# Classification of one-types in weakly o-minimal theories and its corollaries*

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**Abstract**

In this paper a class of theories which had been picked out by Maximo Dickmann [D] and which he called weakly o-minimal is studied. In sections 1–3 properties of one-types over sets in a weakly o-minimal theory are studied. In particular, in section 1 six basic kinds of one-types over sets are distinguished. In section 2 notions of neighbourhood of a set in one-type, weak and almost orthogonality of one-types are introduced. It is proved that a non-weak (non-almost) orthogonality is equivalence relation and each $\not\perp^w$–class of the equivalence contains one-types of only one kind from six basic ones. In section 3 it is proved that all the one-types of each $\not\perp^w$–class which contains at least one definable one-type are definable. In section 4 it is proved that an expansion of a model of a weakly o-minimal theory by any set of unary convex predicates has a weakly o-minimal theory (Theorem 4.3). This result solves Problem of Cherlin, Macpherson, Marker, Steinhorn from the basic paper on weakly o-minimal structures [MMS].

Some lemmas of the paper have analogies in o-minimal theories. In particular, section 1–2 — in [M], [LM], [B], section 3 — in [MS], section 4 — in [MMS], [B].

The results of the paper have been announced in [B1],[B2].

# 1 Essential kinds of 1-types over sets of models of weakly o-minimal theories

**Definition 1.1** [LvdD] Model $M$ of signature $\Sigma$ is ordered minimal (o-minimal) if it is $\emptyset$-definable totally ordered and the realization of each formula of the signature $\Sigma(M)$ in one free variable is a disjoint union of finitely many of open intervals, points.

**Definition 1.2** Let $A$ be a subset of totally ordered set $B$. Then $A$ is convex if for any $\alpha, \beta \in A$ the following holds:

$$\forall \gamma \in B \ \ [\, \alpha < \gamma < \beta \Rightarrow \gamma \in A \,].$$

**Note 1.1** An intersection of family of convex subsets of arbitrary totally ordered set is convex or empty.

*Supported by European Community Grant INTAS-93-3547.

**Definition 1.3** [D]

(i) A totally ordered model $M$ of signature $\Sigma$ is weakly ordered minimal (w.o.m.) if the realization of each formula of the signature $\Sigma(M)$ in one free variable is a disjoint union of finitely many convex subsets.

(ii) A theory $T$ is said to be weakly ordered minimal (w.o.m.) if each model of $T$ is w.o.m.

**Note 1.2** $T$ is w.o.m. iff for any formula $\Phi(x, \bar{y})$ of signature $\Sigma$ there is a natural number $n_\Phi < \omega$ such that for any model of $T$ for any $\bar{a} \in M^{l(\bar{y})}$, $\Phi(M, \bar{a})$ is an union $\leq n_\Phi$ $\neg\Phi(M, \bar{a})$-separable convex subsets of $M$. Here, $\Phi(M, \bar{a}) := \{b \in M : M \models \Phi(b, \bar{a})\}$.

**Note 1.3** A set of all realizations of any 1-type over set of a model of w.o.m. theory in any model of this theory is convex set, because each complete type is determined by family of formulas, the realizations of which are convex (convex formula).

Everywhere we will hold on the following
**Convention 1.1** Let $\bar{a} \in M^{l(\bar{a})}$, where $l(\bar{a}) = l$, $\bar{a} =< a_1, \ldots, a_l >$ . We will write $\bar{a} \in M$ whenever $\{a_1, \ldots, a_l\} \subset M$. Let $1 \leq m < l$ then $\bar{a}|m =< a_1, \ldots, a_m >$. In this paper $M$ is a model of w.o.m. theory $T$, $M'$ is a sufficiently large saturated elementary extension of $M$, $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in M$, $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\mu} \in M' \setminus M$. For $C, D \subset M$ we write $C < D$ whenever $c \in C$, $d \in D$, we have $c < d$. For any convex $D \subset M'$, $D^- := \{\beta \in M' : \beta < D\}$, $D^+ := \{\beta \in M' : \beta > D\}$.
If $A < B$ then $(A, B) := \{\gamma \in M' \mid A < \gamma < B\}$
We write $A \subseteq B$ whenever $a \in A$ we have $a \in B$.
We write $A \subset B$ if $A \subseteq B$ and $\exists b \in B (b \notin A)$.
We always consider subset $A$ of $M'$, such that $M'$ is $|A|^+$-saturated.
$p, q, r \in S_1(A)$ means always non-algebraic complete 1-types over A. We understand that isolated type is non-algebraic isolated type. $\mathrm{Aut}_A(M')$ is the group of automorphisms of $M'$ such that $\forall f \in \mathrm{Aut}_A(M')\ \forall a \in A\ (f(a) = a)$.
We use following notations $x > F(M', \bar{y}) \Leftrightarrow \forall z[F(z, \bar{y}) \to z < x]$
$\Psi(M', \bar{x}) > F(M', \bar{y}) \Leftrightarrow \forall z \forall t[F(z, \bar{y}) \wedge \Psi(t, \bar{x}) \to z < t]$
$p(M') = \{\gamma \in M' | \forall \phi(x) \in p,\ M' \models \phi(\gamma)\}$, $\alpha \models p \Leftrightarrow \alpha \in p(M')$.
By Notes we always understand historical Notes or immediate corollaries of Definitions or proofs of Lemmas or Claims. Sometimes we choose more long proof for Lemma in order to have a possibility of formulating the Notes. Almost always we do not prove the Notes.

**Definition 1.4** A partition of $A \subset M'$ into two convex subsets $C$ and $D$ ($C < D$, $C \cup D = A$) is said to be $(C, D)$-cut in $A$. If $C$ has supremum or $D$ has infimum in $A \cup \{-\infty, \infty\}$, then $(C, D)$-cut is said to be rational. Otherwise $(C, D)$-cut is irrational. Sometimes, by $(C, D)$-cut we understand the following set of formulas : $\{c \leq x \leq d : c \in C, d \in D\}$.

**Theorem 1.1** *(B. Kulpeshov [K]) Let $M$ be a linearly ordered structure. Then $M$ is w.o.m. iff the following conditions hold in M:*

(i) *Any $(C,D)$-cut in $M$ has at most two complete one-types over $M$ which extend (C,D)-cut,*

(ii) *If a (C,D)-cut in $M$ has two complete one-types over $M$ which extend (C,D)-cut, then each realization of these types is convex set in any elementary extension of $M$.*

**Definition 1.5** [LP] Let $p \in S_1(A)$. We say that $p$ is rational to the right (left) if there is $a \in A$ ($b \in A$) such that for any $\alpha \models p$ the following is true:

$$\forall \beta[\alpha < \beta < a \Rightarrow tp(\beta/A) = p]$$

$$(\forall \beta[b < \beta < \alpha \Rightarrow tp(\beta/A) = p]).$$

**Definition 1.6** (i) Let $p \in S_1(A)$ be non-isolated. We say that $p$ is quasirational to the right (left) if there is a formula $U(x, \bar{a}),\ \bar{a} \in A$ such that for any $\alpha \models p$ the following is true:

$$\forall \beta[\alpha < \beta \wedge M' \models U(\beta, \bar{a}) \Rightarrow tp(\beta/A) = p]$$

$$(\forall \beta[\alpha > \beta \wedge M' \models U(\beta, \bar{a}) \Rightarrow tp(\beta/A) = p]).$$

(ii) A non-isolated, non-quasirational type $p$ is irrational.

**Note 1.4** Let $p \in S_1(A)$. Then the following is true:

(i) If $p$ is rational then it is quasirational.

(ii) If $p$ is quasirational to the right (left) then $U(M', \bar{a})^+ = p(M')^+$ ($U(M', \bar{a})^- = p(M')^-$).

(iii) $p$ is quasirational to the right and to the left iff $p$ is isolated.

**Note 1.5** By Theorem of Kulpeshov we have:

(i) Let $p \in S_1(M)$. Then $p$ is quasirational to the right (left) iff there is a formula $U(x, \bar{a}),\ \bar{a} \in M$ such that the following set of formulas

$$\{x > c : M \models U(c, \bar{a}),\ c \in M\} \cup \{U(x, \bar{a})\}$$

$$(\{x < c : M \models U(c, \bar{a}),\ c \in M\} \cup \{U(x, \bar{a})\})$$

has unique completion $p$ from $S_1(M)$.

(ii) Let $p \in S_1(M)$. Then $p$ is irrational iff $p$ is determinated by irrational cut $(C_p, D_p)$.

**Definition 1.7** We say that formula $F(x, y, \bar{a})$ $(G(x, y, \bar{a})), \bar{a} \in A$ is convex to the right (left) if

$$M' \models \forall y \forall x[(F(x, y, \bar{a}) \to (y \leq x \wedge \forall z(y \leq z \leq x \to F(z, y, \bar{a})]$$

$$(M' \models \forall y \forall x[(G(x, y, \bar{a}) \to (x \leq y \wedge \forall z(x \leq z \leq y \to G(z, y, \bar{a})])$$

**Definition 1.8** Let $p \in S_1(A)$. We say that a formula $\Phi(x, y, \bar{a}), \bar{a} \in A$ is p-stable if $\forall \alpha \models p,\ \exists \gamma_1, \gamma_2 \models p$ such that

$$\gamma_1 < \Phi(M', \alpha, \bar{a}) < \gamma_2$$

**Note 1.6** Let $F_1(x, y, \bar{a}), F_2(x, y, \bar{b}), \bar{a}, \bar{b} \in A$ be two 2-$A$-formulas, $p \in S_1(A)$. Then if $\exists \alpha \in p(M')$ such that $F_1(M', \alpha, \bar{a}) \subset F_2(M', \alpha, \bar{b})$ we have:

$$\forall \beta \models p\ [F_1(M', \beta, \bar{a}) \subset F_2(M', \beta, \bar{b})].$$

**Definition 1.9** Let $p \in S_1(A), F_1(x, y, \bar{a}), F_2(x, y, \bar{b}), \bar{a}, \bar{b} \in A$ be two $p$-stable convex to right (left) 2-$A$-formulas. We say that $F_2(x, y, \bar{b})$ is greater than $F_1(x, y, \bar{a})$ if $\exists \alpha \in p(M')$ $(\equiv \forall \alpha \in p(M')),\ F_1(M', \alpha, \bar{a}) \subset F_2(M', \alpha, \bar{b})$.

**Note 1.7** Let $p \in S_1(A)$. The set of all non-equivalent p-stable convex to the right (left) 2-$A$-formulas is totally ordered.

**Definition 1.10** Let $p \in S_1(A)$ be a non-algebraic type. We say that $p$ is semi-quasisolitary to the right (left) if there is the greatest p-stable convex to the right (left) 2-A-formula $F(x, y, \bar{a})$ $(G(x, y, \bar{a})),\ \bar{a} \in A$.

**Note 1.8** Let $p \in S_1(A)$ be semi-quasisolitary to the right (left). If $\beta > \alpha$ $(\alpha > \beta)$ and $tp(\beta/A) = tp(\alpha/A) = p,\ M' \models \neg F(\beta, \alpha, \bar{a})$ $(M' \models \neg G(\beta, \alpha, \bar{a}))$
then for any formula $\Phi(x, \alpha, \bar{c}), \bar{c} \in A$ the following holds:

$$[M' \models \Phi(\beta, \alpha, \bar{c}) \Rightarrow \forall \beta_1 \in F(M', \alpha, \bar{a})^+ \cap p(M')$$

$$(\forall \beta_1 \in G(M', \alpha, \bar{a})^- \cap p(M'))M' \models \Phi(\beta_1, \alpha, \bar{c}))].$$

**Definition 1.11** Let $p \in S_1(A)$. We say that $p$ is quasisolitary if it is semi-quasisolitary to the right and to the left.

**Definition 1.12** Let $p \in S_1(A)$. We say that $p$-stable convex to the right (left) 2-A-formula $F(x, y, \bar{a})$ is locally $p$-decreasing ($p$-increasing) if there are $\alpha_1, \alpha_2 \in p(M')$ such that

$$M' \models \exists x(F(x, \alpha_2, \bar{a}) \wedge \neg F(x, \alpha_1, \bar{a}) \wedge F(\alpha_1, \alpha_2, \bar{a}) \wedge x > \alpha_1).$$

$$(M' \models \exists x(F(x, \alpha_2, \bar{a}) \wedge \neg F(x, \alpha_1, \bar{a}) \wedge F(\alpha_1, \alpha_2, \bar{a}) \wedge x < \alpha_1)).$$

**Lemma 1.1** *Let $p \in S_1(A)$ be a non-algebraic type. Then*

*(i) If $F(x, y, \bar{a})$ is the greatest p-stable convex to the right (left) 2-A-formula, then $F(x, y, \bar{a})$ is non-locally p-decreasing (p-increasing).*

*(ii) If p is semi-quasisolitary then it is quasisolitary.*

*Proof.* Suppose that $F(x,y,\bar{a})$ is locally $p$-decreasing. Then there are $\alpha,\beta,\gamma\in p(M')$ such that
$$M'\models \alpha<\beta<\gamma\wedge F(\gamma,\alpha,\bar{a})\wedge\neg F(\gamma,\beta,\bar{a})$$
Let $K(y,\alpha,\bar{a}):=\exists x(F(x,\alpha,\bar{a})\wedge F(y,x,\bar{a}))$. Because $F(x,y,\bar{a})$ is $p$-stable we have $F(y,\alpha,\bar{a})\equiv K(y,\alpha,\bar{a})$.

So, $F(M',\gamma,\bar{a})\subset F(M',\alpha,\bar{a})$ and $F(M',\alpha,\bar{a})\backslash F(M',\gamma,\bar{a})\neq\emptyset$.
Let $\theta(\alpha,\beta,\gamma,\bar{a}):=\alpha<\beta<\gamma\wedge F(\gamma,\alpha,\bar{a})\wedge\neg F(\gamma,\beta,\bar{a})\wedge\forall x(F(x,\gamma,\bar{a})\to F(x,\alpha,\bar{a}))$. Then $M'\models\theta(\alpha,\beta,\gamma,\bar{a})$.

Let $f\in \mathrm{Aut}_A(M')$ such that $f(\gamma)=\alpha$. Then for any $n<\omega$
$M'\models\theta(f^n(\alpha),f^n(\beta),f^n(\gamma),\bar{a})$.

So, $M'\models\theta(f^n(\alpha),f^n(\beta),f^{n-1}(\alpha),\bar{a})$ and
$F(M',f^n(\alpha),\bar{a})\supset F(M',f^{n-1}(\alpha),\bar{a}), f^{n-1}(\alpha)\notin F(M',f^n(\beta),\bar{a})$,
$f^n(\alpha)<f^n(\beta)<f^{n-1}(\alpha)$.
Thus, $\forall n<\omega[f^n(\alpha)\in F(\gamma,M',\bar{a})$ and $f^{n-1}(\beta)\notin F(\gamma,M',\bar{a})]$. It contradicts to weak o-minimality of $M'$. So, $F(x,y,\bar{a})$ is a non-locally $p$-decreasing.

(ii) Let $F(x,y,\bar{a})$ from Definition 1.10.

By (i) $\forall\alpha,\beta\in p(M'),\alpha<\beta$ the following holds:
If $\beta\in F(M',y,\bar{a})\Rightarrow M'\models\forall x(x\geq\beta\to(F(x,\alpha,\bar{a})\leftrightarrow F(x,\beta,\bar{a})))$

Consider $G(y,\alpha,\bar{a}):=y\leq\alpha\wedge F(\alpha,y,\bar{a})\wedge\forall x(x\geq\alpha\to(F(x,\alpha,\bar{a})\leftrightarrow F(x,y,\bar{a})))$.
Let $G_0(M',\alpha,\bar{a}):=p(M')\cap G(M',\alpha,\bar{a})$.

We claim that $G_0(y,x,\bar{a})$ is the greatest convex to the left $p$-stable formula. If it is not the greatest convex to the left $p$-stable formula then there is $\Phi(x,y,\bar{c}),\bar{c}\in A$, $p$-stable formula such that $\exists\gamma_1,\gamma_2\in p(M')$,
$$\gamma_1<\Phi(M',\alpha,\bar{c})<G_0(M',\alpha,\bar{c}),\gamma_2\in\Phi(M',\alpha,\bar{c})$$
So, we have $\alpha\notin F(M',\gamma_1,\bar{a})$, $\alpha\notin F(M',\gamma_2,\bar{a})$. Notice,
$\alpha\notin F(M',\gamma_2,\bar{a})\Rightarrow\forall\alpha_1\in[F(M',\gamma_2,\bar{a})^+\cap p(M')],\gamma_2\in\Phi(M',\alpha_1,\bar{c})$, by Note 1.8.
$\alpha\notin F(M',\gamma_1,\bar{a})\Rightarrow\forall\alpha_1\in[F(M',\gamma_1,\bar{a})^+\cap p(M')],\gamma_1\notin\Phi(M',\alpha_1,\bar{c})$, by Note 1.8.

Let $\alpha_0\in[(F(M',\gamma_2,\bar{a})\cup F(M',\gamma_1,\bar{a}))^+\cap p(M')]$, $f\in Aut_A(M')$ such that $f(\gamma_1)=\gamma_2$. Then $f(\alpha_0)\in[(F(M',\gamma_2,\bar{a})\cup F(M',\gamma_1,\bar{a}))^+\cap p(M')]$

We have $M'\models\neg\Phi(\gamma_1,\alpha_0,\bar{c})\Rightarrow M'\models\neg\Phi(\gamma_2,f(\alpha_0),\bar{c})$. Contradiction.

Thus, $G_0(y,x,\bar{a})$ is the greatest convex to the left $p$-stable formula, and the formula $E(x,y,\bar{a}):=F(x,y,\bar{a})\vee G_0(x,y,\bar{a})$ is equivalence relation.

So, the type $p$ is quasisolitary. □

Further this equivalence $E(x,y,\bar{a})$ we denote by $E_p(x,y,\bar{c}_p)$.

**Note 1.9** (i) Let $p\in S_1(A)$, then for any convex to the right (left) $p$-stable 2-$A$-formula $F(x,y,\bar{a})$ there is a $F_1(x,y,\bar{a})$ is a convex to the right (left) $p$-stable formula such that $\forall\alpha\in p(M'),F(M',\alpha,\bar{a})\subset F_1(M',\alpha,\bar{a})$ and $F_1(x,y,\bar{a})$ is non locally $p$-decreasing ($p$-increasing).

(ii) V. Verbovsky constructed an example of w.o.m. theory with 1-type $p$ and $F(x,y,\bar{a})$ that is $p$-stable convex to the right, locally $p$-decreasing formula.

(iii) Let $p\in S_1(A)$ be quasisolitary. A set of all $E_p(x,y,\bar{c}_p)$-classes of equivalence in $M'$ is densely ordered. A set of representatives of $E_p(x,y,\bar{c}_p)$-classes in $M'$ is ordered 2-indiscernible.

(iv) Let $p \in S_1(A)$, $\psi(x, y, \bar{b})$, $\bar{b} \in A$ be p-stable, $\phi(x, \bar{\alpha}), \bar{\alpha} \in M'$ such that $\exists \mu_1, \mu_2 \in p(M')$, $\mu_1 < \phi(M', \bar{\alpha}) < \mu_2$. Then $\exists \mu_1', \mu_2' \in p(M')$ such that for the formula

$$H_{\phi,\psi}(x, \bar{\alpha}, \bar{b}) := \exists y \ (\phi(y, \bar{\alpha}) \wedge \psi(x, y, \bar{b}))$$

the following is true:

$$\mu_1' < H_{\phi,\psi}(M', \bar{\alpha}, \bar{b}) < \mu_2'$$

*Proof.* (i) Suppose, $F(x, y, \bar{a})$ is convex to the right, $p$-stable, locally $p$-decreasing. Let $\alpha, \beta, \gamma \in p(M')$, $\alpha < \beta < \gamma$, $\beta \in F(M', \alpha, \bar{a})$, $\gamma \in F(M', \alpha, \bar{a}) \setminus F(M', \beta, \bar{a})$, $f \in \mathrm{Aut}_A(M')$ such that $f(\alpha) = \gamma$. Then $\exists \beta' \in p(M')$ such that $f(\beta') = \beta$. So, $\alpha \notin F(M', \beta', \bar{a})$ and $\beta \notin F(M', \beta', \bar{a}), \beta' \notin F(\beta, M', \bar{a})$. Let $F_0(\beta, M', \bar{a})$ be a maximal $< \bar{a}, \beta >$-definable convex of $F(\beta, M', \bar{a})$ such that $\alpha \in F_0(\beta, M', \bar{a})$.
Then, $F_0(\beta, M', \bar{a}) \subset p(M')$. From proof of Lemma 1.1(i) follows that $\forall \gamma[\alpha < \gamma < \beta \Rightarrow F(\beta, \gamma, \bar{a})]$.

So, $F_1(x, y, \bar{a}) := \exists z[F_0(y, z, \bar{a}) \wedge F(x, z, \bar{a}) \wedge x \geq y]$. It is clear that $F_1(x, y, \bar{a})$ satisfies the condition of Note 1.9 (i).

**Convention 1.2** Further throughout in this paper we assume that any convex to the right (left) $p$-stable formula is non locally $p$-decreasing ($p$-increasing)

**Definition 1.13** [B] Let $p \in S_1(A)$ be quasisolitary. We say that $p$ is solitary if $E(x, y, \bar{a}) \equiv (x = y)$

**Definition 1.14** ([B] for o-minimal) Any non-algebraic, non-quasisolitary type $p \in S_1(A)$ is social.

**Corollary 1.1** *Let $p \in S_1(A)$ be non-algebraic. Then $p$ is quasisolitary iff family of all convex to the right $p$-stable $2-A$-formulas has supremum iff family of all convex to the left $p$-stable formulas has supremum.*

**Note 1.10** (i) If $T$ is o-minimal, then each quasisolitary type is solitary (uniquely realizable [LS]).

(ii) There exist the following six essential kinds of non-algebraic 1-types over set of models with w.o.m. theory:

1–2) isolated (quasisolitary, social);

3–4) quasirational (quasisolitary, social);

5–6) irrational (quasisolitary, social).

**Lemma 1.2** *Let $p, q \in S_1(A)$ be non-algebraic, $\alpha \in p(M')$, $\Phi(x, y, \bar{a}), \bar{a} \in A$ such that $\Phi(M', \alpha, \bar{a}) \subset q(M')$ and $\exists \gamma \in q(M')$, $\gamma \notin \Phi(M', \alpha, \bar{a})$ (i.e. $p$ isolates $q$). Then the following is true:*

*(i) $p$ is isolated iff $q$ is isolated*

*(ii) $p$ is quasirational iff $q$ is quasirational*

*(iii) $p$ is irrational iff $q$ is irrational*

*(iv) $p$ is quasisolitary iff $q$ is quasisolitary.*

**Note 1.11** (i) $\forall\beta \models p,\ \Phi(M',\beta,\bar{a}) \subset q(M')$.

(ii) If $q$ is irrational to the left (right), i.e. $\neg\exists C(x,\bar{c})$ such that $C(M',\bar{c})^- \cup C(M',\bar{c}) = q(M')^-$ then for any formula $\Phi(x,\bar{\beta}), \bar{\beta} \in M'$ if $\Phi(M',\bar{\beta}) \subseteq q(M')$ then $\exists\gamma \in q(M')$ such that $\gamma < \Phi(M',\bar{\beta})(\Phi(M',\bar{\beta}) < \gamma)$.

*Proof of Lemma 1.2.* (i) If $p$ is isolated then $q$ is isolated. It is true for any types of arbitrary theory. If $q$ is isolated then $q(M') = U(M',\bar{b})$ where $U(M',\bar{b})$ is a complete formula of $q$. Let $\gamma,\beta \in U(M',\bar{b})$ such that $\gamma \notin \Phi(M',\alpha,\bar{a}), \beta \in \Phi(M',\alpha,\bar{a})$. Let $f \in Aut_A(M')$ such that $f(\gamma) = \beta$. Then $\beta \notin \Phi(M', f(\alpha),\bar{a})$. So, $\alpha \in \Phi(\beta, M',\bar{a}) \cap p(M')$ and $f(\alpha) \in \neg\Phi(\beta, M',\bar{a}) \cap p(M')$.

Because $T$ is weakly o-minimal there is a formula $H(x,\beta,\bar{a})$ such that $H(M',\beta,\bar{a}) < \neg H(M',\beta,\bar{a})$ or $\neg H(M',\beta,\bar{a}) < H(M',\beta,\bar{a}),\ \alpha \in H(M',\beta,\bar{a})$, $f(\alpha) \in \neg H(M',\beta,\bar{a})$. Then the following formula $K(x,\bar{a},\bar{b})$ is complete for $p$:

$$K(x,\bar{a},\bar{b}) := \exists y_1,\ \exists y_2\ [U(y_1,\bar{b}) \wedge U(y_2,\bar{b}) \wedge H(x,y_1,\bar{a}) \wedge \neg H(x,y_2,\bar{a})].$$

So, $p$ is isolated.

(ii) ($\Rightarrow$) Suppose $p$ is quasirational to the right with a formula $U(x,\bar{b}),\ \bar{b} \in A$ from Definition 1.6. Let

$$L_q(t,\bar{a},\bar{b}) := \exists z[U(z,\bar{b}) \wedge \forall y((y > z \wedge U(y,\bar{b})) \rightarrow \forall x(\Phi(x,y,\bar{a}) \rightarrow t < x))].$$

Then for any formula in one free variable $D(x,\bar{d}),\ \bar{d} \in A$ the following is true:

$$[D(M',\bar{d}) < q(M') \Rightarrow D(M',\bar{d}) \subset L_q(M',\bar{a},\bar{b})] \wedge$$

$$\wedge [q(M') < D(M',\bar{d}) \Rightarrow D(M',\bar{d}) \cap L_q(M',\bar{a},\bar{b}) = \emptyset].$$

Thus, if $L_q(t,\bar{a},\bar{b}) \in q$ then $q$ is quasirational to the right and

$$q(M')^+ = L_q(M',\bar{a},\bar{b})^+ \text{ and if } \neg L_q(t,\bar{a},\bar{b}) \in q$$

then $q$ is quasirational to the left and

$$q(M')^- = L_q(M',\bar{a},\bar{b})^-,\ q(M') \subset \neg L_q(M',\bar{a},\bar{b}).$$

As $q$ is complete type, so either $L_q(t,\bar{a},\bar{b}) \in q$ or $\neg L_q(t,\bar{a},\bar{b}) \in q$.

($\Leftarrow$) Suppose $q$ is quasirational to the right with a formula

$$U(x,\bar{b}), \bar{b} \in A, U(M',\bar{b})^+ = q(M')^+.$$

We will show an existence of $2-\bar{a}$-formula $\Phi_0(x,y,\bar{a})$ such that

$$\forall\gamma \in q(M'),\ \Phi_0(\gamma, M',\bar{a}) \subset p(M').$$

By the preceding it means that $p$ is quasirational.

Consider two cases:

a) There are $\gamma_1,\gamma_2 \in q(M')$ such that $\gamma_1 < \Phi(M',\alpha,\bar{a}) < \gamma_2$.

b) $\Phi(M',\alpha,\bar{a})^+ = q(M')^+$.

a) Let $f \in \mathrm{Aut}_A(M')$ such that $f(\gamma_1) = \gamma_2$.
If $\alpha_1, \alpha_2 \in p(M')$ such that $f(\alpha_1) = \alpha, f(\alpha) = \alpha_2$, then we have

$$\Phi(M', \alpha_1, \bar{a}) < \gamma_1 < \Phi(M', \alpha, \bar{a}) < \gamma_2 < \Phi(M', \alpha_2, \bar{a}).$$

Let $\gamma \in \Phi(M', \alpha, \bar{a})$ then $\alpha \in \Phi(\gamma, M', \bar{a})$ and ($\alpha \in (\alpha_1, \alpha_2)$ or $\alpha \in (\alpha_2, \alpha_1)$).

Let $\Phi_0(\gamma, M', \bar{a})$ be a maximal convex $< \gamma, \bar{a} >$-subformula of $\Phi(\gamma, y, \bar{a})$ such that $\alpha \in \Phi_0(\gamma, M', \bar{a})$. Clearly that $\Phi_0(\gamma, M', \bar{a}) \subset p(M')$.

b) By Note 1.11(ii) there is $\gamma \in q(M')$ such that $\gamma < \Phi(M', \alpha, \bar{a})$.

Consider the following formula

$$\Phi_0(\gamma, y, \bar{a}, \bar{b}) := \forall z(\Phi(z, y, \bar{a}) \to \gamma < z \wedge U(z, \bar{a})) \wedge \exists z \Phi(z, y, \bar{a})).$$

So, $\alpha \in \Phi_0(\gamma, M', \bar{a}, \bar{b})$. If $\Phi_0(\gamma, M', \bar{a}, \bar{b}) \not\subset p(M')$ then there is $D(x, \bar{d})$, $\bar{d} \in A$ such that $D(M', \bar{d}) \cap p(M') = \emptyset$ and $D(M', \bar{d}) \subset \Phi_0(\gamma, M', \bar{a}, \bar{b})$. Then for a formula $H(x, \bar{d}, \bar{a}) := \exists y(D(y, \bar{d}) \wedge \Phi(x, y, \bar{a}))$ we have $\gamma < H(M', \bar{d}, \bar{a}) \subset q(M')$. Contradiction, because $\bar{d}, \bar{a} \in A$ and $\gamma \in q(M')$, $\forall \gamma_1 \in H(M', \bar{d}, \bar{a}), tp(\gamma/A) \neq tp(\gamma_1/A)$.

Thus, $\Phi_0(\gamma, M', \bar{a}, \bar{b}) \subset p(M')$.

(iii) follows from (i) and (ii).

(iv)

**Claim 1.1** *If $p$ isolates $q$ by a formula $\Phi(x, y, \bar{a})$, $\bar{a} \in A$ such that $\exists \alpha \in p(M') \exists \gamma_1 \gamma_2 \in q(M')$, $\gamma_1 < \Phi(M', \alpha, \bar{a}) < \gamma_2, \Phi(M', \alpha, \bar{a}) \neq \emptyset$ then $\exists \Phi_0(x, y, \bar{a})$ such that $\forall \beta \in q(M') \exists \mu_1, \mu_2 \in p(M')$*

$$\Phi_0(\beta, M', \bar{a}) \neq \emptyset, \;\; \mu_1 < \Phi_0(\beta, M', \bar{a}) < \mu_2$$

*and $p$ is quasisolitary iff $q$ is quasisolitary.*

*Proof of Claim 1.1* Let $f \in \mathrm{Aut}_A(M')$ such that $f(\gamma_1) = \gamma_2$. Let $\alpha_1 := f^{-1}(\alpha), \alpha_2 = f(\alpha)$. We have

$$f^{-1}(\gamma_1) < \Phi(M', \alpha_1, \bar{a}) < \gamma_1 < \Phi(M', \alpha, \bar{a}) < \gamma_2 < \Phi(M', \alpha_2, \bar{a}) < f(\gamma_2).$$

If $\alpha < f(\alpha)$ then $f^{-1}(\alpha) < \alpha$. We have $\alpha_1 < \alpha < \alpha_2$ or $\alpha_2 < \alpha < \alpha_1$.

Let $\beta \in \Phi(M', \alpha, \bar{a})$. Consider the formula $\Phi(\beta, y, \bar{a})$. Let $\Phi_0(\beta, y, \bar{a})$ be a convex subformula of $\Phi(\beta, y, \bar{a})$ such that $\alpha \in \Phi_0(\beta, M', \bar{a})$. We have $\alpha_1 < \Phi_0(\beta, M', \bar{a}) < \alpha_2$ or $\alpha_2 < \Phi_0(\beta, M', \bar{a}) < \alpha_1$.

Suppose $p$ be quasisolitary. Let $E_p(x, y, \bar{c}_p)$, $\bar{c}_p \in A$ be a formula from the proof of Lemma 1.1(ii) which defines quasisolitarity of $p$. Let

$$E(z, t, \bar{c}_p, \bar{a}) := \exists y \exists x(\Phi_0(t, y, \bar{a}) \wedge E_p(x, y, \bar{c}_p) \wedge \Phi(z, x, \bar{a})), \;\; \gamma \in \Phi(M', \alpha, \bar{a}).$$

We claim that for any formula $H(x, y, \bar{d})$, $\bar{d} \in A$ the following is true:
If $H(x, y, \bar{d})$ is $q$-stable then $H(M', \gamma, \bar{d}) \subset E(M', \gamma, \bar{c}_p, \bar{a})$.

In opposite case we can choose $q$-stable $H(x, y, \bar{d})$ such that $H(M', \gamma, \bar{d}) \cap E(M', \gamma, \bar{c}_p, \bar{a}) = \emptyset$.

Consider the folowing formula:

$$K(x, \alpha, \bar{a}, \bar{d}) := \exists y \exists z(\Phi(y, \alpha, \bar{a}) \wedge H(z, y, \bar{d}) \wedge \Phi_0(z, x, \bar{a})).$$

By Note 1.9(iv), $K(x, y, \bar{a}, \bar{d})$ is $p$-stable because

- $\gamma_1 < \Phi(M', \alpha, \bar{a}) < \gamma_2$,
- $H(z, y, \bar{d})$ is $q$-stable,
- $\forall \beta \in q(M') \exists \delta_1, \delta_2 \in p(M'),\ \delta_1 < \Phi_0(\beta, M', \bar{a}) < \delta_2$.

Let $\gamma_0 \in H(M', \gamma, \bar{d})$ then $\gamma_0 \notin E(M', \gamma, \bar{c}_p, \bar{a})$ and $\Phi_0(\gamma_0, M', \bar{a}) \cap E_p(M', \alpha, \bar{c}_p) = \emptyset$. Let $\mu \in \Phi_0(\gamma_0, M', \bar{a}) \subset K(M', \alpha, \bar{a}, \bar{d})$ then $\mu \notin E_p(M', \alpha, \bar{c}_p)$. Contradiction. Thus $q$ is quasisolitary. By the same consideration from quasisolitarity of $q$ follows the quasisolitarity of $p$. So, Claim 1.1 is proved. □

By Note 1.11 (ii), Lemma 1.2 (i-iii) we must consider the following case:

- $p, q$ are quasirational or isolated.
- There exists $\Phi(x, y, \bar{a}), \bar{a} \in A$, such that $\forall \alpha \in p(M'), \Phi(M', \alpha, \bar{a}) \subset q(M')$, $\Phi(M', \alpha, \bar{a})^+ = q(M')^+$ or $\Phi(M', \alpha, \bar{a})^- = q(M')^-$.
- $\forall \theta(x, y, \bar{b}),\ \bar{b} \in A,\ \forall \alpha \in p(M')$
$$\theta(M', \alpha, \bar{b}) \subset q(M') \Rightarrow \theta(M', \alpha, \bar{b}) = \Phi(M', \alpha, \bar{a})].$$

We claim that in this case $p, q$ are quasisolitary. Without loss of generality we assume $\Phi(M', \alpha, \bar{a}) = q(M')^+$ and $p$ is quasirational to the right.

Notice that,

$$\forall \beta, \alpha \in p(M')[\beta > \alpha \Rightarrow \Phi(M', \beta, \bar{a}) \subseteq \Phi(M', \alpha, \bar{a})].$$

Let $K(z, y, \bar{a}) := \forall t\ (z \leq t \leq y \rightarrow \Phi(M', z, \bar{a}) \subseteq \Phi(M', \alpha, \bar{a}))$. We claim that $K(z, y, \bar{a})$ is maximal convex to the left $p$-stable formula. If not there is $p$-stable $G(x, y, \bar{c}),\ \bar{c} \in A$, such that

- $G(M', \alpha, \bar{c}) < K(M', \alpha, \bar{a})$
- $\exists \mu \in G(M', \alpha, \bar{c}), \Phi(M', \alpha, \bar{a}) \subset \Phi(M', \mu, \bar{a})$

Let $\Phi_1(x, \alpha, \bar{a}, \bar{c}) := \exists y(G(y, \alpha, \bar{c}) \wedge \Phi(x, y, \bar{a})) \wedge \neg \Phi(x, \alpha, \bar{a})$. By Note 1.9 (iv) $\Phi_1(M', \alpha, \bar{a}, \bar{c}) \subset q(M')$, by construction $\Phi_1(M', \alpha, \bar{a}, \bar{c}) \neq \Phi(M', \alpha, \bar{a})$. Contradiction. So, $p$ is quasisolitary. The same consideration of $\Phi_0(x, y, \bar{a}, \bar{b})$ from Lemma 1.2(ii), $(\Leftarrow)$(b) gives us the quasisolitarity of $q$. □

**Corollary 1.2** *Let $p, q \in S_1(A)$. Then $q$ isolates $p$ iff $p$ isolates $q$.*

# 2 Orthogonality of 1-types over sets, neighbourhoods of sets in 1-types

**Definition 2.1** Let $p \in S_1(A), B \subset M'$ such that $M'$ is $|B \cup A|^+$-saturated. Then a neighbourhood of set $B$ in the type $p$ is the following set:

$$V_p(B) := \{\gamma \in M' | \exists \gamma_1, \gamma_2 \in p(M'), \exists H(x, \bar{b}, \bar{c}), \bar{b} \in B, \bar{c} \in A,$$
$$\gamma_1 < H(M', \bar{b}, \bar{c}) < \gamma_2,\ \gamma \in H(M', \bar{b}, \bar{c})\}$$

. Let $\bar{\alpha} =< \alpha_1, \alpha_2, \ldots, \alpha_k >$ then $V_p(\bar{\alpha}) := V_p(\{\alpha_1, \alpha_2, \ldots, \alpha_k\})$.

**Note 2.1** Let $p \in S_1(A)$.

(i) Let $\beta \in p(M')$. Then $|V_p(\beta)| = 1$ iff $p$ is solitary.

(ii) $p$ is quasisolitary iff $\exists \beta \in p(M')$, $V_p(\beta)$ is $(A \cup \beta)$-definable set. (In fact, $E_p(M', \beta, \bar{c}_p) = V_p(\beta)$).

**Lemma 2.1** *Let $p \in S_1(A)$, $B \subset M'$ such that $M'$ is $|A \cup B|^+$-saturated. Then the following holds:*

*(i) $V_p(B)$ is convex or empty.*

*(ii) Let $p$ be irrational. Then $\forall \bar{\alpha} \in M' \setminus M$ $\forall H(x, \bar{\alpha}, \bar{a})$, $\bar{a} \in A$*

$$[H(M', \bar{\alpha}, \bar{a}) \neq \emptyset,\ H(M', \bar{\alpha}, \bar{a}) \subseteq p(M') \Rightarrow H(M', \bar{\alpha}, \bar{a}) \subseteq V_p(\bar{\alpha})]$$

.

*(iii) $V_p(B) \neq \emptyset \Rightarrow \exists \gamma_1, \gamma_2 \in p(M')$, $\gamma_1 < V_p(B) < \gamma_2$.*

*(iv) $\forall \beta \in p(M')$ $[V_p(B) \neq \emptyset,\ \beta \notin V_p(B) \Rightarrow$*

$$\Rightarrow \forall q \in S_1(A),\ V_q(\beta) \cap V_q(B) = \emptyset].$$

*(v) $\forall \alpha \in p(M') :\ V_p(B) < \alpha\ \exists \alpha_0 \in p(M')[V_p(B) < \alpha_0 < V_p(\alpha)]$.*

*Proof.* (i) By Definition 2.1.

(ii) By Definition 2.1 and by Note 1.11(ii).

(iii) By Definition 2.1 and by Theorem of compactness.

(iv) Suppose that there exists $q \in S_1(A)$ such that $V_q(\beta) \cap V_q(B) \neq \emptyset$. Then there exists $\Phi(x, \beta, \bar{b})$, $\bar{b} \in A$, $\exists H(x, \bar{\alpha}, \bar{c})$, $\bar{\alpha} \in B$, $\bar{c} \in A$ such that
$\exists \gamma \in \Phi(M', \beta, \bar{b}) \cap H(M', \bar{\alpha}, \bar{c}), \Phi(M', \beta, \bar{b}) \subset V_p(\beta),\ H(M', \bar{\alpha}, \bar{c}) \subset V_p(B)$.

By Definition 2.1 $\exists \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in q(M')$ such that
$\gamma_1 < \Phi(M', \beta, \bar{b}) < \gamma_2,\ \gamma_3 < H(M', \bar{\alpha}, \bar{c}) < \gamma_4$.

Let $\mu_1 = \min\{\gamma_1, \gamma_3\}$, $\mu_2 = \max\{\gamma_2, \gamma_4\}$, $f \in \mathrm{Aut}_A(M')$ such that $f(\mu_2) = \mu_1$.

Let $\beta_1 := f(\beta)$, $\beta_2 := f^{-1}(\beta)$. So we have

$$\Phi(M', \beta_1, \bar{b}) < \mu_1 < H(M', \bar{\alpha}, \bar{c}) < \mu_2 < \Phi(M', \beta_2, \bar{b})$$

and ($\beta_1 < \beta < \beta_2$ or $\beta_2 < \beta < \beta_1$).

Let $K(y, \bar{\alpha}, \bar{b}, \bar{c}) := \exists x (H(x, \bar{\alpha}, \bar{c}) \wedge \Phi(x, y, \bar{b}))$. Then $\beta \in K(M', \bar{\alpha}, \bar{b}, \bar{c})$ and $\beta_1, \beta_2 \notin K(M', \bar{\alpha}, \bar{b}, \bar{c})$.

Let $K_1(x, \bar{\alpha}, \bar{b}, \bar{c})$ be a convex subformula of $K(x, \bar{\alpha}, \bar{b}, \bar{c})$ such that
$\beta \in K_1(M', \bar{\alpha}, \bar{b}, \bar{c})$. Thus

$$\beta_1 < K_1(M', \bar{\alpha}, \bar{b}, \bar{c}) < \bar{\beta}_2 \mathrm{or} \bar{\beta}_2 < K_1(M', \bar{\alpha}, \bar{b}, \bar{c}) < \beta_1.$$

Thus $\beta \in V_p(\bar{\alpha})$ and $\beta \in V_p(B)$. Contradiction.

(v) Consider two cases:

a) $\exists U(M', \bar{\beta}, \bar{a}), \bar{\beta} \in B, \bar{a} \in A, U(M', \bar{\beta}, \bar{a})^+ = V_p(B)^+$.

b) $\equiv \neg a$).

a) If $p$ is social, then it is true by Theorem of compactness. Let $p$ be quasisolitary. Consider the following formula

$$K(x, \bar{\beta}, \bar{a}, \bar{c}_p) := U(M', \bar{\beta}, \bar{a}) < x \wedge \ \forall y \ (U(M', \bar{\beta}, \bar{a}) < y < x \rightarrow E_p(x, y, \bar{c}_p).$$

If there is not such $\alpha_0$, then $K(M', \bar{\beta}, \bar{a}, \bar{c}_p) = E_p(M', \alpha, \bar{c}_p)$. So $E_p(M', \alpha, \bar{c}_p) \subseteq V_p(B)$ and $\alpha \in V_p(B)$. Contradiction.

b) It follows from Theorem of compactness. □

**Note 2.2** Let $p, q \in S_1(A)$.

(i) If $\alpha \in p(M'), \beta \in q(M')$ then $\alpha \in V_p(\beta)$ iff $\beta \in V_q(\alpha)$ iff $V_p(\alpha) = V_p(\beta)$ iff $V_q(\alpha) = V_q(\beta)$.

(ii) If $\alpha_1, \alpha_2, \alpha_3 \in p(M')$ such that $\alpha_1 < V_p(\alpha_2) < \alpha_3, V_q(\alpha_1) \neq \emptyset$ then $V_q(\alpha_1) < V_q(\alpha_2) < V_q(\alpha_3)$ or $V_q(\alpha_3) < V_q(\alpha_2) < V_q(\alpha_1)$.

*Proof.* (i) is immediate corollary of proof of Lemma 2.1(iv).

(ii) By Lemma 2.1(iv) the following is true:

$$\forall q \in S_1(A), V_q(\alpha_i) \cap V_q(\alpha_j) = \emptyset, i \neq j \in 1, 2, 3 \text{ and } V_p(\alpha_1) < V_p(\alpha_2) < V_p(\alpha_3).$$

Then

$$tp(\alpha_1/A \cup \alpha_3) = tp(\alpha_2/A \cup \alpha_3), \ tp(\alpha_2/A \cup \alpha_1) = tp(\alpha_3/A \cup \alpha_1).$$

Suppose there is $q \in S_1(A)$ such that $V_q(\alpha_1) < V_q(\alpha_3) < V_q(\alpha_2)$. Let $H(x, y, \bar{a}), \bar{a} \in A$ be a formula such that $H(M', \alpha_1, \bar{a}) \subset V_q(\alpha_1)$. Consider the following formula:

$$G(y, \alpha_3, \bar{a}) := \ y < \alpha_3 \wedge \ \forall x \ \forall z \ ((H(x, y, \bar{a}) \wedge H(z, \alpha_3, \bar{a})) \rightarrow z < x).$$

Then $\alpha_1 \notin G(M', \alpha_3, \bar{a}), \alpha_2 \in G(M', \alpha_3, \bar{a})$. Then $\alpha_2 \in V_p(\alpha_3)$. Contradiction. Consideration of other cases is the same. □

For notations of our notions we will use the analogous ones from Theory of stability ([S], [Ba]).

**Definition 2.2** Let $p, q \in S_1(A)$. We say that $p$ is weakly orthogonal to $q$ ($p \perp^w q$) if for any $H(x, y, \bar{a}), \ \bar{a} \in A$, for any $\alpha \in p(M')$ the following holds:

$$[H(M', \alpha, \bar{a}) \cap q(M') \neq \emptyset \Rightarrow q(M') \subseteq H(M', \alpha, \bar{a})].$$

If $p$ is not weakly orthogonal to $q$, we will denote this fact by $p \not\perp^w q$.

**Note 2.3** (i) $p \not\perp^w q \Rightarrow q \not\perp^w p$.

(ii) $p \perp^w q \Leftrightarrow p(x) \cup q(y)$ is complete $2-A$-type.

(iii) If $p$ is algebraic then $\forall q \in S_1(A), \ p \perp^w q$.

**Definition 2.3** Let $p, q \in S_1(A)$. We say that $p$ is almost orthogonal to $q$ ($p \perp^a q$), if $\exists \alpha \in p(M')$ ($\equiv \forall \alpha \in p(M')$), $V_q(\alpha) = \emptyset$. If $p$ is not almost orthogonal to $q$, then we denote this fact by $p \not\perp^a q$.

**Note 2.4** Let $p, q \in S_1(A)$. Then the following propositions are true:

(i) $p \perp^w q \Rightarrow p \perp^a q$.

(ii) There exists $T$ — w.o.m. theory such that $p \not\perp^w q,\ p \perp^a q$.

(iii) Let $T$ be o-minimal. Then $(p \perp^a q \Leftrightarrow p \perp^w q)$.

**Lemma 2.2** *Let $p, q \in S_1(A)$. Then the following propositions are true:*

*(i) Let $p \not\perp^w q$. If $p$ is social then $q$ is social and $p \not\perp^a q$.*

*(ii) $p \not\perp^a q \Rightarrow q \not\perp^a p$.*

*(iii) $\not\perp^a$ is relation of equivalence on $S_1(A)$.*

*(iv) $\not\perp^w$ is relation of equivalence on $S_1(A)$.*

*Proof.* (i) Consider two cases:
a) $p \not\perp^a q$,
b) $p \not\perp^w q,\ p \perp^a q$.

If $p \not\perp^a q$ then by Claim 1.1 if $p$ is social then $q$ is social.

Consider the case $p \not\perp^w q,\ p \perp^a q$. We can construct the $2-A$-formula $H(x, y, \bar{a})$, $\bar{a} \in A$ such that for any $\alpha \in p(M'),\ H(M', \alpha, \bar{a})$ — convex, $\neg H(M', \alpha, \bar{a})$ — convex,

$$H(M', \alpha, \bar{a}) \cup \neg H(M', \alpha, \bar{a}) = M',\ \exists \beta_1 \in H(M', \alpha, \bar{a}) \cap q(M'),$$
$$\exists \beta_2 \in \neg H(M', \alpha, \bar{a}) \cap q(M'),\ H(M', \alpha, \bar{a}) < \neg H(M', \alpha, \bar{a}).$$

**Note 2.5** $\forall \alpha, \beta \in p(M')$

$$[\exists \gamma \in H(M', \alpha, \bar{a}) \setminus H(M', \beta, \bar{a}) \Rightarrow H(M', \beta, \bar{a}) \subset H(M', \alpha, \bar{a})].$$

Let $\alpha \in p(M')$. Consider $f \in \mathrm{Aut}_A(M')$, such that $f(\beta_1) = \beta_2$. Then

$$H(M', \alpha, \bar{a}) \subset H(M', f(\alpha), \bar{a})$$

**Claim 2.1** If $f(\alpha) > \alpha$ $(f(\alpha) < \alpha)$ then for any $\beta < \alpha$ $(\beta > \alpha)$,
$\beta \in p(M')$ $H(M', \beta, \bar{a}) \subseteq H(M', \alpha, \bar{a})$ and there exists $U(x, \alpha, \bar{c}),\ \bar{c} \in A$,
$U(M', \alpha, \bar{c}) < \alpha$ $(U(M', \alpha, \bar{c}) > \alpha)$ such that

$$\forall \beta < \alpha, \beta \in p(M')[H(M', \beta, \bar{a}) \subset H(M', \alpha, \bar{a}) \Rightarrow \beta \in U(M', \alpha, \bar{c})].$$

*Proof of Claim 2.1.* We suppose that $f(\alpha) > \alpha$, a consideration of the case $f(\alpha) < \alpha$ is the same. Let $\alpha_0 = f^{-1}(\alpha)$ then

$$H(M', \alpha_0, \bar{a}) \subset H(M', \alpha, \bar{a}).$$

Suppose there exists $\beta \in p(M'),\ \beta < \alpha,\ H(M', \alpha, \bar{a}) \subset H(M', \beta, \bar{a})$.

Consider three $<\alpha, \bar{a}>$-definable sets:

$$\begin{aligned} K_1(M', \alpha, \bar{a}) &:= \{\gamma \in M' : \gamma < \alpha, H(M', \alpha, \bar{a}) = H(M', \gamma, \bar{a})\}, \\ K_2(M', \alpha, \bar{a}) &:= \{\gamma \in M' : \gamma < \alpha, H(M', \alpha, \bar{a}) \subset H(M', \gamma, \bar{a})\}, \\ K_3(M', \alpha, \bar{a}) &:= \{\gamma \in M' : \gamma < \alpha, H(M', \gamma, \bar{a}) \subset H(M', \alpha, \bar{a})\}. \end{aligned}$$

$$\alpha_0 \in K_3(M', \alpha, \bar{a}) \cap p(M'),\ \beta \in K_2(M', \alpha, \bar{a}) \cap p(M').$$

Because $T$ is w.o.m. $K_i(M', \alpha, \bar{a})$, $i = 1, 2, 3$ is union of convex $\neg K_i(M', \alpha, \bar{a})$-separable subsets, there are $i \in \{1, 2, 3\}$, $K_i^j(M', \alpha, \bar{a})$ is the maximal convex $<\alpha, \bar{a}>$-definable subset such that

$$\exists \gamma \in K_i^j(M', \alpha, \bar{a}) \cap p(M') \Rightarrow \forall \mu [p(M')^- < \mu < \gamma \Rightarrow \mu \in K_i^j(M', \alpha, \bar{a})].$$

Consider three cases:
$i = 1$. We have two possibilities for $p$.
a) $p$ is irrational to the left. So, $\exists C(x, \bar{c})$, $\bar{c} \in A$

$$C(M', \bar{c}) \subset K_i^j(M', \alpha, \bar{a}),\ C(M', \bar{c}) < p(M').$$

b) $p$ is quasirational to the left, then $\exists C(x, \bar{c})$, $\bar{c} \in A$

$$C(M', \bar{c})^- \cup C(M', \bar{c}) = p(M')^-.$$

Thus we have $M' \models \exists z(C(M', \bar{c}) < z < \alpha \wedge \forall x(C(M', \bar{c}) < x < z \rightarrow$
$\rightarrow H(M', \alpha, \bar{a}) \leftrightarrow H(M', x, \bar{a})))$.

Consider the following formula:

$$\Phi(y, \bar{c}, \bar{a}) := \exists z[C(M', \bar{c}) < z \wedge \forall x(C(M', \bar{c}) < x < z \rightarrow \neg H(y, x, \bar{a})].$$

Then $\beta_2 \in \Phi(M', \bar{c}, \bar{a})$ and $\beta_1 \notin \Phi(M', \bar{c}, \bar{a})$. Contradiction.

$i = 2$. Then $\exists K_3^m(M', \alpha, \bar{a})$ — maximal $<\alpha, \bar{a}>$-definable subset of $K_3(M', \alpha, \bar{a})$ such that $K_3^m(M', \alpha, \bar{a}) \subset p(M')$,
$K_2^j(M', \alpha, \bar{a}) < K_3^m(M', \alpha, \bar{a})$.

Let $L(x, \alpha, \bar{a}) := \exists y\ (K_3^m(y, \alpha, \bar{a}) \wedge \neg H(x, y, \bar{a}) \wedge H(x, \alpha, \bar{a}))$. So,
$L(M', \alpha, \bar{a}) \subset q(M')$. If $\exists \mu \in q(M')$ such that $\mu < L(M', \alpha, \bar{a})$, then
$\mu < L(M', \alpha, \bar{a}) < \beta_2$. Contradiction with $p \perp^a q$.

Thus $L(M', \alpha, \bar{a})^- = q(M')^-$. It is possible by Note 1.11(ii) if $q$ is quasirational to left or isolated. Then there is $1 - A-$formula $G(x, \bar{g})$,
$\bar{g} \in A$ such that $G(M', \bar{g})^- \cup G(M', \bar{g}) = q(M')^-$. Let
$\Theta_1(\alpha, \bar{a}, \bar{g}) := \exists z(G(M', \bar{g}) < z \wedge H(z, \alpha, \bar{a}) \wedge \forall x(G(M', \bar{g}) < x < z \rightarrow$
$\rightarrow \exists y(K_3^m(y, \alpha, \bar{a}) \wedge \neg H(x, y, \bar{a}))))$. It is clear that $M' \models \Theta_1(\alpha, \bar{a}, \bar{g})$.
Notice that $\forall \delta \in p(M')\ M' \models \Theta_1(\delta, \bar{a}, \bar{g})$.
Consider arbitrary $\delta \in K_2^j(M', \alpha, \bar{a}) \cap p(M')$ then $H(M', \alpha, \bar{a}) \subset H(M', \delta, \bar{a})$ and $K_3^m(M', \delta, \bar{a}) \subset K_2^j(M', \alpha, \bar{a})$.
So, $\forall \alpha_1 \in K_3^m(M', \delta, \bar{a})\ H(M', \alpha, \bar{a}) \subset H(M', \alpha_1, \bar{a})$.
It is contradiction with $M' \models \Theta_1(\delta, \bar{a}, \bar{g})$.

$i = 3$. Then $\exists K_2^m(M', \alpha, \bar{a})$ — maximal $<\alpha, \bar{a}>$-definable subset of $K_2(M', \alpha, \bar{a})$ such that $K_2^m(M', \alpha, \bar{a}) \subset p(M')$,
$K_3^j(M', \alpha, \bar{a}) < K_2^m(M', \alpha, \bar{a})$.

Let $R(x,\alpha,\bar{a}) := \exists y\ (K_2^m(y,\alpha,\bar{a}) \wedge H(x,y,\bar{a}) \wedge \neg H(x,\alpha,\bar{a}))$. So, $R(M',\alpha,\bar{a}) \subset q(M')$. If $\exists \mu \in q(M')\ R(M',\alpha,\bar{a}) < \mu$, then
$\beta_1 < R(M',\alpha,\bar{a}) < \mu$. Contradiction with $p \perp^a q$.

Thus $R(M',\alpha,\bar{a})^+ = q(M')^+$. It is possible by Note 1.11(ii) if $q$ is quasirational to the left or isolated. Then there is $1-A-$formula $D(x,\bar{d})$,
$\bar{d} \in A$ such that $D(M',\bar{d})^+ = q(M')^+$.

Let $\Theta_2(\alpha,\bar{a},\bar{d}) := \forall x(D(x,\bar{d}) \wedge \neg H(x,\alpha,\bar{a}) \rightarrow$
$\rightarrow \exists y(K_2^m(y,\alpha,\bar{a}) \wedge \neg H(x,y,\bar{a}))))$.

Consider arbitrary $\delta \in K_3^j(M',\alpha,\bar{a}) \cap p(M')$ then $M' \models \Theta_2(\delta,\bar{a},\bar{d})$,
$K_2^m(M',\delta,\bar{a}) \subset K_3^j(M',\alpha,\bar{a}) \cap p(M')$.

So, for $\beta_2$ there exists $\alpha_1 \in K_2^m(M',\delta,\bar{a})$ such that $\beta_2 \in H(M',\alpha_1,\bar{a})$. Then by Note 2.5 $H(M',\alpha,\bar{a}) \subset H(M',\alpha_1,\bar{a})$. Contradiction because $\alpha_1 \in K_3^j(M',\alpha,\bar{a})$.

So, $p(M') \cap K_2(M',\alpha,\bar{a}) = \emptyset$ and $K_1(M',\alpha,\bar{a}) \cap p(M') > K_3(M',\alpha,\bar{a})$.

Thus $U(M',\alpha,\bar{a})$ is the maximal $< \alpha,\bar{a} >$-definable subset of $K_3(M',\alpha,\bar{a})$. It is clear that $U(x,\alpha,\bar{a})$ is the required formula. So, Claim 2.1 is proved. $\Box$

Let $G(x,\alpha,\bar{a}) := U(M',\alpha,\bar{a}) < x \leq \alpha$. Then $G(x,y,\bar{a})$ is maximal convex to the left $p$-stable $2-A$-formula. It means that $p$ is quasisolitary. So, if $p$ is social and $p \not\perp^w q$, then $p \not\perp^a q$ and by Claim 1.1 $q$ is social.

**Note 2.6** Let $p,q \in S_1(A)$, $p \not\perp^w q$. Then the following hold:

(i) If $\alpha \in p(M')$ then $[p \perp^a q \Leftrightarrow V_q(\alpha) = \emptyset]$.

(ii) If $p \perp^a q, \alpha,\beta \in p(M')$ then

$$[H(M',\alpha,\bar{a}) = H(M',\beta,\bar{a}) \Leftrightarrow \models E_p(\alpha,\beta,\bar{c}_p)].$$

(iii) If $p \perp^a q$, $B \subset M'$ such that $V_p(B) \neq \emptyset$ then

$$[V_q(B) = \emptyset \Leftrightarrow \exists \alpha \in p(M'),\ V_p(B) = E_p(M',\alpha,\bar{c}_p)].$$

*Proof* (ii) follows from Claim 1.1.

(iii) $p \not\perp^a p$ for any $p \in S_1(A)$ by Definition 2.3. If $p \not\perp^a q$ then $q \not\perp^a p$ by Lemma 2.2(ii). Suppose $r \not\perp^a p$, $p \not\perp^a q$.

**Claim 2.2** Let $p \in S_1(A)$, $\alpha_1,\alpha_2 \in p(M')$, $\Phi(x,\bar{\beta}),\bar{\beta} \in M'$ such that $V_p(\alpha_1) < \Phi(M',\bar{\beta}) < V_p(\alpha_2)$. Then for any $q \in S_1(A)(p \not\perp^a q)$, for any $\Psi(x,y,\bar{a})$, $\bar{a} \in A$ such that $\Psi(M',\alpha_1,\bar{a}) \subset V_q(\alpha_1)$ for the formula $K_{\Phi,\Psi}(y,\bar{\beta},\bar{a}) := \exists x(\Phi(x,\bar{\beta}) \wedge \Psi(y,x,\bar{a}))$ the following is true:

$$V_q(\alpha_1) < K_{\Phi,\Psi}(M',\bar{\beta},\bar{a}) < V_q(\alpha_2) \quad \text{or} \quad V_q(\alpha_2) < K_{\Phi,\Psi}(M',\bar{\beta},\bar{a}) < V_q(\alpha_1).$$

*Proof of Claim 2.2.* By Note 2.2(ii) for any $\alpha_0,\alpha_0' \in \Phi(M',\bar{\beta})$

$$V_q(\alpha_1) < V_q(\alpha_0) < V_q(\alpha_2) \Leftrightarrow V_q(\alpha_1) < V_q(\alpha_0') < V_q(\alpha_2).$$

Then suppose that for any $\alpha_0 \in \Phi(M',\bar{\beta})$

$$V_q(\alpha_1) < \Psi(M',\alpha_0,\bar{a}) < V_q(\alpha_2).$$

Thus, $V_q(\alpha_1) < \bigcup_{\alpha_0 \in \Phi(M',\bar{\beta})} \Psi(M', \alpha_0, \bar{a}) < V_q(\alpha_2)$. Then

$$V_q(\alpha_1) < K_{\Phi,\Psi}(M', \bar{\beta}, \bar{a}) < V_q(\alpha_2).$$

So, Claim 2.2 is proved. □

**Note 2.7** Let $p \in S_1(A),\ B \subset M',\ V_p(B) \neq \emptyset$ such that $M'$ is $|A \cup B|^+$-saturated.

(i) If $\alpha_1 < V_p(B) < \alpha_2,\ \alpha_1, \alpha_2 \in p(M')$ then for $q \in S_1(A)$ such that $q \not\perp^a p$ the following is true:

$$V_q(B) \neq \emptyset \text{ and } V_q(\alpha_1) < V_q(B) < V_q(\alpha_2) \text{ or } V_q(\alpha_2) < V_q(B) < V_q(\alpha_1).$$

(ii) $\forall \alpha_0 \in V_p(B) \forall q \in S_1(A),\ q \not\perp^a p$ the following is true:

$$V_q(\alpha_0) \subset V_q(B).$$

Consider $\beta \in r(M')$, then $V_p(\beta) \neq \emptyset$ because $r \not\perp^a p$. By Note 2.7(i) and Lemma 2.1(iii) $V_q(\beta) \neq \emptyset$. So, $r \not\perp^a q$.

(iv) $p \not\perp^w p$ for any $p \in S_1(A)$ by Definition 1.2. If $p \not\perp^w q$ then $q \not\perp^w p$ by Note 2.3(i). Suppose $r \not\perp^w p, p \not\perp^w q, p \perp^a q$. Then $p, q$ are quasisolitary by Lemma 2.2(i). Let $H(x, y, \bar{a})$ be a formula from Claim 2.1. Then from Note 2.6(ii), it follows:
$\forall \alpha_1, \alpha_2 \in p(M')[\models \neg E_p(\alpha_1, \alpha_2, \bar{c}_p) \Rightarrow$
$\Rightarrow H(M', \alpha_1, \bar{a}) \subset H(M', \alpha_2, \bar{a})$ or $H(M', \alpha_2, \bar{a}) \subset H(M', \alpha_1, \bar{a})]$.

Let $\alpha \in p(M')$. If $\exists \alpha_1 \in p(M')$ such that $\alpha_1 < \alpha,\ M' \models \neg E_p(\alpha_1, \alpha, \bar{c}_p)$, $H(M', \alpha_1, \bar{a}) \subset H(M', \alpha, \bar{a})$, then for any $\alpha_2, \alpha_3 \in p(M')$
$M' \models \neg E_p(\alpha_2, \alpha_3, \bar{c}_p) \wedge \alpha_2 \wedge \alpha_3 \Rightarrow H(M', \alpha_2, \bar{a}) \subset H(M', \alpha_3, \bar{a})$.

Without loss of generality suppose that $H(x, y, \bar{a})$ is increasing on classes of equivalence $E_p(x, y, \bar{c}_p)$ of elements from $p(M')$.

Consider the following formula:
$K(x, \alpha, \bar{c}_p, \bar{a}) := \forall y[x < y < \alpha \wedge \neg E_p(x, y, \bar{c}_p) \wedge \neg E_p(y, \alpha, \bar{c}_p) \rightarrow$
$\rightarrow H(M', x, \bar{a}) \subset H(M', y, \bar{a}) \subset H(M', \alpha, \bar{a})]$.

If $p$ is quasirational to the left then there is $U_p(x, \bar{b})$ such that
$U_p(M', \bar{b})^- = p(M')^-$,

$$M' \models \forall x[U_p(M', \bar{b}) < x < \alpha \rightarrow K(x, \alpha, \bar{c}_p, \bar{a})].$$

If $p$ is non-quasirational to the left then $\exists C(M', \bar{e})$ such that
$C(M', \bar{e}) \subset K(M', \alpha, \bar{c}_p, \bar{a}),\ C(M', \bar{e}) < p(M')$. So, we have:

$$M' \models \forall x[C(M', \bar{e}) < x < \alpha \rightarrow K(x, \alpha, \bar{c}_p, \bar{a})].$$

The same consideration of the formula
$K_1(x, \alpha, \bar{c}_p, \bar{a}) := \forall y[(\alpha < y < x \wedge \neg E_p(x, y, \bar{c}_p) \wedge \neg E_p(y, \alpha, \bar{c}_p)) \rightarrow$
$\rightarrow H(M', \alpha, \bar{a}) \subset H(M', y, \bar{a}) \subset H(M', x, \bar{a})]$.
gives the formula $D(x, \bar{d})$ such that $C(M', \bar{e}) < p(M') < D(M', \bar{d})$ and
$M' \models \forall x \forall y[C(M', \bar{e}) < x < y < D(M', \bar{d}) \wedge \neg E_p(x, y, \bar{c}_p) \rightarrow$
$\rightarrow H(M', x, \bar{a}) \subset H(M', y, \bar{a})]$.

Let $\beta \in r(M')$, then there is the formula $\Phi(x, y, \bar{b})$ such that $\Phi(M', \beta, \bar{b}) \cap p(M') \neq \emptyset$ and $\neg\Phi(M', \beta, \bar{b}) \cap p(M') \neq \emptyset$. Suppose $\gamma_1 \in \Phi(M', \beta, \bar{b}) \cap p(M')$ and $\gamma_2 \in \neg\Phi(M', \beta, \bar{b}) \cap p(M')$ and $\gamma_1 < \gamma_2$. Let $H_1(x, \beta, \bar{b})$ be the maximal convex subformula of $\Phi(x, \beta, \bar{b})$ such that $\gamma_1 \in H_1(M', \beta, \bar{b})$. So, $\gamma_2 > H_1(M', \beta, \bar{b})$. For $\gamma_2 \in p(M')$ there is $\mu \in q(M')$ such that $\mu > H(M', \gamma_2, \bar{a})$. Then $\forall \alpha \in H(M', \beta, \bar{a}) \cap C(M', \bar{e})^+ \cap D(M', \bar{d})^-$ we have $H(M', \alpha, \bar{a}) < \mu$.

Consider the formula

$$H_2(x, \beta, \bar{a}, \bar{e}, \bar{b}) := \exists y(H_1(y, \beta, \bar{b}) \wedge C(M', \bar{e}) < y \wedge H(x, y, \bar{a})).$$

Then $H_2(M', \beta, \bar{b}, \bar{d}, \bar{e}, \bar{a}) \cap q(M') \neq \emptyset$, because $H(M', \gamma_1, \bar{a}) \cap q(M') \neq \emptyset$, $\neg H_2(M', \beta, \bar{b}, \bar{d}, \bar{e}, \bar{a}) \cap q(M') \neq \emptyset$, because $\mu > H_2(M', \beta, \bar{b}, \bar{d}, \bar{e}, \bar{a})$. Thus, $r \not\perp^w q$. □

**Corollary 2.1** *The equivalence relations $\not\perp^a$ and $\not\perp^w$ partition the set of non-algebraic types from $S_1(A)$ into the classes of equivalence as follows:*

*(i) Every $\not\perp^w$-class contains $\not\perp^a$-classes or it coincides with a $\not\perp^a$-class.*

*(ii) Every $\not\perp^w$-class contains types only one kind from six basic kinds of Note 1.10.*

*(iii) Every $\not\perp^w$-class, which contains social types, is $\not\perp^a$-class.*

**Lemma 2.3** *Let $A, B, C \subset M'$ such that $M'$ is $|A \bigcup B \bigcup C|^+$-saturated, $p, q \in S_1(A)$, $p \not\perp^w q$. Then the following hold:*

*(i) If $p \not\perp^a q$, then $V_p(B) \bigcap V_p(C) = \emptyset$ iff $V_q(B) \bigcap V_q(C) = \emptyset$*

*(ii) If $p \perp^a q$, $V_p(B) \bigcap V_p(C) = \emptyset$, $V_q(B) \bigcap V_q(C) \neq \emptyset$, then $\exists \delta \in q(M')$ such that $V_q(B) \bigcap V_q(C) = V_q(\delta) = E_q(M', \delta, \bar{c}_q)$, $(V_p(B), V_p(C)) = \emptyset$ or $(V_p(C), V_p(B)) = \emptyset$.*

*(iii) If $p \perp^a q, \exists \alpha \in p(M')$ such that $V_p(B) < V_p(\alpha) < V_p(C)$, then $V_q(B) \bigcap V_q(C) = \emptyset$.*

*Proof.* (i) It follows from Claim 2.2 and Note 2.7 (ii).
(ii) Let $H(x, y, \bar{b})$ be a formula from Claim 2.1, which was obtained from fact $q \not\perp^w p, q \perp^a p$ such that $\forall \alpha_1, \alpha_2 \in q(M')$ the following hold:

- $H(M', \alpha_1, \bar{b}) < \neg H(M', \alpha_1, \bar{b})$
- $\exists \beta_1, \beta_2 \in p(M'), \beta_1 \in H(M', \alpha_1, \bar{b}), \beta_2 \in \neg H(M', \alpha_2, \bar{b})$
- $H(M', \alpha_1, \bar{b}) \subset H(M', \alpha_2, \bar{b}) \Rightarrow \neg E_q(\alpha_1, \alpha_2, \bar{c}_q)$

Without loss of generality as in proof of Claim 2.1 we suppose:

$$H(M', \alpha_1, \bar{b}) \subset H(M', \alpha_2, \bar{b}) \Leftrightarrow \models \alpha_1 < \alpha_2 \& \neg E_q(\alpha_1, \alpha_2, \bar{c}_q)$$

Suppose $\emptyset \neq V_q(B) \bigcap V_q(C) \neq E_q(M', \delta, \bar{c}_q)\ \forall \delta \in q(M')$

Let $\delta \in V_q(B) \bigcap V_q(C)$. Then there are two formulas $\phi(x, \bar{\beta}), \psi(x, \bar{\gamma}), \bar{\beta} \in B \bigcup A, \bar{\gamma} \in C \bigcup A$ such that $\delta \in \phi(M', \bar{\beta}) \bigcap \psi(M', \bar{\gamma})$ and $\exists \mu_1, \mu_2, \mu_3, \mu_4 \in q(M')$ such that

$$\mu_1 < \phi(M', \bar{\beta}) < \mu_2, \mu_3 < \psi(M', \bar{\gamma}) < \mu_4.$$

Consider the formula $\phi_1(x, \bar{\beta}, \bar{c}_q) := \exists y(\phi(y, \bar{\beta}) \& E_q(x, y, \bar{c}_q))$.

Because $E_q(x, y, \bar{c}_q)$ is $p$-stable it follows by Note 1.9 (iv) the existence $\mu_1', \mu_2'$ such that $\mu_1' < \phi_1(M', \bar{\beta}, \bar{c}_q) < \mu_2'$ and $\forall \delta \in \phi_1(M', \bar{\beta}, \bar{c}_q)\ E_q(M', \delta, \bar{b}) \subseteq \phi_1(M', \bar{\beta}, \bar{c}_q)$
Let $\psi_1(x, \bar{\gamma}, \bar{c}_q) := \exists y(\psi(y, \bar{\gamma}) \& E_q(x, y, \bar{c}_q))$. Then for any
$\delta \in \phi_1(M', \bar{\beta}, \bar{c}_q) \bigcap \psi_1(M', \bar{\gamma}, \bar{c}_q)$ we have $E_q(M', \delta, \bar{c}_q) \subseteq \phi_1(M', \bar{\beta}, \bar{c}_q) \bigcap \psi(M', \bar{\gamma}, \bar{c}_q)$.
So, $\exists \alpha_1, \alpha_2 \in q(M')$ such that $\alpha_1, \alpha_2 \in \phi_1(M', \bar{\beta}, \bar{c}_q) \bigcap \psi_1(M', \bar{\gamma}, \bar{c}_q)$ and
$\models \neg \varepsilon_q(\alpha_1, \alpha_2, \bar{c}_q)$. Suppose $\alpha_1 < \alpha_2$. Let

$$K_\phi(\bar{y}, \bar{\beta}, \bar{b}, \bar{c}_q) := \exists x_1\ \exists x_2\ (\phi_1(x_1, \bar{\beta}, \bar{c}_q) \wedge \phi_1(x_2, \bar{\beta}, \bar{c}_q) \wedge \neg E_q(x_1, x_2, \bar{c}_q) \wedge$$

$$\wedge x_1 < x_2 \wedge \neg H(y, x_1, \bar{b}) \wedge \neg H(y, x_2, \bar{b})).$$

Then there are $\beta_1, \beta_2 \in p(M')$ such that

$$p(M')^- < \beta_1 < K_\phi(M', \bar{\beta}, \bar{b}, \bar{c}_q) < \beta_2 < p(M')^+$$

because there are $\alpha_3, \alpha_4 \in q(M')$ such that

$$\alpha_3 < V_p(B) < \alpha_4 \text{ and } \models \neg E_q(\alpha_3, \alpha_1, \bar{c}_q) \wedge \neg E_q(\alpha_2, \alpha_4, \bar{c}_q).$$

$K_\phi(M', \beta, \bar{b}, \bar{c}_q) \neq \emptyset$. because $\exists \mu_0 \in H(M', \alpha_2, \bar{b}) \setminus H(M', \alpha_1, \bar{b})$. Thus, $\mu_0 \in K_\phi(M', \bar{\beta}, \bar{b}, \bar{c}_q)$.

Consider

$$K_\psi(y, \bar{\gamma}, \bar{b}, \bar{c}_q) := \exists x_1\ \exists x_2\ (\psi_1(x_1, \bar{\gamma}, \bar{c}_q) \wedge \psi_1(x_2, \bar{\gamma}, \bar{c}_q) \wedge$$

$$\wedge \neg E_q(x_1, x_2, \bar{c}_q) \wedge x_1 < x_2 \wedge \neg H(y, x_1, \bar{b}) \wedge \neg H(y, x_2, \bar{b})).$$

Then $\mu_0 \in K_\psi(M', \bar{\gamma}, \bar{b}, \bar{c}_q)$ by the same consideration as for
$K_\phi(M', \bar{\beta}, \bar{b}, \bar{c}_q)$. So, $\mu_0 \in V_p(B) \cap V_p(C)$. Contradiction.

Thus, $\exists \delta \in q(M')$ such that $V_q(B) \cap V_q(C) = E_q(M', \delta, \bar{c}_q)$. For quasisolitary type $q$, $E_q(M', \delta, \bar{c}_q) = V_q(B)$. So, (ii) is proved.

(iii) By Lemma 2.1(v) we have $V_p(B) < V_p(\alpha) < V_p(c)$. such that $\phi(M', \bar{\beta}) \subseteq V_q(B)$, $\psi(M', \bar{\gamma}) \subseteq (C), \phi(M', \bar{\beta}) \cap \psi(M', \bar{\gamma} = E_q(M', \delta, \bar{c}_q)$. The existence of these $\phi, \psi, \delta$ follows from proof (ii). Let $\theta(M', \bar{\beta}_1), \bar{\beta}_1 \in B$ such that $\Theta(M', \beta_1) \subset V_p(B)$ and $V_p(B) \subset H(M', \delta, \bar{b})$. Suppose $V_p(B) \subset H(M', \delta, \bar{b})$. Then $V_p(B)^+ = H(M', \delta, \bar{\beta})^+$. If $V_p(B)^+ \neq H(M', \delta, \bar{\beta})^+$, then there is $\theta(x, \bar{\beta}_1), \bar{\beta}_1 \in B$, $\exists \in H(M', \delta, \bar{b})$ such that $\theta(M', \bar{\beta}_1) < \mu$. Consider the following formula

$$R(x, \bar{\beta}, \bar{\beta}_1, \bar{b}) := \exists y\ (\phi(y, \bar{\beta}) \wedge H(x, y, \bar{b}) \wedge \theta(M', \bar{\beta}_1) < x).$$

So, $R(M', \bar{\beta}, \bar{\beta}_1, \bar{b}) \subset V_p(B), R(M', \bar{\beta}, \bar{\beta}_1, \bar{b})^+ = H(M', \delta, \bar{b})^+$.

Suppose $V_p(C) \subset \neg H(M', \delta, \bar{b})$. Then $V_p(C)^- = \neg H(M', \delta, \bar{b})^-$.
If $V_p(C)^- \neq \neg H(M', \delta, \bar{b})^-$, then there is $\theta_1(x, \bar{\gamma}_1), \bar{\gamma}_1 \in C, \exists \mu_1 \in \neg H(M', \delta, \bar{b})$ such that $\theta_1(M', \gamma_1) < \mu_1$. Consider the following formula

$$L(x, \bar{\gamma}, \bar{\gamma}_1, \bar{b}) := \exists y\ (\psi(y, \bar{\gamma}) \wedge \neg H(x, y, \bar{b}) \wedge x < \theta(M', \bar{\gamma}_1)).$$

$L(M', \bar{\gamma}, \bar{\gamma}_1, \bar{b}) \subset V_p(C), L(M', \bar{\gamma}, \bar{\gamma}_1, \bar{b}) = \neg H(M', \delta, \bar{b})^-$.

Thus, if $V_p(B)^+ \cap \neg H(M', \delta, \bar{b}) \neq \emptyset$, then $V_p(C) \subseteq \neg H(M', \delta, \bar{b})$,
$V_p(C)^- = \neg H(M', \delta, \bar{b})^-, V_p(B) \cap V_p(C) \neq \emptyset$. Contradiction. So, $V_p(B)^+ = \neg H(M', \delta, \bar{b})$ and $V_p(C)^- = H(M', \delta, \bar{b})$. Contradiction with the existence of $\alpha$. Thus,
$V_q(B) \cap V_q(C) = \emptyset$. □

**Note 2.8** Let $A, B, C \subset M'$ such that $M'$ is $|A \cup B \cup C|^+$-saturated, $p, q \in S_1(A)$, $p \not\perp^w q, p \perp^a q$. Then the following is true:

(i) $\exists \alpha \in p(M'), [V_p(B) < V_p(\alpha) < V_p(C) \text{ or } V_p(C) < V_p(\alpha) < V_p(B)]$ iff

$$\exists \beta \in q(M')[V_q(B) < V_p(\beta) < V_q(C) \quad \text{or} \quad V_q(C) < V_q(\beta) < V_q(B)].$$

(ii) Let $p$ be quasirational to the right. Then the following is true:

$$\exists \alpha \in p(M'), [V_p(B) < V_p(\alpha) < V_p(C) < U_p(M')^+ \Rightarrow$$

$$\Rightarrow \exists \beta \in q(M') \; V_q(B) < V_q(\beta) < V_q(C) < q(M')^+$$

if $q$ is quasirational to the right or

$$q(M')^- < V_q(C) < V_q(\beta) < V_q(B)$$

if $q$ is quasirational to the left]]. Here, $U_p(x)$ is $A$-definable formula such that $U_p(M')^+ = p(M')^+$.

(iii) Let $p$ be quasirational to the right.

$$\exists \alpha \in p(M'), \; [V_p(B) < V_p(\alpha) < V_p(C) < U_p(M')^+].$$

Then

$$\forall \Phi(M', \bar{\beta}) \subset q(M'), \; \bar{\beta} \in B, \forall \theta(M', \bar{\gamma}) \subset q(M'), \; \bar{\gamma} \in C$$

the following is true:

$$[\Phi(M', \bar{\beta})^+ = q(M')^+ \Rightarrow \theta(M', \bar{\gamma}) \subset \Phi(M', \bar{\beta})].$$

# 3 Definability and strictly definability of one-types

**Definition 3.1** Let $p \in S_n(A), \phi(\bar{x}, \bar{y}), l(\bar{x}) = n$.

(i) Type $p$ is said to be $\phi(\bar{x}, \bar{y})$–definable if there is $\theta_\phi(\bar{y}, \bar{c}), \bar{c} \in A$ such that for any $\bar{a} \in A^{l(\bar{y})}$ the following is true:

$$[M' \models \theta_\phi(\bar{a}, \bar{c}) \Leftrightarrow \phi(\bar{x}, \bar{a}) \in p]$$

(ii) Type $p$ is said to be definable if for any $\phi(\bar{x}, \bar{y}), p$ is $\phi(\bar{x}, \bar{y})$–definable

**Note 3.1** Let $p \in S_1(A)$. Then the following hold:

(i) If $p$ is quasirational or isolated then $p$ is definable

(ii) If $A = M$, where $M$ is a model, then $p$ is definable iff $p$ is quasirational.

**Lemma 3.1** *Let $p$ be an irrational type from $S_1(A)$. Then we have (i) $\Rightarrow$ (ii).*

*(i) $p$ is non-definable.*

*(ii) There is a $\bar{y}$–convex formula $G(x,\bar{y})$ (i.e. $\forall\bar{\gamma}\in M'\quad G(M',\bar{\gamma})$ is convex set), such that the following is true:*

*(a)* $\forall D(x,\bar{d})\in p\exists\gamma_1,\gamma_2\in D(M',\bar{d})\exists\bar{g}\in A^{l(\bar{y})}$

$$[G(x,\bar{g})\in p\wedge\gamma_1<G(M',\bar{g})<\gamma_2]$$

*(b)* $\forall C(x,\bar{c})\forall S(x,\bar{s}),\ \bar{c},\bar{s}\in A$

$$[C(M',\bar{c})<p(M')<S(M',\bar{s})\Rightarrow\exists\bar{g}_1,\bar{g}_2\in A^{l(\bar{y})}$$

$$C(M',\bar{c})<G(M',\bar{g}_1)<p(M')<G(M',\bar{g}_2)<S(M',\bar{s})].$$

*Proof.* Let $\phi(x,\bar{y})$ be a $\bar{y}$–convex formula such that $p$ is non–$\phi(x,\bar{y})$–definable. Suppose there is $D(x,\bar{d}),\bar{d}\in A,D(x,\bar{d})\in p$ such that $\forall\bar{g}\in A^{l(\bar{y})}\forall\gamma_1,\gamma_2\in D(M',\bar{d})$ the following holds:

$$[\gamma_1<\phi(M',\bar{g})<\gamma_2\Rightarrow\phi(x,\bar{g})\notin p]$$

Consider $\phi_1(x,\bar{y},\bar{d}):=\neg\exists z(\phi(M',\bar{y})<z\wedge D(z,\bar{d}))\wedge\phi(x,\bar{y})\wedge x<D(M',\bar{d})^+$
If $\forall C(x,\bar{c})\forall S(x,\bar{s}),\ \bar{c},\bar{s}\in A$

$$D(M',\bar{d})^-<C(M',\bar{c})<p(M')<S(M',\bar{s})<D(M',\bar{d})^+$$

there are $\bar{b}_1,\bar{b}_2\in A^{l(\bar{y})}$ such that

- $p(M')\subset\phi_1(M',\bar{b}_1,\bar{d})(\equiv\phi_1(x,\bar{b}_1,\bar{d})\in p)$
- $C(M'.\bar{c})<\phi_1(M',\bar{b}_2,\bar{d})$
- $S(M',\bar{s})\subset\phi_1(M',\bar{b}_2,\bar{d})$
- $p(M')<\phi_1(M',\bar{b}_2,\bar{d})$

then the formula

$$G(x,\bar{y}_1,\bar{y}_2,\bar{d}):=\phi(x,\bar{y}_1,\bar{d})\wedge\neg\phi(x,\bar{y}_2,\bar{d})$$

satisfies the conditons a) and b).
Thus, there is $C_1(x,\bar{c}_1),\bar{c}_1\in A:$

$$D(M',\bar{d})^-<C_1(M',\bar{c}_1)<p(M')$$

such that

$$\forall\bar{b}\in A^{l(\bar{y})}[C_1(M',\bar{c}_1)<\phi_1(M',\bar{b},\bar{d})\Rightarrow p(M')<\phi_1(M',\bar{b},\bar{d})]$$

or (and) there is $S_1(x,\bar{s}_1),\bar{s}_1\in A$ $[p(M')<S_1(M',\bar{s}_1)<D(M',\bar{d})^+]$ such that

$$\forall\bar{b}\in A^{l(\bar{y})}[S_1(M',\bar{s}_1)\subset\phi_1(M',\bar{b},\bar{d})\Rightarrow p(M')\subset\phi_1(M',\bar{b},\bar{d})]$$

The same consideration for the formula

$$\phi_2(x,\bar{y},\bar{d}):=\neg\exists z(D(z,\bar{d}))\wedge z<\phi(M',\bar{y})\wedge\phi(x,\bar{y})\wedge D(M',\bar{d})<x$$

gives the following:
There is $C_2(x,\bar{c}_2), \bar{c}_2 \in A, D(M',\bar{d}) < C_2(M',\bar{c}_2) < p(M')$
such that
$$\forall \bar{b} \in A^{l(\bar{y})}[C_2(M',\bar{c}_2) \subset \phi_2(M',\bar{b},\bar{d}) \Rightarrow \phi_2(M',\bar{b},\bar{d}) \in p]$$
or (and) there is $S_2(x,\bar{s}_2), \bar{s}_2 \in A:$
$$p(M') < S_2(M',\bar{s}_2) < D(M',\bar{d})^+$$
such that
$$\forall \bar{b} \in A^{l(\bar{y})}[\phi_2(M',\bar{s}_2) < S_2(M',\bar{s}_2) \Rightarrow \phi_2(M',\bar{b},\bar{d}) < p(M')]$$
Suppose there are such $C_1(x,\bar{c}_1), S_2(x,\bar{s}_2)$, then the formula
$$\theta_\phi(\bar{y},\bar{d},\bar{c}_1,\bar{s}_2) := [\exists x(\phi_1(x,\bar{y},\bar{d}) \to \exists x(C_1(x,\bar{c}_1) \wedge \phi(x,\bar{y}))$$
$$\wedge \exists x(\phi_2(x,\bar{y},\bar{d}) \to \exists x(S_2(x,\bar{s}_2) \wedge \phi(x,\bar{y}))$$
$$\wedge \exists x \phi_1(x,\bar{y},\bar{d}) \vee \exists x \phi_2(x,\bar{y},\bar{d}))$$
has the following property:
$$\forall \bar{a} \in A^{l(\bar{y})}[M' \models \theta_\phi(\bar{a},\bar{d},\bar{c}_1,\bar{c}_2) \Leftrightarrow \phi(x,\bar{a}) \in p]$$
Then $p$ is $\phi(x,\bar{y})$–definable. Contradiction.
The same consideration of the cases of the existence of $(C_1,C_2),(S_1,C_2),(S_1,S_2)$ gives us the $\phi(x,\bar{y})$–definability of $p$.
Thus, $\phi(x,\bar{y})$ satisfies the condition a).
If $\forall S(x,\bar{s}), \bar{s} \in A, p(M') < S(M',\bar{s})$, there is $\bar{g} \in A$ such that
$$p(M') < \phi(M',\bar{g}) < S(M',\bar{s}),$$
then the formula
$$H(x,\bar{y}_1,\bar{y}_2) := x < \phi(M',\bar{y}_2) \wedge \phi(x,\bar{y}_1)$$
is required for fairness of ii).
If $\forall C(x,\bar{c}), \bar{c} \in A, C(M',\bar{c}) < p(M')$ there is $\bar{g} \in A$ such that
$$C(M',\bar{c}) < \phi(M',\bar{g}) < p(M'),$$
then the formula
$$H(x,\bar{y}_1,\bar{y}_2) := x > \phi(M',\bar{y}_2) \wedge \phi(x,\bar{y}_1)$$
is required for fairness of (ii).
If there are $C_3(x,\bar{c}_3), S_3(x,\bar{s}_3), \bar{c}_3, \bar{s}_3 \in A$
$$C_3(M',\bar{c}_3) < p(M') < S_3(M',\bar{s}_3)$$
such that
$$\forall \bar{g} \in A[C_3(M',\bar{c}_3) < \phi(M',\bar{g}) < S_3(M',\bar{s}_3) \Rightarrow \phi(x,\bar{g}) \in p],$$
then $p$ is $\phi(x,\bar{y})$–definable. Contradiction. So, the fairness of ii) is proved.

**Lemma 3.2** *Let $p, q \in S_1(A), p \not\perp^w q$.*
*Then $p$ is definable iff $q$ is definable.*

*Proof.* By Corollary 2.1 and Note 3.1(ii) we can assume that $p, q$ are irrational. Suppose $p$ is non-definable, $q$ is definable.
Let $H(x, y, \bar{n}), C_1(y, \bar{c}_1), D_1(y, \bar{d}_1), C_2(x, \bar{c}_2), D_2(x, \bar{d}_2), \bar{n}, \bar{c}_1, \bar{c}_2, \bar{d}_1, \bar{d}_2 \in A$ such that

a) $H(x, y, \bar{c})$ from Claim 2.2.
b) $C_1(M', \bar{c}_1) < p(M') < D_1(M', \bar{d}_1)$
c) $C_2(M', \bar{c}_2) < q(M') < D_2(M', \bar{d}_2)$
d)

$$\forall \gamma_1, \gamma_2 \in M'[C_1(M', \bar{c}_1) < \gamma_1 < \gamma_2 < D_1(M', \bar{d}_1) \Rightarrow$$
$$C_2(M', \bar{c}_2) \subset H(M', \gamma_1, \bar{n}) \subseteq H(M', \gamma_2, \bar{n}) < D_2(M', \bar{d}_2)$$
$$(C_2(M', \bar{c}_2) < H(M', \gamma_1, \bar{n}) \supseteq H(M', \gamma_2, \bar{n}) \supset D_2(M', \bar{d}_2))$$

without loss of generality we can assume that $H(x, y, \bar{n})$ is increasing on $(C_1(M', \bar{c}_1), D_1(M', \bar{d}_1))$. Let

$$\phi(y, \bar{z}_1, \bar{z}_2) := G(M', \bar{z}_1) < y < G(M', \bar{z}_2),$$

where $G(y, \bar{z})$ is formula from Lemma 3.1(ii). Then $p$ is non–$\phi(y, \bar{z}_1, \bar{z}_2)$–definable. Consider the formula

$$\psi(x, \bar{z}_1, \bar{z}_2, \bar{n}, \bar{c}_1, \bar{c}_2, \bar{d}_1, \bar{d}_2) := \exists y(H_0(x, y, \bar{n}) \wedge C_1(M', \bar{c}_1) < y < D_1(M', \bar{d}_1) \wedge$$
$$C_2(M\text{''}, \bar{c}_2) < x < D_2(M', \bar{d}_2) \wedge \phi(y, \bar{z}_1, \bar{z}_2))$$

Here $H_0(x, y, \bar{n})$ is the formula from proof of Claim 2.2.

**Claim 3.1** $\forall \bar{a}_1, \bar{a}_2 \in A^{l(\bar{z})}$ *the following is true:*

$$[\phi(y, \bar{a}_1, \bar{a}_2) \in p \Leftrightarrow \psi(x, \bar{a}_1, \bar{a}_2, \bar{c}_1, \bar{c}_2, \bar{d}_1, \bar{d}_2) \in q]$$

*Proof of Claim 3.1.* Suppose $\phi(y, \bar{a}_1, \bar{a}_2) \in p$.
Let $\gamma \in p(M') \bigcap \phi(M', \bar{a}_1, \bar{a}_2)$, then

$$\forall \mu \in M'[C_2(M', \bar{c}_2) < \mu < q(M') \Rightarrow \mu \in H(M', \gamma, \bar{n})]$$

Because $q$ is irrational and $C_1(M\text{''}, \bar{c}_1) < \gamma < D_1(M', \bar{d}_1)$ we have $\psi(x, \bar{a}_1, \bar{a}_2, \ldots) \in q$.
Suppose $\psi(x, \bar{a}_1, \bar{a}_2, \ldots) \in q$.
Let $\mu \in q(M') \bigcap \psi(M', \bar{a}_1, \bar{a}_2, \ldots)$, then there is $\gamma \in p(M')$ such that $\mu \in H(M', \gamma, \bar{n})$ and $\gamma \in \phi(M', \bar{a}_1, \bar{a}_2)$.
So, because $p$ is irrational and $C_2(M', \bar{c}_2) < \mu < D_2(M', \bar{d}_2)$ we have $\phi(y, \bar{a}_1, \bar{a}_2) \in p$.
Thus, Claim 3.1 is proved.

Because $q$ is definable there is $\theta_\psi(\bar{z}_1, \bar{z}_2, \bar{c}), \bar{c} \in M$ such that $\forall \bar{a}_1, \bar{a}_2 \in A$ the following is true:

$$M' \models \Theta_\Phi(\bar{a}_1, \bar{a}_2, \bar{c}) \Leftrightarrow \Psi(x, \bar{a}_1, \bar{a}_2, \ldots) \in q.$$

Thus, $[\phi(y, \bar{a}_1, \bar{a}_2) \in p \Leftrightarrow M' \models \Theta_\Phi(\bar{a}_1, \bar{a}_2, \bar{c})]$. Contradiction.

**Definition 3.2** Let $p \in S_1(A)$ be irrational. We say that $p$ is quasimodel if $p$ is determinated by cut in $A$ i.e. $\forall G(x,\bar{c})\forall D(x,\bar{d}), \bar{c},\bar{d} \in A$ such that

$$G(M',\bar{c}) < p(M') < D(M',\bar{d})$$

there are $g, d \in A$ such that the following is true:

$$G(M',\bar{c}) < g < p(M') < d < D(M',\bar{d}).$$

**Note 3.2** Let $p, q \in S_1(A)$, $p$ be quasimodel, $p \not\perp^w q$. Then $q$ is non-definable.

**Notation 3.1** Let $B, C \subseteq M'$. We denote $B \leq C \Leftrightarrow B < C$ or $B < C,\ (B,C) = \emptyset$.

**Definition 3.3** (i) Let $p \in S_1(A)$. We say that $p$ is strictly definable over $A$ if

$$\forall \Psi(x,\bar{z}) \exists G^1_\Psi(x,\bar{b}), G^2_\Psi(x,\bar{b}),\ \bar{b} \in A,\ G^1_\Psi(M',\bar{b}) \leq p(M') \leq G^2_\Psi(M',\bar{b})$$

such that for any $\bar{a} \in A^{l(\bar{z})}$ the following is true:

$$M' \models \exists x (G^1_\Psi(M',\bar{b}) < x < G^2_\Psi(M',\bar{b}) \wedge \Psi(x,\bar{a})) \rightarrow$$

$$\forall x (G^1_\Psi(M',\bar{b}) < x < G^2_\Psi(M',\bar{b}) \rightarrow \Psi(x,\bar{a})).$$

or $M' \models \exists x (G^1_\Psi(M',\bar{b}) < x < G^2_\Psi(M',\bar{b}) \wedge \Psi(x,\bar{a})) \Leftrightarrow \Psi(x,\bar{a}) \in p.$

(ii) Let $\bar{\gamma} \in M'$. We say that $p$ is strictly definable over $A \cup \bar{\gamma}$

$$\forall \Psi(x,\bar{z}) \exists G^1_\Psi(x,\bar{b},\bar{\gamma}), G^2_\Psi(x,\bar{b},\bar{\gamma}),\ \bar{b} \in A,$$

$$G^1_\Psi(M',\bar{b},\bar{\gamma})^+ \cap p(M') \cap G^2_\Psi(M',\bar{b},\bar{\gamma})^- \neq \emptyset$$

such that for any $\bar{a} \in A^{l(\bar{z})}$ the following is true:

$$M' \models \exists x (G^1_\Psi(M',\bar{b},\bar{\gamma}) < x < G^2_\Psi(M',\bar{b},\bar{\gamma}) \wedge \Psi(x,\bar{a})) \rightarrow$$

$$\forall x (G^1_\Psi(M',\bar{b},\bar{\gamma}) < x < G^2_\Psi(M',\bar{b},\bar{\gamma}) \rightarrow \Psi(x,\bar{a})).$$

**Note 3.3** Let $p \in S_1(A)$. Then the following are true:

(i) If $p$ is isolated then $p$ is strictly definable.

(ii) If $p$ is quasirational, non-strictly definable over $A$ then for any $\gamma \in p(M')$, $p$ is strictly definable over $A \cup \gamma$.

(iii) If $p$ is irrational, non-definable then for any $\gamma_1 \neq \gamma_2 \in p(M')$, $p$ is strictly definable over $A \cup \{\gamma_1, \gamma_2\}$.

(iv) If $A = M$, where $M$ is a model, then any non-algebraic type is non-strictly definable.

**Lemma 3.3** Let $p \in S_1(A)$. Then the following is true:

(i) Let $p$ be quasirational to the right (left), then $p$ is non-strictly definable over $A$ iff $\exists G_p(x, \bar{b}_p, \bar{a}), \bar{b}_p \in$ such that

$$\forall D(M', \bar{d}) < p(M'),\ \bar{d} \in A,\ \exists \bar{a}_D \in A D(M', \bar{d}) < G_p(M', \bar{b}_p, \bar{a}_D) < p(M').$$

$$(\forall D(M', \bar{d}) > p(M'),\ \bar{d} \in A,\ \exists \bar{a}_D \in A p(M') < G_p(M', \bar{b}_p, \bar{a}_D) < D(M', \bar{d})).$$

(ii) Let $p$ be irrational. Then $p$ is non-strictly definable over $A$ if $\exists G_{p,1}(x, \bar{b}_{p,1}, \bar{u}_1)$, $\bar{b}_{p,1} \in A$ such that

$$\forall D(M', \bar{d}) < p(M'),\ \exists \bar{a}_D \in A^{l(\bar{u}_1)} D(M', \bar{d}) < G_{p,1}(M', \bar{b}_{p,1}, \bar{a}_D) < p(M')$$

or (and) $\exists G_{p,2}(x, \bar{b}_{p,2}, \bar{u}_2),\ \bar{b}_{p,2} \in A$ such that

$$\forall D(M', \bar{d}) > p(M'),\ \exists \bar{a}_D \in A^{l(\bar{u}_1)}[p(M') < G_{p,2}(M', \bar{b}_{p,2}, \bar{a}_D) < D(M', \bar{d})].$$

(iii) Let $p$ be irrational, non-strictly definable then there is $\bar{\gamma} \in M' \setminus A$ which is determinated by $G_{p,1},\ G_{p,2}$ or only by $G_{p,1}$ or only by $G_{p,2}$ such that $p$ is strictly definable over $A \cup \bar{\gamma}$.

*Proof.* (i),(ii) are true by Definition 3.3.
(iii) is true by Theorem of compactness. □

# 4 Expansions of models of weakly o-minimal theories by unary predicates

**Definition 4.1** Let $p \in S_1(A), A, B \subset M'$ such that $M'$ is $|A \bigcup B|^+$-saturated model of $T$.

(i) We say that $B$ is weakly orthogonal to $p$ ($B \perp^w p$), if for any $\Phi(x, \bar{\beta}, \bar{a}), \beta \in B, \bar{a} \in A$ the following is true:

$$\Phi(M', \bar{\beta}, \bar{a}) \bigcap p(M') \neq \emptyset \Rightarrow p(M') \subseteq \Phi(M', \bar{\beta}, \bar{a}).$$

(ii) We say that $B$ is almost orthogonal to $p(B \perp^a p)$, if $V_p(B) = \emptyset$.

(iii) Let $B \perp^w p, B \perp^w q, \alpha \in p(M'), \beta \in q(M'), p' := tp(\alpha | A \cup B),\ q' := tp(\beta | A \cup B)$. Then $p(M') = p'(M'),\ q(M') = q'(M')$. We say that $p$ is $B$-weakly orthogonal to $q$ ($p(\perp^w, B)q$), if $p' \perp^w q'$. In opposite case, we say $p$ is non $B$-weakly orthogonal to $q$. We denote this fact by $p(\not\perp^w, B)q$.

**Note 4.1** Let $p, q \in S_1(A), B \subset M'$. Then the following holds:

(i) $B \perp^w p \Rightarrow B \perp^a p$

(ii) $p, q \in S_1(A)$. If $p \perp^w q$ ($p \perp^a q$), then $\forall \alpha \in p(M') \quad \alpha \perp^w q(\alpha \perp^a q)$

(iii) If $B \perp^w p(B \perp^a p)$, then $\forall C \subseteq B, C \perp^w p(C \perp^a p)$.

(iv) $p \not\perp^w q$ $[B \perp^w p \Rightarrow B \perp^w q]$.

**Note 4.2** Let $p, q \in S_1(A), B \subset M'$. Then the following is true:

- $\bar\beta \in M' \backslash A,\ \bar\beta \perp^w p$. Then $tp(\bar\beta/A) \bigcup p(x)$ is complete $l(\bar\beta) + 1 - A$-type.
- $p \not\perp^w q, B \perp^w p$. Then $p(\not\perp^w, B)q$.

**Note 4.3** Let $p, q \in S_1(A),\ B \subset C_1,\ B \subset C_2,\ B \perp^w p,\ B \perp^w q,\ p(\not\perp^w, B)q$. Then the following is true:

$$\exists\gamma(V_p(C_1) < \gamma < V_p(C_2) \quad \text{or} \quad V_p(C_2) < \gamma < V_p(C_1))$$

iff

$$\exists\eta(V_q(C_1) < \eta < V_q(C_2) \ \text{ or } \ V_q(C_2) < \eta < V_q(C_1)).$$

Proof of Note 4.3. Notice that $V_p(C_1) = V_{p'}(C_1 \backslash B)$. The fairness of Note 4.3 follows from Note 2.8(i).

**Theorem 4.1** *Let $M$ be a model of w.o.m. theory $T$, $M'$ be an elementary extension of $M$ such that $M'$ is $|M|^+$–saturated. Then for any $\bar\alpha \in M' \backslash M$ $\quad \exists\bar\beta \in M' \backslash M, \exists\bar\gamma \in M' \backslash M$ such that $\bar\beta =<\bar\beta_1, \ldots, \bar\beta_{l(\bar\alpha)}>$, $tp(\bar\beta_m|M) = tp(\bar\alpha|M)$, $1 \le m \le l(\bar\alpha)$ and for any $\phi(x, \bar y, \bar\alpha)$ there is $K_\phi(\bar y, \bar\alpha, \bar\beta, \bar\gamma, \bar c), \bar c \in M$ such that for any $\bar a \in M^{l(\bar y)}$ the following holds:*

$$[\exists b \in M : M' \models \phi(b, \bar a, \bar\alpha) \Leftrightarrow M' \models K_\phi(\bar a, \bar\alpha, \bar\beta, \bar\gamma, \bar c)].$$

*Proof.* It contains three stages:

(i) Choice of $\bar\beta$.

(ii) Choice of $\bar\gamma$ and elimination of non-strictly definability.

(iii) Construction of $K_\phi$ and conclusion.

(1) Choice of $\bar\beta$.
Let $\bar\alpha =<\alpha_1, \ldots, \alpha_n>$. Let $p_m = tp(\alpha_m/M \bigcup \bar\alpha|m-1), m \in \{1, 2, \ldots, n\}$.
Consider the sequence of one-types over $S_1(M)$, which is determinated by the following considerations:
$r_1 := p_1$
Let $O_{m+1} := \{\ r \in S_1(M)|\bar\alpha|m+1 \not\perp^\omega r, \bar\alpha|m \perp^w r\ \}$.
If $O_{m+1} = \emptyset$, then $r_{m+1} := r_m,\ \bar\beta_{m+1} := \bar\beta_m$
If $O_{m+1} \neq \emptyset$, then $r_{m+1}$ is arbitrary type from $O_{m+1}$.
Notice that, $\forall r \in O_{m+1}, r(\not\perp^w, \bar\alpha|m)r_{m+1}$. Let $\mu^1_{m+1}, \mu^2_{m+1} \in r_{m+1}(M')$ such that $\mu^1_{m+1} < V_{r_{m+1}}(\bar\alpha) < \mu^2_{m+1}$ if $V_{r_{m+1}}(\bar\alpha) \neq \emptyset$ and

$$\mu^1_{m+1} \in H_{m+1}(M', \bar\alpha, \bar a), \mu^2_m + 1 \in \neg H_{m+1}(M', \bar\alpha, \bar a) \ \text{ if } \ V_{r_{m+1}}(\bar\alpha) = \emptyset.$$

Consider $fm + 1 \in Aut_{M \cup \alpha|m}(M')$ such that:

- $f_{m+1}(\mu^1_{m+1}) = \mu^2_{m+1}$, if $r_{m+1}$ is quasirational to the right or irrational

- $f_{m+1}(\mu^2_{m+1}) = \mu^1_{m+1}$, if $r_{m+1}$ is quasirational to the left.

Let $\bar{\beta}_{m+1} :=< f_{m+1}(\alpha_1), f_{m+1}(\alpha_2), \ldots, f_{m+1}(\alpha_n) >$ . Then $tp(\bar{\beta}_{m+1}|M) = tp(\bar{\alpha}|M)$, $\bar{\beta}_{m+1}|m = \bar{\alpha}|m$. Let $< r_1, r_{i_2}, \ldots, r_{i_s} >$ be a sequence of all different types from $\bar{r} :=< r_1, r_2, \ldots, r_n >$, then $\bar{\beta} :=< \bar{\beta}_1, \bar{\beta}_{i_2}, \ldots, \bar{\beta} i_s >$ .

**Lemma 4.1** *If for* $\forall j \in \{1, i_2, \ldots, i_s\}$, $r_j$ *is irrational, then* $\forall \phi(x, \bar{y}, \bar{\alpha}), \forall \bar{a} \in M^{l(\bar{y})}$ *the following holds:*

$$[\exists b \in M, M' \models \phi(b, \bar{a}, \bar{\alpha}) \Leftrightarrow M' \models \exists x(\phi(x, \bar{a}, \bar{\alpha}) \ \wedge \bigwedge_{j\in\{1,i_2,\ldots,i_s\}} \phi(x, \bar{a}, \bar{\beta}_j))].$$

*Proof of Lemma 4.1.*
($\Rightarrow$) It is clear because $tp(\bar{\beta}_j/M) = tp(\bar{\alpha}/M)$
($\Leftarrow$) Let $M' \models \exists x(\phi(x, \bar{a}, \bar{\alpha}) \ \wedge \wedge \wedge_{j\in\{1,i_2,\ldots,i_s\}} \phi(x, \bar{a}, \bar{\beta}_j))$.
Suppose $\phi(M', \bar{a}, \bar{\alpha}) \bigcap M = \emptyset, \phi(M', \bar{a}, \bar{\alpha}) = \bigcup_{k=1}^n \phi_k(M', \bar{a}, \bar{\alpha})$, where $\{\phi_k | k \le n_\phi\}$ is maximal $\neg\phi(M', \bar{a}, \bar{\alpha})$–separable convex family of subformulas of $\phi(x, \bar{a}, \bar{\alpha})$ (Note 1.2).
Let $k \le n_\phi$ such that $\phi_k(M', \bar{a}, \bar{\alpha}) \ne \emptyset$
Then there is $r \in S_1(M')$ such that $\phi_k(M', \bar{a}, \bar{\alpha}) \subset r(M')$.
Let $j \in \{1, i_2, \ldots, i_s\}$ such that $r \in O_j$.
By Corollary 2.1(ii) and because $r(\not\perp^w, \alpha|j-1)r_j$ and $r_j$ is irrational, $r$ is irrational.
By Note 1.11(ii) $\phi_k(M', \bar{a}, \bar{\alpha}) \subset V_r(\bar{\alpha})$. By choice of $\bar{\beta}_j$ and by Note 4.3 $V_r(\bar{\alpha}) \bigcap V_r(\bar{\beta}_j) = \emptyset$. Then $\phi_k(M', \bar{a}, \bar{\alpha}) \bigcap V_r(\bar{\beta}_j) = \emptyset$. So, $\phi(M', \bar{a}, \bar{\alpha}) \bigcap \wedge_{j\in\{1,i_2,\ldots,i_s\}} \phi(M', \bar{a}, \bar{\beta}_j) = \emptyset$.
Contradiction. Thus, $\phi(M', \bar{a}, \bar{\alpha}) \bigcap M \ne \emptyset$. □

**Note 4.4** Lemma 4.1 answers to B. Poizat question on number of copies of $\bar{\alpha}$ for irrational types in o-minimal case. In [BP] a number of such copies is $2^{l(\bar{\alpha})}$. Lemma 4.1 strengthens their result because it holds for weakly o-minimal theories and the number of such copies is $\le l(\bar{\alpha})$.

(2) Choice of $\bar{\gamma}$ and elimination of non-strictly definability.
Let $j \in \{1, 2, \ldots, n\}$ such that:

- $r_j$ is quasirational
- $r_j \ne r_{j-1}$ or $r_j = r_1$
- $\forall k \in \{1, 2, \ldots, n\}$ [$r_k$ is quasirational, $r_k \ne r_{k-1} \Rightarrow k \ge j$]

We will choice $\bar{\gamma}$ by back induction on $m : n, n-1, \ldots, j$ such that the following holds:

$$\bar{\gamma}_n \subseteq \bar{\gamma}_{n-1} \subseteq \ldots \subseteq \bar{\gamma}_{j+1} \subseteq \bar{\gamma}_j = \bar{\gamma} \in M' \backslash M$$

$\forall i, m \in \{j, j+1, \ldots, n-1, n\}$ we have:

$(a)_m \quad \forall \delta \in V_{p_i}(\alpha_i), tp(\delta/M \bigcup \bar{\alpha}|i-1 \bigcup \bar{\gamma}_m) = tp(\alpha_i/M \bigcup \bar{\alpha}|i-1 \bigcup \bar{\gamma}_m) := q_{i,m}$

$(b)_m \quad \exists \eta(V_{p_i}(\bar{\gamma}_m) < \eta < V_{p_i}(\bar{\alpha})$, if $p_i$ is quasirational to the right
$V_{p_i}(\bar{\alpha}) < \eta < V_{p_i}(\bar{\gamma}_m)$, if $p_i$ is quasirational to the left.
$(c)_m \quad q_{m,m+1}$ is strictly definable over $M \cup \alpha|(m-1) \cup \bar{\gamma}_m$
Choice of $\bar{\gamma}_m$. Suppose $\bar{\gamma}_{m+1}$ has been determined already.

Consider $q_{m,m+1}$. If $q_{m,m+1}$ is strictly definable over $M \bigcup \bar{\alpha}|(m-1) \bigcup \bar{\gamma}_{m+1}$ then we pose $\bar{\gamma}_m = \bar{\gamma}_{m+1}$
If $q_{m,m+1}$ is non-strictly definable over $M \bigcup \bar{\alpha}|(m-1) \bigcup \bar{\gamma}_{m+1}$ then by Lemma 3.3(ii), (iii) and Note 3.3(i) it needs to consider two cases:
a) $p_m$ is quasirational
b) $p_m$ is irrational
Let $p_i$ be irrational, then

$$I_{p_i}(\bar{u}) := \{\exists x(\phi_1(x, \alpha_i, \bar{\alpha}|(i-1), \bar{a}) \wedge \psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b})) \rightarrow$$

$$\rightarrow \forall x(\phi_2(x, \alpha_i, \bar{\alpha}|i-1, \bar{a}) \rightarrow \psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b}))|$$

$$\phi_1(x, y, \bar{\alpha}_{i-1}, \bar{a}_j), \phi_2(x, y, \bar{\alpha}_{i-1}, \bar{a}) - p_i - \text{stable formulas},$$

$$\psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b}) \text{ is } l(\bar{u}) + 1 - M \bigcup \bar{\alpha}|i-1 \bigcup \bar{\gamma}_{m+1} - \text{formulas}\}.$$

Let $p_i$ be a quasirational to the right, $U_i(x, \bar{\alpha}|i-1, \bar{c}_i), \bar{c}_i \in M$ be a formula such that $p_i(M')^+ = U_i(M')^+$, then

$$\begin{aligned} QRR_{p_i}(\bar{u}) \ := \ & \{\exists x(\psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b}) \wedge \phi(x, \bar{\alpha}, \bar{a})) \rightarrow \\ & \rightarrow \exists x_1[x_1 < \phi(M', \bar{\alpha}, \bar{a}) \wedge \forall x(x_1 < x < U_i(M')^+ \rightarrow \\ & \rightarrow \psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b}) : \\ & \phi(M', \bar{\alpha}, \bar{a}) \subset p_i(M'), \psi(x, \bar{u}, \bar{\alpha}|i-1, \bar{\gamma}_{m+1}, \bar{b}) \text{ is} \\ & l(\bar{u}) + 1 - (M \bigcup \bar{\alpha}|i-1 \cup \bar{\gamma}_{m+1}) - \text{formula}\}. \end{aligned}$$

If $p_i$ is quasirational to the left, then $QRL_{p_i}(\bar{u})$ is definable as $QRR_{p_i}(\bar{u})$.

**Claim 4.1** *(i) $I_{p_i}(\bar{u}), QRR_{p_i}(\bar{u}), QRLp_i(\bar{u})$ are consistent for suitable $p_i$, $i \in \{j, j+1, \ldots, n\}$.*

*(ii) For any $\bar{\mu} \models I_{p_i}$ we have :*

$$\forall \delta \in V_{p_i}(\alpha_i)[tp(\delta/M \cup \bar{\alpha}|i-1 \cup \bar{\gamma}_{m+1} \cup \bar{\mu}) = tp(\bar{\alpha}_i/M \cup \bar{\alpha}|i-1 \cup \bar{\gamma}_{m+1} \cup \bar{\mu})].$$

*(iii) For any $\bar{\mu} \models QRR_{p_i}(\bar{u})$ the following is true:*

$$\exists \eta \in p_i(M') \ V_{p_i}(\bar{\gamma}_{m+1}, \bar{\mu}) < \eta < V_{p_i}(\bar{\alpha}).$$

*(iv) For any $\bar{\mu} \models QRL_{p_i}(\bar{u})$ the following is true:*

$$\exists \eta \in p_i(M') \ [V_{p_i}(\bar{\alpha}) < \eta < V_{p_i}(\bar{\gamma}_{m+1}, \bar{\mu})].$$

*Proof of Claim 4.1.* Proof follows from the fairness $(a)_{m+1}, (b)_{m+1}$.
a) $p_m$ is quasirational. Without loss of generality we assume that $p_m$ is quasirational to the right. Because $q_{m,m+1}$ is non-strictly definable, by Lemma 3.3(i) there is a formula $G(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{u})$ such that $\forall D(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}), \bar{d} \in M \quad D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}) < p(M')$ there is $\bar{a}_D \in M^{l(u)}$ such that

$$D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}) < G(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{a}_D) < p(M').$$

Consider

$$Q_m(\bar{u}) := \bigcup_{i\in\{j,j+1,\ldots,n\}} I_{p_i}(\bar{u}) \bigcup \bigcup_{p_i-\text{quasirational}} QRR_{p_i}(\bar{u}) \bigcup QRL_{p_i}(\bar{u}) \bigcup$$

$$\bigcup \{D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}) < G(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{u}) < \alpha_m |$$
$$D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}) < p(M')\}$$

$Q_m(\bar{u})$ is consistent set of $l(\bar{u})$-formulas because for any finite subset of $Q_m(\bar{u})$ we can choice suitable $\bar{c} \in M^{l(\bar{u})}$ which satisfies it.

Let $\bar{\mu}_m \in M'$ such that $\bar{\mu}_m \models Q_m(\bar{u})$, then $\bar{\mu}_m =< \bar{\gamma}'_m, \bar{e}_m >$ where $\bar{e}_m \in M, \bar{\gamma}'_m \in M' \setminus M$.
Thus, $\bar{\gamma}_m :=< \bar{\gamma}_{m+1}, \bar{\gamma}'_m >$ .
b) $p_m$ is irrational. From non-strictly definability of $q_{m,m+1}$ by Lemma 3.3 (iii) follows the existence of $G_1(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}^1_m, \bar{u}_1), G_2(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}^2_m, \bar{u}_2)$ or only $G_1(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}^1_m, \bar{u}_1)$ or only $G_2(x, \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}^2_m, \bar{u}_2)$. Without loss of generality we suppose the existence of $G_1, G_2$. Let

$$Q_m(\bar{u}_1, \bar{u}_2) := \bigcup_{i\in\{1,\ldots,n\}} I_{p_i}(\bar{u}_1, \bar{u}_2) \cup \bigcup_{p_i-\text{quasirational}} QRR_{p_i}(\bar{u}_1, \bar{u}_2) \bigcup QRL_{p_i}(\bar{u}_1, \bar{u}_2) \bigcup$$

$$\bigcup \{C(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}) < G_1(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{u}_j) < \alpha_m <$$
$$< G_2(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{u}_2) < D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d})|$$
$$C(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{c}) < q_{m,m+1}(M') < D(M', \bar{\alpha}|m-1, \bar{\gamma}_{m+1}, \bar{d}),$$
$$C(x), D(x) \text{ are } M \bigcup \bar{\alpha}|m-1 \bigcup \bar{\gamma}_{m+1}\text{-definable formulas }\}.$$

Let $\bar{\mu}_m \in M'$ such that $\bar{\mu}_m \models Q_m(\bar{u}_1, \bar{u}_2)$, then $\bar{\mu}_m =< \bar{\gamma}'_m, \bar{e}'_m)$, where $\bar{\gamma}'_m \in M' \setminus M, \bar{e}'_m \in M$.
So, pose $\bar{\gamma}_m =< \bar{\gamma}_{m+1}, \bar{\gamma}'_m >$,
$\bar{e}_m :=< \bar{e}_{m+1}, \bar{e}'_m >$ . It is clear that $\bar{\gamma}_m$ satisfies the conditions $(a)_m, (b)_m, (c)_m$ by construction of $Q_m$.

(3) Construction of $K_\phi$ and conclusion.
Let $\phi(x, \bar{z}, \bar{\alpha})$ be an arbitrary formula.
Definition of $\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m),\ m = n+1, n, \ldots, j$.
Denote by $\phi_{n+1}(x, \bar{z}, \bar{\alpha}) := \phi(x, \bar{z}, \bar{\alpha})$.

Consider the type $q_{m,m+1}$. Because $q_{m,m+1}$ is strictly definable over $M \cup \bar{\alpha}|m-1 \cup \bar{\gamma}_m$, for the formula $\phi_{m+1}(x, \bar{z}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1}, \bar{c}_{m+1})$ there are $G^m_1(y_m, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{e}_m)$, $G^m_2(y_m, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{e}_m),\ \bar{e}_m \in M$ such that

$$M' \models G^m_1(y_m, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{e}_m) < \alpha_m < G^m_2(y_m, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{e}_m)$$

and for $b, \bar{a} \in M$ we have:

$$c)\ M' \models \exists y_m(G^m_1(M') < y_m < G^m_2(M') \wedge \phi_{m+1}(b, \bar{a}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1}, \bar{c}_{m+1})) \rightarrow$$
$$\forall y_m(G^m_1(M') < y_m < G^m_2(M') \rightarrow \phi_{m+1}(b, \bar{a}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1})).$$

*d)* $M' \models \exists y_m(G_1^m(M') < y_m < G_2^m(M') \wedge \phi_{m+1}(b, \bar{a}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1}, \bar{c}_{m+1}))$

$$\Rightarrow \phi_{m+1}(b, \bar{a}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1}) \in q_{m,m+1}.$$

Denote by

$$\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) := \forall y_m(G_1^m(M') < y_m < G_2^m(M') \to$$

$$\to \phi_{m+1}(x, \bar{z}, \bar{\alpha}|m-1, y_m, \bar{\gamma}_{m+1}, \bar{c}_{m+1})).$$

**Claim 4.2** *(i)* $\forall b, \bar{a} \in M\ [M' \models \phi_{m+1}(b, \bar{a}, \bar{\alpha}|m, \bar{\gamma}_{m+1}, \bar{c}_{m+1}) \Leftrightarrow$

$$\Leftrightarrow M' \models \phi_m(b, \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m)].$$

*(ii)* $M' \models \forall x \forall \bar{z}[\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) \to \phi_{m+1}(x, \bar{z}, \bar{\alpha}|m, \bar{\gamma}_{m+1}, \bar{c}_{m+1})]$.

*(iii)* $\forall \bar{a} \in M[M' \models \exists x \phi_m(x, \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) \to$

$$\Rightarrow M' \models \exists x \phi_{m+1}(x, \bar{a}, \bar{\alpha}|m, \bar{\gamma}_{m+1}, \bar{c}_{m+1})].$$

*Proof of Claim 4.2.* (i) follows from c).
(ii) follows from definition of $\phi_m$.
(iii) follows from (ii).

**Note 4.5** $M' \models \forall x \forall \bar{z}[\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) \to \phi(x, \bar{z}, \bar{\alpha})]$.

**Claim 4.3** *Let* $\bar{a} \in M^{l(\bar{z})}, \delta \in \phi_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m)$. *Then* $\forall \bar{\mu} \in M' \backslash M,\ tp(\bar{\mu}/M \cup \bar{\alpha}|m-1 \cup \bar{\gamma}_m) = tp(\bar{\alpha}/M \cup \bar{\alpha}|m-1 \cup \bar{\gamma}_m))$ *the following is true:*

$$\delta \in \phi(M', \bar{a}, \bar{\mu}),\ \textit{and}\ \phi_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) \subset \phi(M', \bar{a}, \bar{\mu}).$$

*Proof of Claim 4.3.* By construction of $\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m)$ we have:

$$\phi_m(x, \bar{z}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) \equiv$$

$$\forall y_m \forall y_{m+1} \ldots \forall y_n[\bigwedge_{j=m}^{n} G_1^j(M') < y_j < G_2^j(M') \to \phi(x, \bar{z}, \bar{\alpha}|m-1, y_m, y_{m+1}, \ldots, y_n)].$$

By Note 4.5, $M' \models \phi(\delta, \bar{a}, \bar{\alpha})$.
Because $\bigwedge_{j=m}^{n} G_1^j(M') < y_j < G_2^j(M') \in tp(<\alpha_m, \alpha_{m+1}, \ldots, \alpha_n>/M \cup \bar{\alpha}|m-1, \bar{\gamma})$, we have $M' \models \phi(\delta, \bar{a}, \bar{\mu})$. □
Conclusion. Let $\bar{\gamma} := \bar{\gamma}_j,\ \bar{c} := \bar{c}_j$.

$$K_\phi(\bar{z}, \bar{a}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{c}) := \exists x(\phi(x, \bar{z}, \bar{\alpha}) \wedge \bigwedge_{m \in \{1, i_2, \ldots, i_s\}} \phi(x, \bar{z}, \bar{\beta}_m) \wedge \phi_j(x, \bar{z}, \bar{\alpha}|j-1, \bar{\gamma}, \bar{c})).$$

Let $\bar{a} \in M^{l(\bar{z})}$, suppose $\exists b \in M,\ M' \models \phi(b, \bar{a}, \bar{\alpha})$.
Then $M' \models \phi(b, \bar{a}, \bar{\beta}_m)$ because $tp(\bar{\beta}_m/M) = tp(\bar{\alpha}/M),\ \forall m \in \{1, i_1, \ldots, i_s\}$.
By Claim 4.2(i) we have

$$M' \models \phi_j(b, \bar{a}, \bar{\alpha}|j-1, \bar{\gamma}, \bar{c}).$$

Thus, $M' \models K_\phi(\bar{a}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{c})$.

Suppose $\phi(M', \bar{a}, \bar{\alpha}) \cap M = \emptyset$.

Let $r \in S_1(M)$, $\phi(M', \bar{a}, \bar{\alpha}) \cap r(M') \neq \emptyset$. Denote by $\phi^r(M', \bar{a}, \bar{\alpha}) := \phi(M', \bar{a}, \bar{\alpha}) \cap r(M')$.

So, $\phi^r(M', \bar{a}, \bar{\alpha})$ is $< \bar{a}, \bar{\alpha} >$-definable by Note 1.2 and because $\phi(M', \bar{a}, \bar{\alpha}) \cap M = \emptyset$.

Let $m \in \{1, 2, \ldots, n\}$ such that $r \in O_m$ then $r(\not\perp^w, \bar{\alpha}|m-1)r_m$.

By construction of $\bar{\beta}_m$ there is $\eta$, $V_{r_m}(\bar{\alpha}) < \eta < V_{r_m}(\bar{\beta}_m)$ or $V_{r_m}(\bar{\beta}_m) < \eta < V_{r_m}(\bar{\alpha})$. Then by Note 4.3 we have $V_r(\bar{\alpha}) \cap V_r(\bar{\beta}_m) = \emptyset$.

**Note 4.6** (i) If $\phi^r(M', \bar{a}, \bar{\alpha}) \subseteq V_r(\bar{\alpha})$ then $\phi^r(M', \bar{a}, \bar{\alpha}) \cap \phi^r(M', \bar{a}, \bar{\beta}_m) = \emptyset$.

(ii) If $r$ is irrational then $\phi^r(M', \bar{a}, \bar{\alpha}) \cap \phi^r(M', \bar{a}, \bar{\beta}_m) = \emptyset$.

$\phi^r_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) := \phi_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m) \cap r(M')$ is $< \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m >$-definable by Note 4.5 and because $\phi(M', \bar{a}, \bar{\alpha}) \cap M = \emptyset$.

**Claim 4.4** *Let $r$ be quasirational. Then $\phi^r_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m) = \emptyset$.*

*Proof of Claim 4.4* Without loss of generality we suppose that $r, p_m$ are quasirational to the right.

By construction of $\bar{\gamma}$ we have:

$$V_{p_m}(\bar{\gamma}_m) < \eta < V_{p_m}(\bar{\alpha}) < U_{p_m}(M')^+.$$

By Notes 4.3, 2.8(ii)

$$V_r(\bar{\gamma}_m, \bar{\alpha}_{m-1}) < \eta_1 < V_r(\bar{\alpha}) < U_r(M')^+.$$

Suppose, $\phi^r_m(M', \bar{a}, \bar{\alpha}|m-1, \bar{\gamma}_m) \neq \emptyset$. By Note 2.8(iii)

$$\phi^r(M', \bar{\alpha}, \bar{a}) \subset \phi^r_m(M', \bar{\alpha}|m-1, \bar{\gamma}_m, \bar{c}_m).$$

It is contradiction with Note 4.5 and Note 1.3. □

**Note 4.7** If $r$ is quasirational then $\phi^r_j(M', \bar{a}, \bar{\alpha}|j-1, \bar{\gamma}, \bar{c}) = \emptyset$.

*Proof of Note 4.7.* It follows from Claim 4.2(ii) and Claim 4.4. □

Thus by Note 4.6(ii) and Note 4.7 we have

$$\phi(M', \bar{a}, \bar{\alpha}) \cap \bigcap_{m \in \{1, i_1, \ldots, i_s\}} \phi(M', \bar{a}, \bar{\beta}_m) \cap \phi_j(M', \bar{\alpha}|j-1, \bar{\gamma}, \bar{c}) = \emptyset.$$

It means that $M' \models \neg K_\phi(\bar{a}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{c})$. Thus,

$$M' \models K_\phi(\bar{a}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{c}) \quad \text{iff} \quad \phi(M', \bar{a}, \bar{\alpha}) \cap M \neq \emptyset.$$

So, Theorem 4.1 is proved. □

**Theorem 4.2** *Let $M$ be a model of w.o.m. theory $T$, $M'$ be an elementary extension of $M$. Then for any irrational quasisolitary one-type $p \in S_1(M), p(M') \neq \emptyset$, for any formula $\psi(x, \bar{z}, y)$ there exists the formula $K_\psi(\bar{z}, \bar{c}_p, y), \bar{c}_p \in A$ such that for any $\bar{a} \in M^{l(\bar{z})}$ the following holds:*

$$[\exists b \in M,\ M' \models \psi(b, \bar{a}, \alpha) \Leftrightarrow M' \models K_\psi(\bar{a}, \bar{c}_p, \alpha)].$$

*Here, $\alpha$ is arbitrary element from $p(M')$.*

*Proof.* Let $E_p(x, y, \bar{c}_p)$ from Lemma 1.1(ii),

$$\begin{aligned} K_\psi(\bar{z}, \bar{c}_p, y) &:= \exists x \exists y_1, \exists y_2[y_1 < y < y_2 \wedge \neg E_p(y, y_2, \bar{c}_p) \wedge \neg E_p(y, y_1, \bar{c}_p) \wedge \\ &\wedge \quad \forall t(y_1 \le t \le y_2 \to \psi(x, \bar{z}, t))] \end{aligned}$$

By note 1.5(ii) there exists the cut $(C_p, D_p)$ of $M$ such that $C_p < p(M') < D_p$.

*Necessity.* Let $b \in M$ such that $M' \models \psi(b, \bar{a}, \alpha)$. Then $\psi(b, \bar{a}, M') = \bigcup_{i=1}^{n_\psi} \psi_i(b, \bar{a}, M')$, where $\{\psi_i(b, \bar{a}, M') \mid i \le n_\psi\}$ is finite family of $\neg\psi(b, \bar{a}, M')$–separable convex $(b, \bar{a})$–definable subsets of $\psi(b, \bar{a}, M')$.

Let $\psi_i(b, \bar{a}, M')$ such that $\alpha \in \psi_i(b, \bar{a}, M')$. So, $\psi_i(b, \bar{a}, y) \in p$. Because $p$ is determinated by $(C_p, D_p)$, there exists $c \in C_p,\ d \in D_p$ such that $c, d \in \psi_i(b, \bar{a}, M')$ and $\alpha \in (c, d)$ and

$$M' \models c < \alpha < d \wedge \forall t[c \le t \le d \to \psi_i(b, \bar{a}, t) \wedge \neg E_p(c, \alpha, \bar{c}_p) \wedge \neg E_p(d, \alpha, \bar{c}_p)]$$

Thus $M' \models K_\psi(\bar{a}, \bar{c}_p, \alpha)$.

*Sufficity.* $M' \models K_\psi(\bar{a}, \bar{c}_p, \alpha)$.Let

$$\begin{aligned} L(y_1, \alpha, \bar{a}, \bar{c}_p) &:= \exists x \exists y_2[y_1 < \alpha < y_2 \wedge \neg E_p(y_1, \alpha, \bar{c}_p) \wedge \neg E_p(y_2, \alpha, \bar{c}_p) \wedge \\ &\wedge \quad \forall t(y_1 \le t \le y_2 \to \psi(x, \bar{a}, t))] \end{aligned}$$

$L(M', \alpha, \bar{a}, \bar{c}_p) \cap M \neq \emptyset$ because $\forall \gamma \in L(M', \alpha, \bar{a}, \bar{c}_p)$ the following holds:

$$\forall \gamma_1 \in M'[(\gamma < \gamma_1 < \alpha \wedge \neg E_p(\gamma_1, \alpha, \bar{c}_p)) \Rightarrow M' \models L(\gamma_1, \alpha, \bar{a}, \bar{c}_p)]$$

Because $p$ is quasisolitary there is $e \in C_p$ such that $M' \models L(e, \alpha, \bar{a}, \bar{c}_p)$.

$$\begin{aligned} \text{Let } R(y_2, e, \alpha, \bar{a}, \bar{c}_p) &:= \exists x[c < \alpha < y_2 \wedge \neg E_p(y_2, \alpha, \bar{c}_p) \wedge \\ &\wedge \quad \forall t(c \le t \le y_2 \to \psi(x, \bar{a}, t))] \end{aligned}$$

By the same consideration as left, there exists $d \in D_p$ such that $M' \models R(d, c, \alpha, \bar{a}, \bar{c}_p)$.
So, $M' \models \exists x \forall t[c \le t \le d \to \psi(x, \bar{a}, t)]$.
Since $M \prec M'$, $M \models \exists x \forall t[c \le t \le d \to \psi(x, \bar{a}, t)]$.
So, there is $b \in M$ such that

$$M \models \forall t[c \le t \le d \to \psi(b, \bar{a}, t)],$$

$\alpha \in (c, d)$ thus $M' \models \psi(b, \bar{a}, \alpha)$. □

**Note 4.8** Let $M$ be a model of w.o.m. theory $T$, $M'$ be an elementary extension of $M$. Then for any $p_1, p_2, \dots, p_n \in S_1(M)$ irrational, quasisolitary types such that $p_1(x_1) \cup \dots \cup p_n(x_n)$ complete $n$-type over $M$, $p_i(M') \neq \emptyset,\ 1 \le i \le n$ for any formula $\psi(x, \bar{z}, y_1, \dots, y_n)$ there exists the formula $K_\psi(\bar{z}, \bar{c}, y_1, \dots, y_n),\ \bar{c} \in A$ such that for any $\bar{a} \in M^{l(\bar{z})}$ the following holds:

$$[\exists b \in M,\ M' \models \psi(b, \bar{a}, \bar{\alpha}) \Leftrightarrow M' \models K_\psi(\bar{a}, \bar{c}, \bar{\alpha})].$$

Here, $\bar{\alpha} =< \alpha_1, \dots, \alpha_n >,\ \alpha_i$ is arbitrary element from $p_i(M'),\ 1 \le i \le n$.

**Note 4.9** Theorem 4.2 was proved by Macpherson, Marker, Steinhorn [MMS] for irrational solitary one-type in o-minimal model. (In fact, in real closed field).

**Theorem 4.3** *Let $M$ be a model of w.o.m. theory, $M^+$ be an expansion of $M$ by the family of unary predicates. Then $M^+$ is a model of w.o.m. theory iff the realization of each predicate is finite number of convex subsets.*

**Note 4.10** Let $M$ be a model of w.o.m. theory, $M^+$ be an expansion of M such that $M^+$ is weakly o-minimal.

If the theory of $M^+$ is not weakly o-minimal then this expansion of $M$ must be contained some $n$-ary ($n \geq 2$) predicate which is non-$M$-definable in $M$.

*Proof.* Let $M'$ be an elementary extension of $M$ such that $M'$ is $|M|^+$-saturated.

Let $M^+ =< M, \Sigma, Q^1_\lambda >_{\lambda \in S}$, where $Q'_\lambda(M')$ is finite number of convex sets. Let $L_1 := \{\beta_{\lambda,i} | \beta_{\lambda,i} \in (C_{\lambda,i}, D_{\lambda,i})$, where $(C_{\lambda,i}, D_{\lambda,i})$ is irrational cut in M, which is determined by $Q^1_\lambda\}$, $Z(L_1) := \{\mu \in M' \setminus M : \exists \bar{\alpha} \in L_1, \bar{\beta}, \bar{\gamma} \in M' \setminus M, \bar{\beta}, \bar{\gamma}$ were chosen on $\bar{\alpha}$ by Theorem 4.1, $\mu \in \bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma}\}$.

Let $L_s := \cup_{n<\omega} Z^n(L_1)$.

Denote by

$$\Sigma^+ := \Sigma \cup \{Q^1_\lambda|_{\lambda \in S}\},\ \Sigma^+(M \cup L_s) := \Sigma^+ \cup \{c_a | a \in M \cup L_s\}$$

**Fact 4.1** *For any quantifier-free formula $\theta(\bar{y})$ of signature $\Sigma^+$ there is $\Phi_\theta(\bar{y}, \bar{\alpha})$ of $\Sigma(L_s)$ such that*

$$\forall \bar{b} \in M[M^+ \models \theta(\bar{b}) \Leftrightarrow M' \models \phi_\theta(\bar{b}, \bar{\alpha})]$$

*Proof.* We replace "$Q^1_\lambda(y)$" by " $\vee (\beta_{\lambda,2k-1} < y < \beta_{\lambda,2k})$".

**Fact 4.2** *If for the formula $\theta(x, \bar{y})$ of signature $\Sigma^+$ there is the formula $\phi_\theta(x, \bar{y}, \bar{\alpha})$ of signature $\Sigma(M \bigcup L)$ such that*

$$\forall b, \bar{a} \in M[M^+ \models \theta(b, \bar{a}) \Leftrightarrow M' \models \phi_\theta(b, \bar{a}, \bar{\alpha})]$$

*then for the formulas $K_{\phi_\theta}(\bar{y}, \bar{\alpha}', \bar{c}_1), K_{\neg\phi_\theta}(\bar{y}, \bar{\alpha}', \bar{c}_2)$ from Theorem 4.1, the following is true:*

(i) $\forall \bar{a} \in M[M^+ \models \exists x \theta(x, \bar{a}) \Leftrightarrow M' \models K_{\phi_\theta}(\bar{a}, \bar{\alpha}', \bar{c}_1)]$,

(ii) $\forall \bar{a} \in M[M^+ \models \forall x \theta(x, \bar{a}) \Leftrightarrow M' \models \neg K_{\neg\phi_\theta}(\bar{a}, \bar{\alpha}', \bar{c}_2)]$.

*Proof.* (i) follows from Theorem 4.1 and Definition $K_\theta$. (ii) follows from (i) and from the equivalence of formulas

$$\neg\phi_\theta(x, \bar{y}, \bar{\alpha}) \equiv \phi_{\neg\theta}(x, \bar{y}, \bar{\alpha}).$$

□

Denote by

$$\phi_{\exists x \theta(x,\bar{y})}(\bar{y}, \bar{\alpha}', \bar{c}) := K_{\phi_\theta}(\bar{y}, \bar{\alpha}', \bar{c})$$

$$\phi_{\forall x \theta(x,\bar{y})}(\bar{y}, \bar{\alpha}', \bar{c}) := \neg K_{\neg\phi_\theta}(\bar{y}, \bar{\alpha}', \bar{c})$$

**Fact 4.3** *For any formula* $\theta(\bar{y})$ *of signature* $\Sigma^+$ *there is a formula* $\phi_\theta(\bar{y}, \bar{\alpha}, \bar{c}), \bar{c} \in M$ *of signature* $\Sigma(M \bigcup L_s)$ *such that*

$$\forall \bar{a} \in M^{l(\bar{y})}[M^+ \models \theta(\bar{a}) \Leftrightarrow M' \models \phi_\theta(\bar{a}, \bar{\alpha}, \bar{c})].$$

*Proof.* It follows from Fact 4.1, Fact 4.2 and from the representation of $\theta(\bar{y})$ in the following form:

$$\theta(\bar{y}) \equiv R_1 x_1 \dots R_n x_n (\theta_1(x_1, \dots, x_n, \bar{y})),$$

where $R_i \in \{\exists, \forall\}, \theta_1(x_1, \dots, x_n, \bar{y})$ is quantifier-free formula of signature $\Sigma^+$.

**Fact 4.4** $M^+$ *is model of weakly o-minimal theory.*

*Proof.* Let $\theta(z, \bar{a})$ be an arbitrary formula with one free variable of signature $\Sigma^+(M)$. By Fact 4.3 there is the formula $\phi_\theta(x, \bar{y}, \bar{\alpha}, \bar{c}), \bar{c} \in M$ such that for any $b \in M$, any $\bar{a} \in M^{l(\bar{y})}$ the following is true:

$$M^+ \models \theta(b, \bar{a}) \Leftrightarrow M' \models \phi_\theta(b, \bar{a}, \bar{\alpha}, \bar{c})]$$

Consider the formula $\phi_\theta(x, \bar{a}, \bar{\alpha}, \bar{c})$. By weak o-minimality of $M'$
$\phi_\theta(M', \bar{a}, \bar{\alpha}, \bar{c}) = \bigcup_{i=1}^{n_\phi} \phi_\theta^i(M', \bar{a}, \bar{\alpha}, \bar{c})$.

Let $a_1, a_2 \in \phi_\theta^i(M', \bar{a}, \bar{\alpha}, \bar{c}) \bigcap M$.
Then $(a_1, a_2) \subseteq \phi_\theta^i(M', \bar{a}, \bar{\alpha}, \bar{c})$ because $\phi_\theta^i(M', \bar{a}, \bar{\alpha}, \bar{c})$ is convex.
Here $(a_1, a_2) := \{\gamma \in M' | a_1 < \gamma < a_2\}$.
Thus for any $a \in (a_1, a_2) \cap M$ follows $a \in \psi_\theta^i(M', \bar{a}, \bar{\alpha}, \bar{c})$.
So, the realization of the formula $\theta(x, \bar{a})$ is union of finite number of $\neg\theta(M^+, \bar{a})$-separable convex subsets of $M$.
This number does not depend from $\bar{a}$, only from the formula $\phi_0(x, \bar{y}, \bar{\alpha}, \bar{c})$ of the weakly o-minimal theory by Note 1.2. Thus the theory $Th(M^+)$ is weakly o-minimal. □

Sufficity of Theorem 4.3 follows from Fact 4.4.

Necessity of Theorem 4.3 follows from Definition 1.3.

**Note 4.11** In paper [MMS] Theorem 4.2, Facts 4.1, 4.2, 4.3, 4.4 are represented for $L_1 = \{\alpha_1, \dots, \alpha_n, \dots\}, p_i = tp(\alpha_i/M)$ — solitary, $V_{p_i}(L_1 \setminus \alpha_i) = \emptyset, M \preceq < R, <, +, \cdot, 0, 1 >$.


**Acknowledgements**

I thank W. Hodges for his organization and skilful guidance of Grant INTAS-93-3547, Ch. Steinhorn for his interesting report on III Kazakh-French Colloqium on Model Theory (June 1994) and for close collaboration, B. Poizat, B. Kulpeshov, R. Aref'ev, V. Verbovsky for useful discussion and valuable remarks.

I also thank participants of the seminars on Model Theory of Universities of Lyon-1 and Paris-7 on which has been doing a report on a preliminary version of this paper for their hospitality in May 1996.

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