tag{10}$$

where $y_i(t)$, $i = 1, 2, 3, 4$, are concentrations of components in **A, B, C, D** respectively, at the moment of time $t \in \big[0, T\big]$, are the coefficients (parameters) of the velocities in the individual reaction stages, $y_1(0) = y_0$ is the given initial concentration.

Some ill-posed inverse problems with respect to the initial conditions of the heat equation were considered in [9].

2 General mathematical formulation of the problem

Let us be given the experimental data of the parameters k_i, $i = 1, 2, 3, 4$, and the initial condition y_0 for $y_1(t)$, as well as restrictions on the parameters k_i, $i = 1, 2, 3, 4$, of the velocities: $0 < k_i \le \overline{k_i}$, $i = 1, 2, 3, 4$, for given $\overline{k_i}$, $i = 1, 2, 3, 4$. The given experimental data are: $y_0 = 50$, $k_1 = 5$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$ and also $\overline{k_i} = 10$, $i = 1, 2, 3, 4$. We denote by $y_i^{exp}(t, k_j, y_0)$, $i, j = 1, 2, 3, 4$, the solutions with experimental data $y_0 = 50$, $k_1 = 5$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$ for $t \in \big[0, 7\big]$.

I. It is required to find the optimal coefficients (parameters) k_i, $i = 1, 2, 3, 4$, from the condition of minimization of the functional

$$\sum_{i=1}^{4} \Big(y_i^{exp}\big(t, k_j, y_0\big) - y_i\big(t, k_j, y_0\big)\Big)^2 \longrightarrow \min_{t,\, j=1,2,3,4}. \tag{11}$$

II. Investigate the effect of the initial data $y_1(0) = y_0$, $y_i(0) = 0$, $i = 2, 3, 4$, on the experimental solution $y_i^{exp}\big(t, k_j, y_0\big)$, $i, j = 1, 2, 3, 4$.

3 Solution

We solve system (3)–(10) considering k_i, $i = 1, 2, 3, 4$, as parameters of the solution of system (3)–(10). We denote this solution by $y_i\big(t, k_1, k_2, k_3, k_4, y_0\big)$, $i = 1, 2, 3, 4$. The initial condition y_0 is also considered as the solution parameter. We need to find optimal parameters y_0, k_i, $i = 1, 2, 3, 4$, with least squares constraints, that is (11)

$$\sum_{i=1}^{4} \Big(y_i^{exp}\big(t, k_j, y_0\big) - y_i\big(t, k_j, y_0\big) \Big)^2 \longrightarrow \min_{t,\, j=1,2,3,4}.$$

Let the parameters k_i, $i = 1, 2, 3, 4$, be are given experimental data. We consider the Cauchy problem for these given parameters $k_1 = 5$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$ and for the initial value $y_1(0) = 50$. So we have the ODEs system

$$\begin{aligned} \frac{y_1^{exp}(t)}{dt} &= -5y_1^{exp}(t), \\ \frac{y_2^{exp}(t)}{dt} &= 5y_1^{exp}(t) - 5y_2^{exp}(t) + 4y_4^{exp}(t), \\ \frac{y_3^{exp}(t)}{dt} &= 2y_2^{exp}(t), \\ \frac{y_4^{exp}(t)}{dt} &= 3y_2^{exp}(t) - 4y_4^{exp}(t), \\ y_1^{exp}(0) &= 50, \\ y_2^{exp}(0) &= 0, \\ y_3^{exp}(0) &= 0, \\ y_4^{exp}(0) &= 0. \end{aligned} \tag{12}$$

The exact solution of this problem (12) has the form $y_1^{exp}(t) = 50e^{-5t}$, $y_2^{exp}(t) = -\frac{1000}{21}e^{-8t} + \frac{375}{14}e^{-t} + \frac{125}{6}e^{-5t}$, $y_3^{exp}(t) = -\frac{375}{7}e^{-t} + \frac{250}{21}e^{-8t} - \frac{25}{3}e^{-5t} + 50$, $y_4^{exp}(t) = \frac{375}{14}e^{-t} + \frac{250}{7}e^{-8t} - \frac{125}{2}e^{-5t}$, and their graphs are plotted.

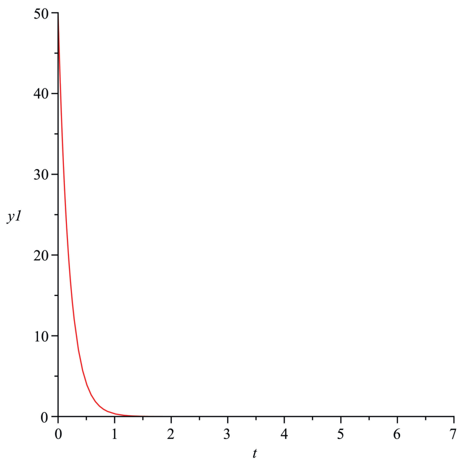

Figure 1 – $y_1^{exp}(t)$

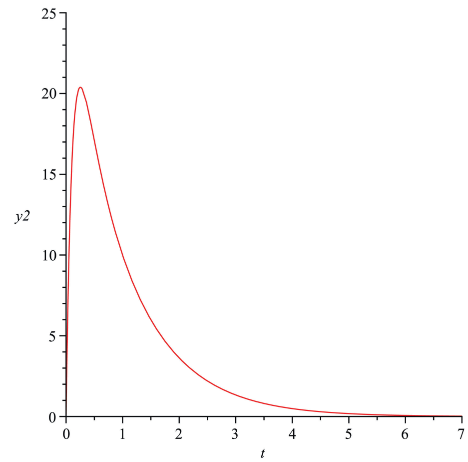

Figure 2 – $y_2^{exp}(t)$

Next, let us consider only the concentration $y_2(t)$ (the blood concentration) and the parameter k_2 (the speed parameter corresponding to this concentration). Let now k_2 be an unknown parameter. Consider the same problem, i.e. for $k_1 = 5$, $k_2 = x$, $k_3 = 3$, $k_4 = 4$ the following Cauchy problem with the same initial condition $y_1(0) = y_0 = 50$ for the system of differential equations:

$$\begin{aligned} \frac{y_1(t)}{dt} &= -5y_1(t), \\ \frac{y_2(t)}{dt} &= 5y_1(t) - xy_2(t) + 4y_4(t), \\ \frac{y_3(t)}{dt} &= xy_2^{exp}(t), \\ \frac{y_4(t)}{dt} &= 3y_2(t) - 4y_4(t), \\ y_1(0) &= 50, \\ y_2(0) &= 0, \\ y_3(0) &= 0, \\ y_4(0) &= 0. \end{aligned} \tag{13}$$

Solving the problem (13), we find $y_2\big(t, k_1, x, k_3, k_4, y_0\big)$. Now we minimize the functional for $y_0 = 50$:

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big) - y_2\big(t, k_1, x, k_3, k_4, y_0\big)\Big)^2 \longrightarrow \min_{t\in[0,7],\, x}. \tag{14}$$

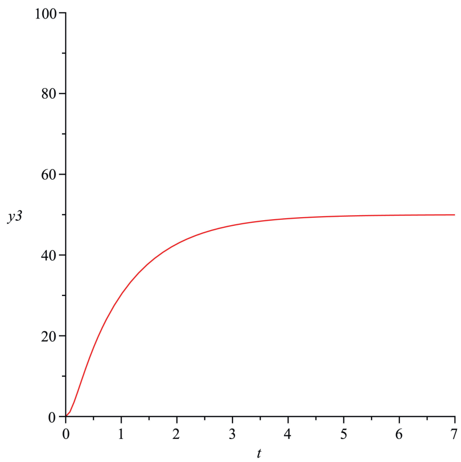

Figure 3 – $y_3^{exp}(t)$

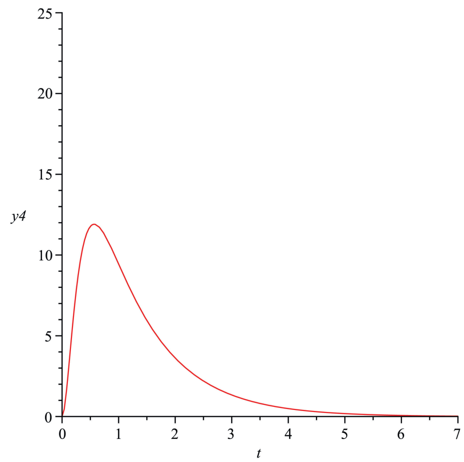

Figure 4 – $y_4^{exp}(t)$

As a result of minimization, we obtained min $= 5.70287319210706536l0 \cdot 10^{-25}$ $t = 2.61941764237572228$, $x = 1.99999999999967138$.

Hence it is clear that $x = k_2 = 2$ and indeed $k_2 = 2$ is an optimal parameter. Graphs of two functions $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and $y_2\big(t, k_1, x, k_3, k_4, y_0\big)$ almost coincide.

Next, we investigate the influence of the initial condition. We consider the same problem with initial conditions **1.** $y_1(t) = y_0 = 55$ and **2.** $y_1(t) = y_0 = 45$. For this, solving the corresponding Cauchy problems with the corresponding parameters and initial data $\big(k_1, x, k_3, k_4, 55\big)$ and $\big(k_1, x, k_3, k_4, 45\big)$, we find $y_2\big(k_1, x, k_3, k_4, 55\big)$ and $y_2\big(k_1, x, k_3, k_4, 45\big)$.

1. As a result of minimizing the functional

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, 50\big) - y_2\big(t, k_1, x, k_3, k_4, 55\big)\Big)^2 \longrightarrow \min_{t\in[0,7],\, x} \tag{15}$$

for $y_1(t) = y_0 = 55$ we find min $= 8.79178738841243450 \cdot 10^{-22}$, $t = 3.44254019385341570$, $x = 2.06251146631368298$.

2. As a result of minimizing the functional

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, 50\big) - y_2\big(t, k_1, x, k_3, k_4, 45\big)\Big)^2 \longrightarrow \min_{t\in[0,7],\, x} \tag{16}$$

for $y_1(t) = y_0 = 45$ we find min $= 1.20124267659072705 \cdot 10^{-24}$, $t = 0.558371362474722188$, $x = 1.62283110817454901$.

The influence of the initial condition on k_2 for $y_1(t) = y_0 = 55$ is less than for $y_1(t) = y_0 = 45$. But the impact is palpable.

Further, without changing the initial condition $y_1(t) = y_0 = 50$ and assuming the two parameters to be unknown $k_1 = u$ and $k_2 = v$, we solve the corresponding Cauchy problem and find $y_2\big(t, u, v, k_3, k_4, y_0\big)$. Then we minimize the functional

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big) - y_2\big(t, u, v, k_3, k_4, y_0\big)\Big)^2 \longrightarrow \min_{t\in[0,7],\, u,\, v}. \tag{17}$$

As a result, we obtained the following optimal parameters:
$k_1 = u = 0.914833719634728482$, $k_2 = v = 0.984369761310716518$ for $t = 0.819339355572412$ 776, and at these values $\min = 1.52723471250787886 \cdot 10^{-27}$.

Further, also without changing the initial condition $y_1(t) = y_0 = 50$ and considering three variables as unknown $k_1 = u$, $k_2 = v$ and $k_3 = w$, we solve the corresponding Cauchy problem and find $y_2\big(t, u, v, w, k_4, y_0\big)$. Then we minimize the functional

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big) - y_2\big(t, u, v, w, k_4, y_0\big)\Big)^2 \longrightarrow \min_{t\in[0,7],\, u,\, v,\, w}. \tag{18}$$

As a result, we obtained the following optimal parameters:
$k_1 = u = 0.899429404474657690$, $k_2 = v = 1.00326990573297814$ and $k_3 = w = 0.746891529$ 779095676 for $t = 0.661375832090767335$ and at these values $\min = 1.33177371090744570 \cdot 10^{-21}$.

Finally, without changing the initial condition $y_1(t) = y_0 = 50$ and considering four parameters as unknown $k_1 = u$, $k_2 = v$, $k_3 = w$ and $k_4 = z$, we solve the corresponding Cauchy problem and find $y_2\big(t, u, v, w, z, y_0\big)$. Then we minimize the functional

$$\Big(y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big) - y_2\big(t, u, v, w, z, y_0\big)\Big)^2 \longrightarrow \min\nolimits_{t\in[0,7],\, u,\, v,\, w,\, z}. \tag{19}$$

As a result, we obtained the following optimal parameters:
$k_1 = u = 0.870035291306964886$, $k_2 = v = 1.04421255971011662$, $k_3 = w = 0.984995940722$ 765974 and $k_4 = z = 1.16027804673082446$ for $t = 0.781021983948520848$ and at these values $\min = 4.77387065403547504 \cdot 10^{-26}$.

Then we compare graphically $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and $y_2\big(t, u, v, w, z, y_0\big)$ for the initial condition $y_0 = 50$ and for $t \in [0,\, 7]$ for parameters $k_1 = 5$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$ in $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and for parameters $k_1 = u = 0.870035291306964886$, $k_2 = v = 1.04421255971011662$, $k_3 = w = 0.984995940722765974$, $k_4 = z = 1.16027804673082446$ in $y_2\big(t, u, v, w, z, y_0\big)$.

Graphical solution of $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and $y_2\big(t, u, v, w, z, y_0\big)$. (See Figure 5). Red line is $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and blue line is $y_2\big(t, u, v, w, z, y_0\big)$. Further, since we are more interested in the concentration $y_2\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and parameter k_2, for different values of only k_2 we plotted the solution y_2 in order to compare them with $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$, that is, we plotted the graphs of

$$\overline{y}_2(t) = \Big|y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big) - y_2\big(t, k_1, j, k_3, k_4, y_0\big)\Big| \quad \text{for} \quad j = 5, 4, 3, 1.5, 1.2.$$

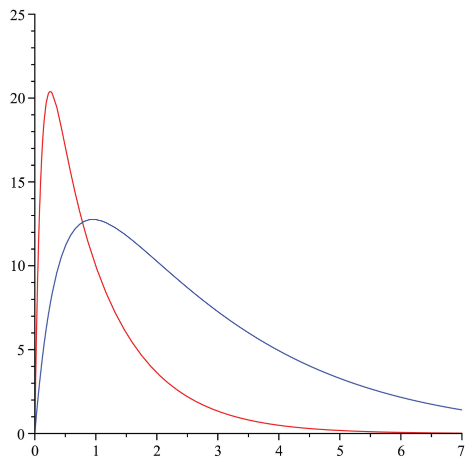

Figure 5 – Red line is $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and blue line is $y_2\big(t, u, v, w, z, y_0\big)$

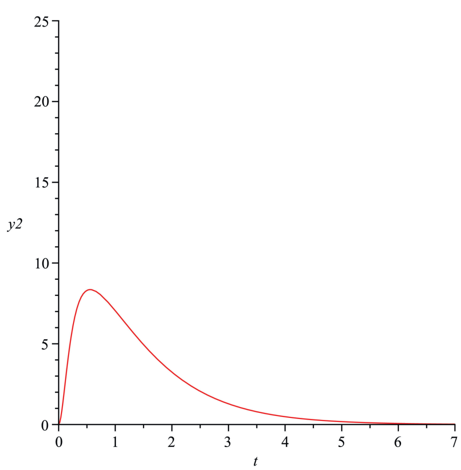

Figure 6 – $\overline{y}_2(t)$ for $k_2 = 5$

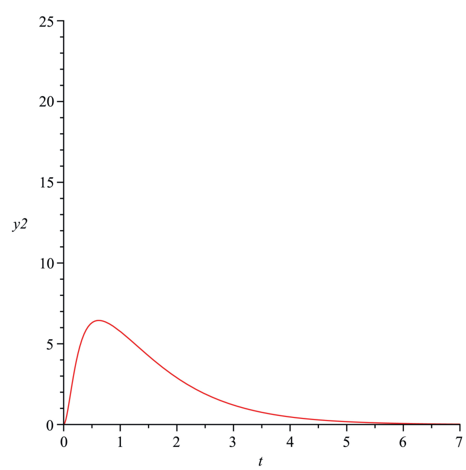

Figure 7 – $\overline{y}_2(t)$ for $k_2 = 4$

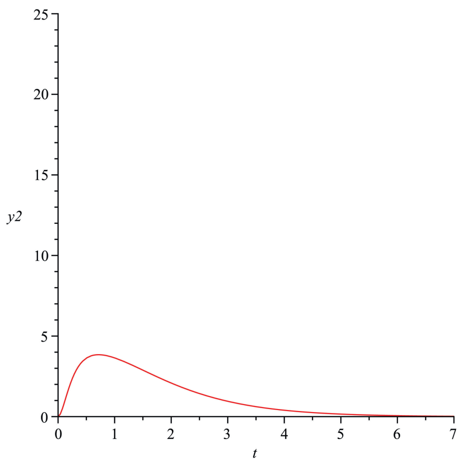

Figure 8 – $\overline{y}_2(t)$ for $k_2 = 3$

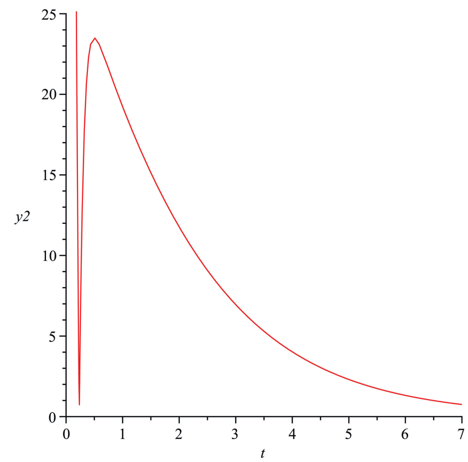

Figure 9 – $\overline{y}_2(t)$ for $k_2 = 1.5$

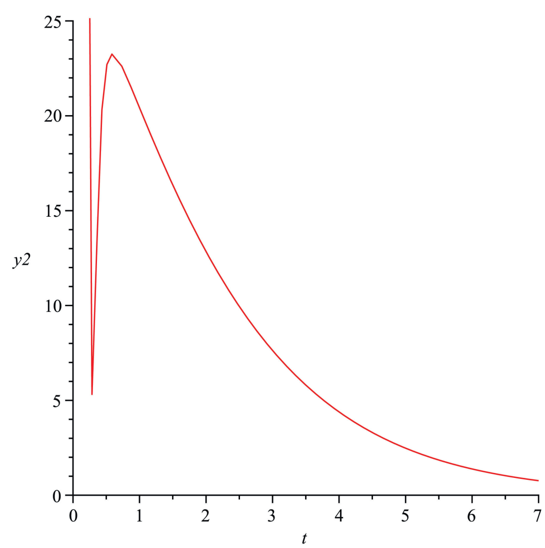

Figure 10 – $\overline{y}_2(t)$ for $k_2 = 1.2$

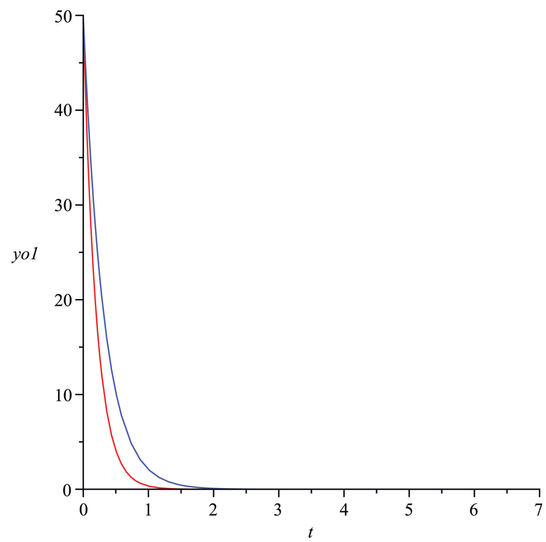

Figure 11 – Red line is $y_1^{exp}\big(t,k_1,k_2,k_3,k_4,y_0\big)$ and blue line is $y_1^{opt}\big(t,u,v,w,z,y_0\big)$

3 Conclusion

An analytic investigation of the solution $y_2(t)$ showed that the solution $y_2(t)$ is stable for $k_2 \longrightarrow_{+0} 2$ and unstable for $k_2 < 2$. Note that this result does not affect the finding of the actual values of the optimal parameters k_i, $i = 1,2,3,4$. Optimal parameters k_i, $i = 1,2,3,4$, may be different.

Finally, we consider the general problem. Without changing the initial condition $y_1(t) = y_0 = 50$ and considering four parameters as unknown $k_1 = u$, $k_2 = v$, $k_3 = w$ and $k_4 = z$, we solve the corresponding Cauchy problem and find $y_i\big(t,u,v,w,z,y_0\big)$, $i = 1,2,3,4$. Then we minimize the functional

$$\sum_{i=1}^{4}\Big(y_i^{exp}\big(t,k_1,k_2,k_3,k_4,y_0\big) - y_i\big(t,u,v,w,z,y_0\big)\Big)^2 \longrightarrow \min_{t\in[0,7],u,v,w,z}. \qquad (20)$$

As a result, we obtained the following optimal parameters:
$k_1 = u = 3.17221257981978$, $k_2 = v = 1.0712973632105791$, $k_3 = w = -0.0019837608531703696$ and $k_4 = z = 0.6859574034351494$ for $t = 9.85692203652167$ and at these values $\min = 1.22371719355872573 \cdot 10^{-17}$. But, for these optimal parameters, the solution $y_4^{opt}\big(t,u,v,w,z,y_0\big)$ turned out to be exponentially fast growing, and the solution $y_3^{opt}\big(t,u,v,w,z,y_0\big)$ is negative, and parameter $k_3 = w = -0.0019837608531703696$ is negative. Therefore, we compared only solutions $y_1^{exp}\big(t,k_1,k_2,k_3,k_4,y_0\big)$ and $y_1^{opt}\big(t,u,v,w,z,y_0\big)$, $y_2^{exp}\big(t,k_1,k_2,k_3,k_4,y_0\big)$ and $y_2^{opt}\big(t,u,v,w,z,y_0\big)$.

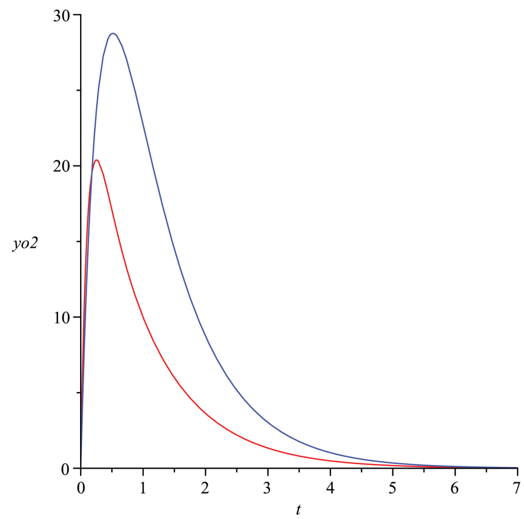

Figure 12 – Red line is $y_2^{exp}\big(t, k_1, k_2, k_3, k_4, y_0\big)$ and blue line is $y_2^{opt}\big(t, u, v, w, z, y_0\big)$

Remark 1. *The effect of the initial condition* y_0 *on the solution* y_2^{exp} *is investigated numerically. See formulas (15) and (16).*

Remark 2. *Analytic expressions for solutions* $y_2\big(t, k_1, x, k_3, k_4, y_0\big)$, $y_2\big(t, u, v, k_3, k_4, y_0\big)$, $y_2\big(t, u, v, w, k_4, y_0\big)$ $y_2\big(t, u, v, w, z, y_0\big)$ *are exponential and very cumbersome, and therefore, they and their difference from* y_2^{exp} *are investigated numerically and graphically. See formulas (17), (18), (19) and Figure 5.*

Remark 3. *The influence of the parameter* k_2 *(different values of* k_2*) on the solution* y_2^{exp} *is investigated graphically. See Figures 6, 7, 8, 9, 10.*

Remark 4. *Analytic expressions for solutions* $y_i\big(t, u, v, w, z, y_0\big)$ *for all* $i = 1, 2, 3, 4$ *are also exponential and very cumbersome, and therefore, these solutions are compared with experimental solutions* y_i^{exp} *for all* $i = 1, 2, 3, 4$ *graphically. See Figures 11 and 12.*

The obtained results are quite acceptable for two concentrations $y_1(t)$ and $y_2(t)$.

Some other models of the process of chemical kinetics in pharmacokinetics and pharmacodynamics can be found in [11], [12].

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Шакенов Қ.Қ., Шакенов И.Қ. ХИМИЯЛЫҚ РЕАКЦИЯЛАР КИНЕТИКАСЫН ТИІМДЕНДІРУДІҢ БІР ЕСЕБІ ТУРАЛЫ

Бұл мақалада химиялық реакциялар кинетикасының детерминдендірілген бір моделі қарастырылады. Осы химиялық кинетиканың математикалық моделі құрылды. Реакциялар жылдамдықтарының келтірілген эксперименттік мәліметтері – моделдің параметрлері (коэффициенттері) үшін – ең аз шаршылар әдісін пайдалану арқылы моделдің нақты тиімді параметрлері (коэффициенттері) табылды. Бір концентрация – қандағы концентрация үшін – және осы концентрацияның реакция жылдамдығы үшін тиімді параметр – осы концентрацияның реакция жылдамдығы табылды. Жұмыс практикалық болып саналады және химиялық кинетика мен медицинада үлкен маңызы бар.

Кілттік сөздер. Химиялық кинетика, ЖДТ жүйесі, тиімдендіру, ең кіші шаршылар әдісі, реакция, концентрация, жылдамдық.

Шакенов К.К., Шакенов И.К. ОБ ОДНОЙ ЗАДАЧЕ ОПТИМИЗАЦИИ КИНЕТИКИ ХИМИЧЕСКИХ РЕАКЦИЙ

В данной статье рассматривается одна детерминированная модель кинетики химических реакций. Построена математическая модель этой химической кинетики. Для приведенных экспериментальных данных скоростей реакций – параметров (коэффициентов) модели – точные оптимальные параметры (коэффициенты) модели были найдены с использованием методов наименьших квадратов. Для одной концентрации – концентрации в крови – и для скорости реакции этой концентрации был найден оптимальный параметр – скорость реакции этой концентрации. Задача является практической и имеет большое значение в химической кинетике и в медицине.

Ключевые слова. Химическая кинетика, система ОДУ, оптимизация, метод наименьших квадратов, реакция, концентрация, скорость.

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