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DEFINABILITY OF 1-TYPES IN WEAKLY *o*-MINIMAL THEORIES

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Abstract

In the article, we prove a criterion for definability of 1-types over sets in weakly o-minimal theories in terms of left and right convergences of a formula to a type.

Van den Dries proved that every type over the field of reals is definable. Marker and Steinhorn strengthened his result. They (and, later, Pillay) proved the following assertion. Let $M \prec N$ be a pair of models of some o-minimal theory. If, for each element of N, the type of this element over M is definable then, for each tuple of elements of N, the type of this tuple over M is definable.

We construct a weakly o-minimal theory for which the Marker–Steinhorn theorem fails; i.e., some pair of models of the theory possesses the following property: For all elements of the larger model, the 1-type over the smaller model is definable but there exists a tuple of elements of the larger model whose 2-type over the smaller model is not definable.

Key words and phrases: definable type, weakly o-minimal theory, nonorthogonality of types.

1. Introduction

The article consists of four sections. In Section 1, we present some general facts about definability of types, introduce the notions of a quasimodel type, weak and strong convergences of a formula to a type, and establish a connection between these notions and the well-known notions of stability theory (Proposition 15). In Section 2, we recall the notations, definitions, and available facts (without proof) about nonorthogonality of 1-types in weakly o-minimal theories. In Section 3, we prove a criterion for definability of a 1-type over a set in a model of a weakly o-minimal theory (Theorem 31) in terms of left and right convergences of a formula to a type (Definition 28).

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Let M be an elementary submodel of N. The pair (M, N) is said to be a D-1-*pair* if, for each $\alpha \in N$, the type $\mathrm{tp}(\alpha/M)$ of α over M is definable. The pair (M, N) is said to be a D-*pair* if, for each finite tuple $\overline{\alpha}$ of elements of N, the type $\mathrm{tp}(\overline{\alpha}/M)$ is definable. We call M a D-1-*model* if every 1-type over M is definable and a D-*model* if every type over M is definable. In [13], Van den Dries showed that every D-1-model of the theory of a real closed field is a D-model. In [17], Marker and Steinhorn proved that, for every o-minimal theory, each D-1-pair is a D-pair and, consequently, each D-1-model is a D-model.

In Section 4, we construct a D-1-pair of models in an ω-categorical weakly o-minimal theory that is not a D-pair (Theorem 37).

Throughout the article, we assume that M is a linearly ordered structure of a fixed language L.

Definition 1. A partition (A, B) of M is called a *cut* if $A < B$, i.e., for all $a \in A$ and $b \in B$, we have $a < b$. A cut is said to be *rational* if either A possesses a maximal element, or B possesses a minimal element, or one of these sets is empty. We say that a cut is *quasirational* if A (and, consequently, B) is definable (with parameters). A cut that is not quasirational is said to be *irrational*. We say that M is *Dedekind complete* if every cut of M is rational. We say that M is *quasi-Dedekind complete* if every cut of M is quasirational. Let M be an elementary submodel of N. We say that a cut (A, B) of M is *realized* in N if $A < \alpha < B$ for some $\alpha \in N \setminus M$.

Definition 2 [12, 20]**.** A linearly ordered structure M is *o-minimal* if every definable (with parameters) subset of M is the union of a finite family of points in M and intervals (a, b), where $a \in M \cup \{-\infty\}$ and $b \in M \cup \{\infty\}$.

Observe that every quasirational cut of an o-minimal model is rational.

Definition 3 [17]**.** Let M be an elementary submodel of N, where $N \models T$ and T is an o-minimal theory. The model M is *Dedekind complete in* N if every cut of M realized in N is rational.

Definition 4. A subset A of a linearly ordered structure M is *convex* if every element of M lying between two elements of A belongs to A. In particular, the empty set and all singletons are convex. We say that a formula $\phi(x)$ is *convex* if the set $\phi(M) := \big\{\alpha \in M \mid M \models \phi(\alpha)\big\}$ is convex.

Definition 5 [11, 15]**.** A linearly ordered structure M is said to be *weakly o-minimal* if every definable (with parameters) subset of M is the union of finitely many convex subsets.

A theory T is *weakly o-minimal* if every model of T is weakly o-minimal.

Observe that each o-minimal model is weakly o-minimal.

Definition 6. Let M be an elementary submodel of N, where $N \models T$ and T is a weakly o-minimal theory. We say that M *is quasi-Dedekind complete in* N if no irrational cut of M is realized in N.

Definition 7. Let A be a set in a model M, where $M \models T$, and let $p \in S_n(A)$ for some $n < \omega$. The type p is $\phi(\overline{x}_n, \overline{v})$*-definable*, where $\phi(\overline{x}_n, \overline{v}) \in L(\overline{x}_n)$, if there exists a formula $R_\phi(\overline{v}) \in L(A)$ (i.e., an A-definable formula) such that, for every $\overline{a} \in A$, we have $\phi(\overline{x}_n, \overline{a}) \in p$ if and only if $M \models R_\phi(\overline{a})$.

Such a formula $R_\phi(\overline{v})$ is called a $\phi(\overline{x}_n, \overline{v})$*-definition* of p. We say that the type p is *definable* if p is $\phi(\overline{x}_n, \overline{v})$-definable for every formula $\phi(\overline{x}_n, \overline{v}) \in L(\overline{x}_n)$, $n < \omega$.

A tuple $\overline{\gamma} \in M$ is said to be ht*-definable over* A if its type over A is definable.

In an o-minimal theory, for every cut of M, there exists a unique 1-type over M extending the cut (see [17, Lemma 2.3]). This type is definable if and only if the cut is rational. Hence, an o-minimal model M is a D-1-model if and only if M is Dedekind complete. A pair (M, N) in an o-minimal theory is a D-1-pair if and only if M is Dedekind complete in N.

Van den Dries studied definable types over a real closed field and proved the following

Theorem 8 [13]**.** *Every type over* $(\mathbb{R}, +, \cdot, 0, 1)$ *is definable, where* $\mathbb{R}$ *stands for the set of reals.*

Marker and Steinhorn generalized this result to the case of o-minimal theories.

Theorem 9 [17]**.** *Let* T *be an o-minimal theory and let* $M \models T$.

(1) *The model* M *is Dedekind complete if and only if every type in* $S(M) := \bigcup_{n<\omega} S_n(M)$ *is definable.*
(2) *Let* M *be an elementary submodel of* N*. Then, for every* $\overline{\alpha} \in N \setminus M$*, the type* $\mathrm{tp}(\overline{\alpha}/M)$ *is definable if and only if* M *is Dedekind complete in* N.

Let T be a weakly o-minimal theory. By Assertion 21, a pair (M, N) of models in T is a D-1-pair if and only if M is quasi-Dedekind complete in N, and a model M of T is a D-1-model if and only if M is quasi-Dedekind complete.

Some general facts about definability of types. The notion of convergence of a formula to a type is central to this article.

Let T be a complete theory of language L, let N be a sufficiently saturated model of T, let $A \subset N$, and let $\overline{\alpha} \in N$. Let $q \in S(A)$ be a nonisolated type and let $\phi(\overline{x}, \overline{y})$ be an A-definable formula. We say that the formula $\phi(\overline{x}, \bar{b})$, $\bar{b} \in N$, *divides* $C \subset N^l$, where l is the length of the tuple $\overline{x}$ and C is not necessarily definable, if $\phi(N^l, \bar{b}) \cap C \neq \varnothing$ and $\neg\phi(N^l, \bar{b}) \cap C \neq \varnothing$. We often write $\phi(N, \bar{b})$ instead of $\phi(N^l, \bar{b})$.

We say that an A-definable formula $\phi(\overline{x}, \overline{y})$ *converges weakly to a type* $q(\overline{x}) \in S(A)$ and write $\mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q(\overline{x})\big)$ if, for every $\Theta \in q$, there exists $\overline{a} \in A$ such that $\phi(\overline{x}, \overline{a})$ divides $\Theta(N)$.

We say that an A-definable formula $\phi(\overline{x}, \overline{y})$ *converges strongly to a type* $q(\overline{x})$ and write $\mathrm{STC}\big(\phi(\overline{x}, \overline{y}), q(\overline{x})\big)$ if, for every $\Theta \in q$, there exists $\overline{a} \in A$ such that $\phi(N, \overline{a}) \subset \Theta(N)$. We usually omit $\overline{x}$ in the notation $q(\overline{x})$.

Assume that $\mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q(\overline{x})\big)$ holds for some $q \in S(A)$. Let $\phi(\overline{x}, \overline{y})$ be the graph of an A-definable function $f(\overline{y})$ $\big($i.e., $\phi(\overline{x}, \overline{y}) \equiv \overline{x} = f(\overline{y})\big)$. Then $\mathrm{STC}\big(\phi(\overline{x}, \overline{y}), q(\overline{x})\big)$ holds. In this case, we say that *the values of* $f(\overline{y})$ *converge to the type* q and write $\mathrm{STC}\big(f(\overline{y}), q\big)$.

We say that a tuple $\overline{\alpha}$ is *weakly orthogonal to a type* q and write $\overline{\alpha} \perp^w q$ if, for every A-definable formula $\phi(\overline{x}, \overline{y})$, the formula $\phi(\overline{x}, \overline{\alpha})$ does not divide $q(N) = \bigcap_{\Theta \in q} \Theta(N)$. We say that $\overline{\alpha}$ is *not weakly orthogonal to a type* q and write $\overline{\alpha} \not\perp^w q$ if there exists an A-definable formula $\phi(\overline{x}, \overline{y})$ such that $\phi(\overline{x}, \overline{\alpha})$ divides $q(N)$.

Observe that $\phi(\overline{x}, \overline{\alpha})$ divides $q(N)$ if and only if, for every $\Theta \in q$, the formula $\phi(\overline{x}, \overline{\alpha})$ divides $\Theta(N)$. In this case, for all $\overline{\alpha}$ and $\overline{\beta}$ with $\mathrm{tp}(\overline{\alpha}/A) = \mathrm{tp}(\overline{\beta}/A)$, we have $\overline{\alpha} \not\perp^w q \Longleftrightarrow \overline{\beta} \not\perp^w q$. We say that a type $p \in S(A)$ is *weakly orthogonal to a type* $q \in S(A)$ and write $p \perp^w q$ if there exists $\overline{\alpha} \in p(N)$ such that $\overline{\alpha} \perp^w q$ or, equivalently (see [21, Definition V. 1.1 (i)]), $p(\overline{x}) \cup q(\overline{y})$ defines a complete $\big(l(\overline{x}) + l(\overline{y})\big)$-type. Observe that $p \not\perp^w q$ implies $q \not\perp^w p$ [21, Lemma V. 1.1 (i)].

Definition 10. Let Γ be a nonisolated and consistent set of A-definable formulas. We say that Γ is a *quasimodel set* if, for every formula $\Theta \in \Gamma$, there exists $\overline{a} \in A$ such that $N \models \Theta(\overline{a})$.

Assertion 11. *Let Γ be a nonisolated quasimodel set of formulas over A. Assume that Γ is closed under formation of finite conjunctions. Then Γ can be extended to a quasimodel type over A.*

Proof. For every A-definable formula $Q(\overline{y})$, at least one of the sets $\Gamma(\overline{y}) \cup \big\{Q(\overline{y})\big\}$, $\Gamma(\overline{y}) \cup \big\{\neg Q(\overline{y})\big\}$ is a quasimodel set. $\Box$

Let $q(\overline{x}) \in S(A)$, where $A \subset N$. We say that q is a *strictly definable* (or *weakly isolated*) type if, for every A-definable formula $\phi(\overline{x}, \overline{y})$, there exists an A-definable formula $\Theta(\overline{x}) \in q$ such that

$$N \models \exists\, \overline{x}\, \big(\Theta(\overline{x}) \wedge \phi(\overline{x}, \overline{a})\big) \to \forall\, \overline{x}\, \big(\Theta(\overline{x}) \to \phi(\overline{x}, \overline{a})\big)$$

for every $\overline{a} \in A^{l(\overline{y})}$.

It is clear that every isolated type is strictly definable. Every strictly definable type $q \in S(A)$ is definable; moreover, there exists a formula $\Theta \in q$ such that the A-definable formula $\Psi_\phi(\overline{y}) := \forall\, \overline{x}\, \big(\Theta(\overline{x}) \to \phi(\overline{x}, \overline{y})\big)$ is a $\phi(\overline{x}, \overline{y})$-definition of $q(\overline{x})$.

Assertion 12. *A type $q \in S(A)$ is strictly definable if and only if, for every formula $\phi(\overline{x}, \overline{y})$ of language L, we have $\neg \mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q\big)$.*

Assertion 13. *Let $q \in S(A)$.*

(1) *If there is an A-definable formula $\phi(\overline{x}, \overline{y})$ such that $\mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q\big)$ holds then there exists a quasimodel type $r \in S(A)$ such that, for every $\overline{\alpha} \in r(N)$, the formula $\phi(x, \overline{\alpha})$ divides $q(N)$, i.e., $r \not\perp^w q$.*

(2) *If $r \not\perp^w q$ for some quasimodel type $r \in S(A)$ then $\mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q\big)$ holds for some A-definable formula $\phi(\overline{x}, \overline{y})$.*

Proof. (1) Denote by Γ the following set of A-definable formulas:

$$\Big\{K(\Theta)(\overline{y}) \mid \Theta \in q,\ K(\Theta)(\overline{y}) := \exists\, x\big[\phi(\overline{x}, \overline{y}) \wedge \Theta(\overline{x})\big] \wedge \exists\, x\big[\neg\phi(\overline{x}, \overline{y}) \wedge \Theta(\overline{x})\big]\Big\}.$$

It is clear that Γ is a quasimodel set; moreover, Γ is closed under formation of finite conjunctions. By Assertion 11, there exists a quasimodel type $r \in S(A)$ extending Γ. Hence, for every $\overline{\gamma} \in r(N) \subseteq \Gamma(N)$, the formula $\phi(x, \overline{\alpha})$ divides $q(N)$. Therefore, $\overline{\gamma} \not\perp^w q$ and, consequently, $r \not\perp^w q$.

(2) Let $\overline{\alpha} \in r(N)$ and let $\overline{\alpha} \not\perp^w q$. Then, for some formula $\phi(\overline{x}, \overline{y})$, the formula $\phi(\overline{x}, \overline{\alpha})$ divides $q(N)$. Hence, $\phi(\overline{x}, \overline{\alpha})$ divides $\Theta(N)$ for every $\Theta(\overline{x}) \in q$. This means that $N \models K(\Theta)(\overline{\alpha})$. Therefore, we have $K(\Theta)(\overline{y}) \in r$. Since r is a quasimodel type, there exists $\overline{a} \in A$ such that $N \models K(\Theta)(\overline{a})$. Thus, $\mathrm{WEC}\big(\phi(\overline{x}, \overline{y}), q\big)$ holds. □

Assertion 14. *Let $r, q \in S(A)$. Assume that r is a quasimodel type and $H(\overline{x}, \overline{y})$ is an A-definable formula. If there exists $\overline{\gamma} \in r(N)$ such that $H(N, \overline{\gamma}) \subset q(N)$ then $\mathrm{STC}\big(H(\overline{x}, \overline{y}), q\big)$ holds.*

Proof. We have $H(N, \overline{\gamma}) \subset q(N)$ if and only if $H(N, \overline{\gamma}) \subset \Theta(N)$ for every formula $\Theta \in q$. The latter condition is equivalent to the fact that $N \models \forall\, \overline{x}\big(H(\overline{x}, \overline{\gamma}) \to \Theta(\overline{x})\big)$ for every formula $\Theta \in q$. Since $\forall\, \overline{x}\big(H(\overline{x}, \overline{y}) \to \Theta(\bar{x})\big) \in r$ and r is a quasimodel type, we conclude that $\mathrm{STC}\big(H(\overline{x}, \overline{y}), q\big)$ holds. □

For a weakly o-minimal theory, the notion of weak convergence of a formula to a 1-type transforms into the notions of left, right, and two-sided convergences (cf. Definition 28 and Lemma 36). These notions are used in the formulations of the criterion for definability of a 1-type over an arbitrary set (Theorem 31) and of its corollary presenting the criterion for definability of a 1-type over the union of a model and a finite sequence that is ht-definable over the model (Proposition 35).

R e m a r k . The notion of convergence of a formula to a nonisolated type is implicitly used in proofs of theorems about ordered models (see [4, 6, 17, 19]) and models of stable theories (see [1, 3, 8, 9, 14, 21]). The notion of a quasimodel type is implicitly used in [2, 4]. To conclude the motivation for introducing these notions, we present a simple fact which explains the nature of convergence of a formula to a type in terms of well-known notions of the stability theory.

Proposition 15. *Let T be a stable theory, let N be a large saturated model of T, let $A \subset N$, let $q \in S(A)$, and let $\phi(\overline{x}, \overline{y})$ be an A-definable formula. Then the following assertions hold*:

(1) *If q is a quasimodel type then q is stationary.*
(2) *If $\mathrm{WEC}(\phi(\overline{x}, \overline{y}), q)$ holds then there exists a quasimodel (stationary) type $r \in S(A)$ such that $r \not\perp^a q$.*
(3) *Let M be an elementary submodel of N. Assume that $p \in S(N)$ and p does not fork over M. Then $p_1 := \big\{\theta(\overline{x}) \in p \;\big|\; \theta(\overline{x})$ is an $(M \cup \overline{\alpha})$-definable formula$\big\}$ is a quasimodel type for every $\overline{\alpha} \in N \setminus M$.*
(4) *Let $\overline{\alpha}$ be a tuple in $N \setminus A$, let $q = \mathrm{tp}(\overline{\alpha}/A)$, and let the formula $\phi(\overline{x}, \overline{\alpha})$ be divided over A* (see [21, Definition V.1.3]). *Then, for every $p \in S(A)$ with $p(N) \cap \phi(N, \overline{\alpha}) \neq \varnothing$, we have $p(N) \cap \neg\phi(N, \overline{\alpha}) \neq \varnothing$, i.e., $\overline{\alpha} \not\perp^w p$ and, consequently, $q \not\perp^w p$.*

Proof. (1) By [21, Definition III. 1.7, Definition III. 4.1, Lemma III. 4.18, Corollary III. 2.9 (ii)], we conclude that q is a stationary type provided, for every A-definable equivalence relation $E(\overline{x}, \overline{z})$ with finitely many cosets, there exists $\overline{a} \in A$ such that $E(\overline{x}, \overline{a}) \in q$.

Let $E(\overline{x}, \overline{y})$ be an A-definable equivalence relation with finitely many cosets. Put $\phi_0(\overline{x}) := (\exists\, \overline{z})E(\overline{x}, \overline{z}) \in q$. Since q is a quasimodel type, there exists $\overline{a}_0 \in A$ such that $N \models (\exists\, \overline{z})E(\overline{z}, \overline{a}_0)$. If $E(\overline{x}, \overline{a}_0) \notin q$ then $\phi_1(\overline{x}) := \exists\, \overline{z}\big[E(\overline{x}, \overline{z}) \wedge \neg E(\overline{x}, \overline{a}_0)\big]$ is an A-definable formula and $\phi_1 \in q$. Consider $\overline{a}_1 \in A$ such that $N \models \phi_1(\overline{a}_1)$. Put $\phi_i(\overline{x}) := \exists\, \overline{z}\big[E(\overline{x}, \overline{z}) \wedge \bigwedge_{j<i} \neg E(\overline{x}, \overline{a}_j)\big]$. This is an A-definable formula; moreover, $\phi_i \in q$. Then $N \models \bigwedge_{j \neq n < i} \neg E(\overline{a}_j, \overline{a}_n)$. If, for every $i < \omega$, we have $\phi_i(\overline{x}) \notin q$ then we arrive at a contradiction, because there exist only finitely many cosets modulo E.

Thus, if $E(\overline{x}, \overline{z})$ is an A-definable equivalence relation with finitely many cosets then $q(N)$ is a subset of one of A-definable cosets modulo E.

(2) By [8, p. 143], two types $p, q \in S(A)$ are almost orthogonal, i.e., $p \perp^a q$, if arbitrary tuples $\overline{\alpha} \in p(N)$ and $\overline{\beta} \in q(N)$ are A-independent; moreover, if $p \perp^a r$ and at least one of these types is stationary then $p \perp^w r$. In view of Assertion 13, there exists a quasimodel type $r \in S(A)$ such that $p \not\perp^w r$. Since r is stationary, Proposition 15 (1) implies $p \not\perp^a r$.

(3) From [21, Theorem III. 0.1. (4), Corollary III. 4.10] it follows that a type $p \in S(N)$ does not fork over M, where $M \prec N$, if and only if p is finitely satisfiable in M (recall that p is finitely satisfiable in M if, for every formula $\phi \in p$, there exists $\overline{a} \in M$ such that $N \models \phi(\overline{a})$). Consider an arbitrary $(M \cup \overline{\alpha})$-definable formula $\theta(\overline{x}, \overline{\alpha}) \in p$. Since p is finitely satisfiable in M,

there exists $\overline{a} \in M$ such that $N \models \theta(\overline{a}, \overline{\alpha})$. This means that p_1 is a quasimodel type.

(4) Let $\overline{\gamma}$ be a tuple in $p(N) \cap \phi(N, \overline{\alpha})$. The definition of division over A implies that there exist $n < \omega$ and $\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n \in q(N)$ such that $N \models \neg\exists\, \overline{x}\big(\phi(\overline{x}, \overline{\alpha}) \wedge \bigwedge_{1\le i\le n} \phi(\overline{x}, \overline{\alpha}_i)\big)$. Then, for some $\overline{\alpha}_i$, we have $\overline{\gamma} \notin \phi(N, \overline{\alpha}_i)$. Since $\mathrm{tp}(\overline{\alpha}/A) = \mathrm{tp}(\overline{\alpha}_i/A) = q$ and $\overline{\gamma} \in p(N)$, $p \in S(A)$, there exists $\overline{\gamma}_i \in p(N)$ such that $\overline{\gamma}_i \notin \phi(N, \overline{\alpha})$. This finishes the proof of Proposition 15 (4). $\square$

2. Notations, definitions, and facts

In the sequel, we assume that M and N are models of a weakly o-minimal theory T such that $M \prec N$ and N is sufficiently saturated. Given $A \subset N$, put

$$A^+ := \{x \in N \mid \forall\, a \in A,\ x > a\},$$

$$A^- := \{x \in N \mid \forall\, a \in A,\ x < a\}.$$

For an arbitrary tuple $\langle y_1, \ldots, y_n\rangle$, we use the abbreviation $\overline{y}$. Let $l(\overline{y})$ denote the length of such a tuple, i.e., $l(\overline{y}) = n$. We often write $\overline{a} \in A$ instead of $\overline{a} \in A^{l(\overline{a})}$.

Let B be a set in a model N. By a *B-definable formula* we mean the formula $\phi(\overline{x}, \bar{b})$ obtained from an L-formula $\phi(\overline{x}, \overline{y})$ by substituting a tuple of parameters $\bar{b} \in B^{l(\overline{y})}$. A subset $X \subset M^l$ is said to be *B-definable* if $X = \phi(M^l, \bar{b}) := \big\{\overline{a} \in M^l \mid M \models \phi(\overline{a}, \bar{b})\big\}$ for some B-definable formula $\phi(\overline{x}, \bar{b})$ with $l(\overline{x}) = l$. We sometimes write "$L(B)$-formula" instead of "B-definable formula." Let B be an arbitrary (not necessarily definable) convex set. We say that a formula $U(x)$ *splits* B if $U(N)$ and $\neg U(N)$ are convex sets, $U(N) \cap B \neq \varnothing$, and $\neg U(N) \cap B \neq \varnothing$. Given $A \subset M$, let $S_n(A)$ denote the set of n-types over A and let $S(A) = \bigcup_{n<\omega} S_n(A)$. We often write first-order formulas as relations between definable sets. For example,

$$x < \phi(N) \equiv \forall\, y\big(\phi(y) \to x < y\big);$$

$$x \in (\beta_1, \beta_2) \equiv \beta_1 < x < \beta_2;$$

$$\phi(N) \cap \theta(N) \neq \varnothing \equiv N \models \exists\, x\big(\phi(x) \wedge \theta(x)\big);$$

$$\phi(N) < \theta(N)^+ \equiv N \models \forall\, t\Big(\forall\, y\big(\theta(y) \to y < t\big) \to \forall\, x\big(\phi(x) \to x < t\big)\Big).$$

We say that convex sets C and D are *separated by an element a* (*a-separated*) if $C < a < D$ or $D < a < C$. A family of convex sets is *E-separated* if the sets of this family are pairwise separated by elements of E.

Assertion 16 [2, 6]**.** *A theory T is weakly o-minimal if and only if, for every formula $\phi(x, \overline{y})$, there exists $n_\phi < \omega$ such that, for every $M \models T$ and every $\overline{a} \in M$, the set $\phi(M, \overline{a})$ is the union of less than n_ϕ convex $\overline{a}$-definable $\neg\phi(M, \overline{a})$-separated subsets.*

Remark 17. The intersection of a family of convex subsets of an arbitrary linearly ordered set is convex. By Assertion 16, for every $p \in S_1(A)$, where $A \subset M \models T$, the set $p(M)$ is convex. If the model M is $|A|^+$-saturated and the type p is nonalgebraic then $p(M)$ possesses neither minimal nor maximal elements.

Assertion 16 and Remark 17 yield

Remark 18. If there exists an $\overline{a}$-definable formula dividing a convex set B then there exists an $\overline{a}$-definable formula splitting B.

Assertion 19. *Let $p \in S_1(A)$, where A is a set in a model M of T. The type p is definable if and only if p is $\phi(x, \overline{y})$-definable for each formula $\phi(x, \overline{y})$ such that, for every $\overline{b} \in M$, the set $\phi(M, \overline{b})$ is convex.*

Definition 20 [4]**.** Let $p \in S_1(A)$, where A is a set in a model M of T. We say that p is *right quasirational* if there exists an A-definable convex formula $U_p(x) \in p$ such that, for every sufficiently saturated model $N \succ M$, we have $U_p(N)^+ = p(N)^+$. We say that p is *left quasirational* if there exists an A-definable convex formula $U_p(x) \in p$ such that, for every sufficiently saturated model $N \succ M$, we have $U_p(N)^- = p(N)^-$. If a type p is both right and left quasirational then p is said to be *isolated*. A nonisolated 1-type is *quasirational* if it is either right or left quasirational. If a 1-type p is neither quasirational nor isolated then p is said to be *irrational*.

Let p be an n-type over A and let $F \subset p$. We say that p is *defined* by F (or F *defines* p) if, for every formula $\phi(\overline{x}) \in p$, there exists $\theta(\overline{x}) \in F$ such that $N \models \forall \overline{x}\big(\theta(\overline{x}) \to \phi(\overline{x})\big)$.

We say that a 1-type $p \in S_1(A \cup B)$ is *defined by a quasirational cut* (A, B) if p is defined by either $\{a < x \wedge U(x) \mid a \in A\}$ or $\{x < b \wedge \neg U(x) \mid b \in B\}$, where $U(x)$ is an $(A \cup B)$-definable formula such that $A \subset U(N)$ and $U(N) \cap B = \varnothing$.

Assertion 21 [4, 6]**.** *Let $p \in S_1(M)$, where $M \models T$. Then*

(1) *p is not definable if and only if p is irrational if and only if p is defined by an irrational cut in M;*

(2) *p is definable if and only if p is quasirational if and only if p is defined by a quasirational cut in M.*

Assertion 21 generalizes a similar fact about o-minimal theories which was proven by Marker and Steinhorn [17, Lemma 2.3]. Observe that, for o-minimal theories, definable 1-types over models are defined by rational cuts.

Proposition 22 [4]**.** *Let $p, r \in S_1(A)$, where A is a set in a model N of T. Assume that $\overline{\gamma} \in N \setminus A$.*

(1) *If $p \not\perp^w r$ then $r \not\perp^w p$.*

(2) *If $\overline{\gamma}$ is ht-definable over A and $\overline{\gamma} \not\perp^w p$ then p is definable.*

(3) *If $p \not\perp^w r$ then p is definable if and only if r is definable.*

Proof. (1) Recall that $p \perp^w r$ if and only if $p(x) \cup r(y)$ is a complete 2-type.

(2) Let $K(x, \overline{y})$ be an A-definable formula such that $K(N, \overline{\gamma}) \cap p(N) \neq \varnothing$ and $\neg K(N, \overline{\gamma}) \cap p(N) \neq \varnothing$. By Remark 18, there exists an $(A \cup \overline{\gamma})$-definable convex formula $H(x, \overline{y})$ such that $H(N, \overline{\gamma}) < \neg H(N, \overline{\gamma})$; moreover, there exist $\beta_1, \beta_2 \in p(N)$ with $\beta_1 < \beta_2$, $\beta_1 \in H(N, \overline{\gamma})$, and $H(N, \overline{\gamma}) < \beta_2$.

Let $\theta(x, \overline{z})$ be an arbitrary formula. In view of Assertion 19, we may assume that, for every $\bar{b} \in N$, the formula $\theta(x, \bar{b})$ is convex. For every $\overline{a} \in A^{l(\overline{z})}$, we have

$$\begin{aligned}\theta(x, \overline{a}) \in p &\iff (\beta_1, \beta_2) \subset p(N) \subset \theta(N, \overline{a}) \\ &\iff N \models \forall x\big(x \in (\beta_1, \beta_2) \to \theta(x, \overline{a})\big) \\ &\iff N \models \exists x_1, x_2\Big(H(x_1, \overline{\gamma}) \wedge \neg H(x_2, \overline{\gamma}) \wedge x_1 < x_2 \\ &\qquad\qquad \wedge \forall x\big(x \in (x_1, x_2) \to \theta(x, \overline{a})\big)\Big).\end{aligned}$$

Let $Q(\theta)(\overline{\gamma}, \overline{a})$ denote the last formula. By Definition 7, there exists an A-definable formula $R_{Q(\theta)}(\overline{z})$ such that

$$N \models Q(\theta)(\overline{\gamma}, \overline{a}) \iff N \models R_{Q(\theta)}(\overline{a}).$$

Put $\mu_\theta(\overline{z}) := R_{Q(\theta)}(\overline{z})$. We obtain

$$\theta(x, \bar{a}) \in p \iff N \models \mu_\theta(\overline{a}).$$

(3) If $p \not\perp^w r$ then $\alpha \not\perp^w r$ for every $\alpha \in p(N)$. Hence, (3) follows from (1) and (2). $\square$

We present an easy consequence of the definitions of a type and a saturated model.

Assertion 23. *Let M be a model of a first-order theory and let M be $|A|^+$-saturated for some $A \subset M$.*

(i) *Let there exist $n, m < \omega$, an A-definable formula $\psi(\overline{x}_1, \ldots, \overline{x}_n)$, and $p \in S_m(A)$, where $l(\overline{x}_1) = \cdots = l(\overline{x}_n) = m$, such that $M \models \psi(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$ for every $\overline{\alpha}_1, \ldots, \overline{\alpha}_n \in p(M^m)$. Then there exists $\theta(\overline{x}) \in p$ such that $M \models \forall \overline{x}_1, \ldots, \forall \overline{x}_n \big(\bigwedge \theta(\overline{x}_i) \to \psi(\overline{x}_1, \ldots, \overline{x}_n)\big)$.*

(ii) *If $p \in S_m(A)$, where $m < \omega$, is a nonisolated type then the set $p(M^m)$ is not M-definable.*

(iii) *Let $p, r \in S(A)$, let $\overline{\alpha} \in p(M)$, let $\overline{\beta}_1, \overline{\beta}_2 \in r(M)$, and let $\phi(\overline{x}, \overline{y})$ be an A-definable formula such that $M \models \phi(\overline{\beta}_1, \overline{\alpha}) \wedge \neg\phi(\overline{\beta}_2, \overline{\alpha})$. Then, for every $\overline{\beta} \in r(M)$, there exist $\overline{\alpha}_1, \overline{\alpha}_2 \in p(M)$ such that $M \models \phi(\overline{\beta}, \overline{\alpha}_1) \wedge \neg\phi(\overline{\beta}, \overline{\alpha}_2)$.*

Definition 24. A formula $K(x, y)$ *increases monotonically in* y *on a convex set* B if the following condition holds:

$$\forall\, b_1, \forall\, b_2\big[(b_1 \in B \wedge b_2 \in B \wedge b_1 < b_2) \to K(N, b_1) < K(N, b_2)^+\big].$$

Monotone decreasing formulas are defined in a similar way.

Theorem 25 [4–6]**.** *If* $p(y) \not\perp^w q(x)$ *then there exists an* A*-definable formula* $K(x, y)$ *satisfying the following conditions*:

(1) $K(x, y)$ *is monotone in* y *on some* $\Theta(N)$, *where* $\Theta(y) \in p$, *and is monotone in* x *on some* $\mu(N)$, *where* $\mu(x) \in q$;

(2) $K(x, \alpha)$ *splits* $q(N)$ *and* $K(\beta, y)$ *splits* $p(N)$ *for all* $\alpha \in p(N)$ *and* $\beta \in q(N)$.

In the case of an o-minimal theory, the formula $K(x, y)$ of Theorem 25 is the graph of a suitable monotone function (see [2, 16, 18]).

Observe that Theorem 25 is valid in a more general case, namely, if A is a subset of a weakly o-minimal model of finite depth whose theory need not be weakly o-minimal [7].

Proposition 26 [4–6]. *Let* $p, q \in S_1(A)$ *and let* $p \not\perp^w q$. *Then the following hold*:

(1) p *is strictly definable if and only if* q *is strictly definable*;

(2) p *is irrational if and only if* q *is irrational*;

(3) p *is quasirational if and only if* q *is quasirational*;

(4) $\not\perp^w$ *is an equivalence relation on* $S_1(A)$.

Proposition 26 generalizes a similar fact about 1-types over o-minimal models which was proven by Marker in [16].

3. Definability of 1-types

The main results of this section are the criterion for undefinability of a 1-type (Theorem 31) and one of its versions (Proposition 35).

Assertion 27. *Every quasirational type* $p \in S_1(A)$ *is definable.*

Proof. For definiteness, assume that p is right quasirational. The case in which p is left quasirational is similar. Let $U_p(x)$ be an A-definable formula such that $U_p(x) \in p$ and $U_p(N)^+ = p(N)^+$. Let $\varphi(x, \overline{y})$ be an arbitrary A-definable $\big(l(\overline{y}) + 1\big)$-formula. Consider the A-definable $l(\overline{y})$-formula

$$\Theta_\varphi(\overline{y}) := \exists\, x\Big(\big(\varphi(x, \overline{y}) \wedge U_p(x)\big) \wedge \forall\, z\big(x < z < U_p(N)^+ \to \varphi(z, \overline{y})\big)\Big).$$

It is clear that, for every $\overline{a} \in A^{l(\overline{y})}$, we have $N \models \Theta_\varphi(\overline{a}) \iff \varphi(x, \overline{a}) \in p$. □

Let $q \in S_1(A)$, where $A \subset N$. We introduce the notations:

$$L(q) := \big\{G(x) \mid G(x) \text{ is an } A\text{-definable formula such that } G(N) < q(N)\big\},$$

$$R(q) := \big\{D(x) \mid D(x) \text{ is an } A\text{-definable formula such that } q(N) < D(N)\big\}.$$

Definition 28. Let $q \in S_1(A)$, where $A \subset N$. Assume that $\theta(\overline{y})$ and $H(x, \overline{y})$ are A-definable formulas. Put $X := \theta\big(N^{l(\overline{y})}\big) \cap A^{l(\overline{y})}$.

We say that the condition of *left convergence* of $H(x, \overline{y})$ to the type q on the set X or $\theta(\overline{y})$ holds and write

$$\mathrm{LC}\big(H(x, \overline{y}), X, q\big) \quad \text{or} \quad \mathrm{LC}\big(H(x, \overline{y}), \theta(\overline{y}), q\big)$$

if the following is satisfied:

$$\forall\, G(x) \in L(q),\ \exists\, \overline{a} \in X\ \ N \models \exists\, x\big(G(N) < x < H(N, \overline{a})^+\big),\ H(N, \overline{a}) < q(N).$$

We say that the condition of *right convergence* of $H(x, \overline{y})$ to q on X or $\theta(\overline{y})$ holds and write $\mathrm{RC}\big(H(x, \overline{y}), X, q\big)$ or $\mathrm{RC}\big(H(x, \overline{y}), \theta(\overline{y}), q\big)$ if the following is satisfied:

$$\forall\, D(x) \in L(q),\ \exists\, \overline{a} \in X\ \ N \models \exists\, x\big(H(N, \overline{a}) < x < D(N)\big),\ q(N) < H(N, \overline{a})^+.$$

We say that the condition of *two-sided convergence* of $H(x, \overline{y})$ to q on X or $\theta(\overline{y})$ holds and write $\mathrm{C}\big(H(x, \overline{y}), X, q\big)$ or $\mathrm{C}\big(H(x, \overline{y}), \theta(\overline{y}), q\big)$ if both $\mathrm{LC}(H, X, q)$ and $\mathrm{RC}(H, X, q)$ are satisfied.

In the definitions of left and right convergences of $H(x, \overline{y})$, we have used the right bound of the formula $H(x, \overline{a})$, $\bar{a} \in A$. It is possible to define convergence using the left bound of the formula but this will not be done in the present article.

Observe that in fact the articles [17, 19] dealt with (left) convergence of the values of a function to a type q.

Remark 29. Let $q \in S_1(A)$. If q is right quasirational then, for all A-definable formulas $H(x, \overline{y})$ and $\theta(\overline{y})$, we have $\neg\mathrm{LC}(H, \theta, q)$. If q is left quasirational then, for all A-definable formulas $H(x, \overline{y})$ and $\theta(\overline{y})$, we have $\neg\mathrm{RC}(H, \theta, q)$.

In view of Remark 29, if $\mathrm{C}(H, \theta, q)$ holds for $L(A)$-formulas $H(x, \overline{y})$ and $\theta(\overline{y})$ and for a 1-type $q \in S_1(A)$ then q must be irrational. In view of Assertion 27, the question on undefinability of a 1-type should be considered for irrational types only.

Remark 30. Let $H(x, \overline{y})$ and $\theta(\overline{y})$ be A-definable formulas such that $\mathrm{C}(H, \theta, q)$ holds for some $q \in S_1(A)$. Then, for every A-definable formula $\theta_1(\overline{y})$, we have

(i) if $\mathrm{LC}(H, \theta_1, q)$ and $\neg\mathrm{RC}(H, \theta_1, q)$ hold then $\mathrm{RC}\big(H, \theta(\overline{y}) \wedge \neg\theta_1(\overline{y}), q\big)$ holds;

(ii) if $\models \forall\, \overline{y}\big(\theta(\overline{y}) \to \theta_1(\overline{y})\big)$ then $\mathrm{C}(H, \theta_1, q)$ holds.

Theorem 31. *Let A be a set in a model M of a weakly o-minimal theory T. Let q be an irrational* 1*-type over A. Then the following conditions are equivalent*:

(i) *q is not definable*;

(ii) *there exists an A-definable formula $H(x, \overline{y})$ such that, for every A-definable formula $\theta(\overline{y})$, we have*

$$\mathrm{C}\big(H(x, \overline{y}), \theta(\overline{y}), q\big) \vee \mathrm{C}\big(H(x, \overline{y}), \neg\theta(\overline{y}), q\big).$$

Proof. The condition "q is irrational" means that there is no greatest formula in $L(q)$ and there is no least formula in $R(q)$. We briefly outline the proof. The crucial point in the proof of necessity is the observation that at least one of the bounds (either left or right) for the formula with undefinability of the type approximates both $L(q)$ and $R(q)$ on using constants in A. To prove sufficiency, we start with a formula approximating both $L(q)$ and $R(q)$ and construct a formula proving that q is not definable.

N e c e s s i t y . Let $\varphi(x, \overline{y})$ be an A-formula such that q is not $\varphi(x, \overline{y})$-definable. In view of Assertion 19, we may assume that, for every $\overline{b} \in M$, the set $\varphi(x, \overline{b})$ is convex. Put

$$H_1(x, \overline{y}) := x < \varphi(N, \overline{y}), \qquad H_2(x, \overline{y}) := \varphi(x, \overline{y}).$$

Then $H_i(x, \overline{y})$, $i = 1, 2$, are A-definable formulas.

Let $\theta(\overline{y})$ be an arbitrary A-formula.

R e m a r k 32. Given $j = 1, 2$, we have $\neg\mathrm{RC}(H_j, \theta, q)$ if and only if there exists $D_j(x) \in R(q)$ such that

$$\forall\, \overline{a} \in \theta\big(N^{l(\overline{y})}\big) \cap A^{l(\overline{y})} \Big[N \models \exists x\big(H_j(N, \overline{a}) < x < D_j(N)\big) \Longleftrightarrow H_j(N, \overline{a}) < q(N)\Big].$$

R e m a r k 33. For $j = 1, 2$, we have $\neg\mathrm{LC}(H_j, \theta, q)$ if and only if there exists $G_j(x) \in L(q)$ such that

$$\forall\, \overline{a} \in \theta\big(N^{l(\overline{y})}\big) \cap A^{l(\overline{y})} \Big[N \models \exists\, x\big(G_j(N) < x < H_j(N, \overline{a})^+\big) \Longleftrightarrow q(N) < H_j(N, \overline{a})^+\Big].$$

We claim that at least one of the formulas H_1, H_2 satisfies (ii). Assume the contrary, i.e., let there exist A-formulas $\theta_1(\overline{y})$ and $\theta_2(\overline{y})$ such that

$$\neg\mathrm{C}(H_1, \theta_1, q), \quad \neg\mathrm{C}(H_1, \neg\theta_1, q), \quad \neg\mathrm{C}(H_2, \theta_2, q), \quad \neg\mathrm{C}(H_2, \neg\theta_2, q).$$

From the definition of two-sided convergence, we obtain

$$\begin{aligned}\big(\neg\mathrm{LC}(H_1, \theta_1, q) &\vee \neg\mathrm{RC}(H_1, \theta_1, q)\big)\\ &\wedge \big(\neg\mathrm{LC}(H_1, \neg\theta_1, q) \vee \neg\mathrm{RC}(H_2, \neg\theta_1, q)\big)\\ &\wedge \big(\neg\mathrm{LC}(H_2, \theta_2, q) \vee \neg\mathrm{RC}(H_2, \theta_2, q)\big)\\ &\wedge \big(\neg\mathrm{LC}(H_2, \neg\theta_2, q) \vee \neg\mathrm{RC}(H_2, \neg\theta_2, q)\big).\end{aligned}$$

For the conditions $\neg\mathrm{RC}(H_i, \theta_i, q)$ and $\neg\mathrm{RC}(H_i, \neg\theta_i, q)$, $i = 1, 2$, let $D_{i,1}(x)$ and $D_{i,2}(x)$ denote the A-formulas whose existence is mentioned in Remark 32.

For the conditions $\neg\mathrm{LC}(H_i, \theta_i, q)$ and $\neg\mathrm{LC}(H_i, \neg\theta_i, q)$, $i = 1, 2$, let $G_{i,1}(x)$ and $G_{i,2}(x)$ denote the A-formulas whose existence is mentioned in Remark 33.

We introduce the notations:

$$\mu_{1,1}(\overline{y}) := \begin{cases} \exists x\big(H_1(N, \overline{y}) < x < D_{1,1}(N)\big) & \text{if } \neg\mathrm{RC}(H_1, \theta_1, q), \\ H(N, \overline{y}) < G_{1,1}(N)^+ & \text{if } \mathrm{RC}(H_1, \theta_1, q) \\ & \text{and } \neg\mathrm{LC}(H_1, \theta_1, q); \end{cases}$$

$$\mu_{1,2}(\overline{y}) := \begin{cases} \exists x\big(H_1(N, \overline{y}) < x < D_{1,2}(N)\big) & \text{if } \neg\mathrm{RC}(H_1, \neg\theta_1, q), \\ H(N, \overline{y}) < G_{1,2}(N)^+ & \text{if } \mathrm{RC}(H_1, \neg\theta_1, q) \\ & \text{and } \neg\mathrm{LC}(H_1, \neg\theta_1, q); \end{cases}$$

$$\mu_{2,1}(\overline{y}) := \begin{cases} D_{2,1}(N)^- < H_2(N, \overline{y})^+ & \text{if } \neg\mathrm{RC}(H_2, \theta_2, q), \\ \exists x\big(G_{2,1}(N) < x < H(N, \overline{y})^+\big) & \text{if } \mathrm{RC}(H_2, \theta_2, q) \\ & \text{and } \neg\mathrm{LC}(H_2, \theta_2, q); \end{cases}$$

$$\mu_{2,2}(\overline{y}) := \begin{cases} D_{2,2}(N)^- < H_2(N, \overline{y})^+ & \text{if } \neg\mathrm{RC}(H_2, \neg\theta_2, q), \\ \exists x\big(G_{2,2}(N) < x < H(N, \overline{y})^+\big) & \text{if } \mathrm{RC}(H_2, \neg\theta_2, q) \\ & \text{and } \neg\mathrm{LC}(H_2, \neg\theta_2, q). \end{cases}$$

It is clear that $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{2,1}$, and $\mu_{2,2}$ are A-formulas. Consider the A-formula

$$\mu(\overline{y}) := \exists x\varphi(x, \overline{y}) \wedge \big[\theta_1(\overline{y}) \to \mu_{1,1}(\overline{y})\big] \wedge \big[\neg\theta_1(\overline{y}) \to \mu_{1,2}(\overline{y})\big] \\ \wedge \big[\theta_2(\overline{y}) \to \mu_{2,1}(\overline{y})\big] \wedge \big[\neg\theta_2(\overline{y}) \to \mu_{2,2}(\overline{y})\big].$$

In view of Remarks 32 and 33, we have

$$\forall\,\overline{a} \in A^{l(\overline{y})}\ \big[N \models \mu(\overline{a}) \iff H_1(N, \overline{a}) < q(N) < H_2(N, \overline{a})^+\big].$$

Recall that

$$\varphi(x, \overline{a}) \in q \iff q(N) \subset \varphi(N, \overline{a}) \iff \varphi(N, \overline{a})^- < q(N) < \varphi(N, \overline{a})^+ \\ \iff H_1(N, \overline{a}) < q(N) < H_2(N, \overline{a})^+.$$

Hence, $\forall\,\overline{a}\big[N \models \mu(\overline{a}) \iff \varphi(x, \overline{a}) \in q\big]$. Therefore, q is $\varphi(x, \overline{y})$-definable, which is a contradiction.

S u f f i c i e n c y . Let $H(x, \overline{y})$ be an A-formula satisfying (ii). Take an arbitrary A-formula $D(x)$ such that $q(N) < D(N)$.

Put $\varphi(x, \overline{y}) := H(N, \overline{y}) < x < D(N)$. We show that the type q is not $\varphi(x, \overline{y})$-definable.

Assume that there exists an A-formula $\mu(\overline{y})$ such that

$$\big[N \models \mu(\overline{a}) \Longleftrightarrow \varphi(x, \overline{a}) \in q\big] \qquad (*)$$

for every $\overline{a} \in A^{l(\overline{y})}$. Since

$$\big[N \models \mu(\overline{a}) \Longleftrightarrow \varphi(x, \overline{a}) \in q \Longleftrightarrow H(N, \overline{a}) < q(N)\big]$$

for every $\overline{a} \in A^{l(\overline{y})}$, we obtain the following condition:

(1) $\mathrm{LC}(H, \mu, q)$, $\neg\mathrm{RC}(H, \mu, q)$.

Consider an arbitrary formula $G \in L(q)$. By Remark 33, we have $\neg\mathrm{LC}(H, \neg\mu, q)$. In view of condition (ii) of the theorem, condition (1) above, and Remark 30 (i), we obtain the following condition:

(2) $\mathrm{RC}(H, \neg\mu, q)$, $\neg\mathrm{LC}(H, \neg\mu, q)$.

If both (1) and (2) hold then

$$\neg\mathrm{C}(H, \mu, q) \wedge \neg\mathrm{C}(H, \neg\mu, q),$$

which contradicts the choice of $H(x, \overline{y})$. Thus, condition $(*)$ cannot hold; hence, the type q is not $\varphi(x, \overline{y})$-definable.

Assertion 34*. *Let $\overline{\alpha}$ be a tuple in $N \setminus M$, let $\mathrm{tp}(\overline{\alpha}/M)$ be definable, and let $\mathrm{LC}\big(H(x, \overline{y}, \overline{\alpha}), \Theta(\overline{y}, \overline{\alpha}), q\big)$ hold. Then there exists an M-formula $\mu(\overline{y})$ such that $\mathrm{LC}\big(H(x, \overline{y}, \overline{\alpha}), \mu(\overline{y}), q\big)$ holds.*

Proof. By the definition of $\mathrm{tp}(\overline{\alpha}/M)$, there exists an M-formula $\mu_\Theta(\overline{y})$ such that $\forall\, \overline{a} \in M\ \big[N \models \Theta(\overline{a}, \overline{\alpha}) \Longleftrightarrow M \models \mu_\Theta(\overline{a})\big]$. To check convergence, it suffices to consider tuples of elements of M only. □

Proposition 35. *Let $\overline{\alpha}$ be a tuple in $N \setminus M$, let $\mathrm{tp}(\overline{\alpha}/M)$ be definable, let $\beta \in N \setminus (M \cup \overline{\alpha})$, and let $q := \mathrm{tp}(\beta/M \cup \overline{\alpha})$ be an irrational type that is not strictly definable. Then the following conditions are equivalent:*

(i) *there exist $(M \cup \overline{\alpha})$-definable formulas $H(x, \overline{y})$ and $\Theta(\overline{y})$ such that either $\mathrm{LC}(H, \Theta, q)$, $\neg\mathrm{RC}(H, \Theta, q)$ or $\neg\mathrm{LC}(H, \Theta, q)$, $\mathrm{RC}(H, \Theta, q)$ hold;*

(ii) *the type q is definable.*

Proof. (i) $\Rightarrow$ (ii). Assume that $\mathrm{LC}(H, \Theta, q)$ and $\neg\mathrm{RC}(H, \Theta, q)$ hold. The case in which $\neg\mathrm{LC}(H, \Theta, q)$ and $\mathrm{RC}(H, \Theta, q)$ hold is similar. We may assume (see Assertion 34) that $\Theta(\overline{y})$ is an M-formula. Put $X := \Theta(M^k)$ and $Z := \Theta(N^k)$. We have $H(N, \overline{a}, \overline{\alpha}) < q(N)$ for every $\overline{a} \in X$. Let $\varphi(x, \overline{y}, \overline{\alpha})$ be an $(M \cup \overline{\alpha})$-formula such that, for every $\overline{\gamma} \in N$, the set $\phi(N, \overline{\gamma}, \overline{\alpha})$ is convex. Put

$$S_1(\overline{z}, \overline{y}_1, \overline{\alpha}) := \varphi(N, \overline{z}, \overline{\alpha})^- < H(N, \overline{y}_1, \overline{\alpha})^+ \wedge \theta_1(\overline{y}_1),$$

$$S_2(\overline{z}, \overline{y}_2, \overline{\alpha}) := \varphi(N, \overline{z}, \overline{\alpha}) \quad < H(N, \overline{y}_2, \overline{\alpha})^+ \wedge \theta_1(\overline{y}_2).$$

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* For the case of o-minimal theories, this assertion was proven in [17].

Since $\mathrm{tp}(\overline{\alpha}/M)$ is definable, there are M-formulas $Q_1(\overline{z}, \overline{y}_1)$ and $Q_2(\overline{z}, \overline{y}_2)$ such that

$$N \models S_1(\overline{a}, \overline{b}_1, \overline{\alpha}) \Longleftrightarrow M \models Q_1(\overline{a}, \overline{b}_1),$$

$$N \models S_2(\overline{a}, \overline{b}_2, \overline{\alpha}) \Longleftrightarrow M \models Q_2(\overline{a}, \overline{b}_2)$$

for all $\overline{a}, \overline{b}_1, \overline{b}_2 \in M$. Put

$$Q(\overline{z}) := \exists\, \overline{y}_1 Q_1(\overline{z}, \overline{y}_1) \wedge \neg\exists\, \overline{y}_2 Q_2(\overline{z}, \overline{y}_2).$$

We have

$$\forall\, \overline{a} \in M^{l(\overline{z})} \quad \big[M \models Q(\overline{a}) \Longleftrightarrow \varphi(x, \overline{a}, \overline{\alpha}) \in q\big].$$

Indeed, $M \models Q(\overline{a})$ implies that there exists $\overline{b}_1 \in M$ such that

$$M \models Q_1(\overline{a}, \overline{b}_1) \ \text{ and } \ M \models \forall\, \overline{y}_2 \neg Q_2(\overline{a}, \overline{y}_2).$$

It remains to notice that the following equivalences are valid:

$$\big[M \models Q_1(\overline{a}, \overline{b}_1) \Longleftrightarrow \varphi(N, \overline{a}, \overline{\alpha})^- < q(N)\big],$$

$$\big[M \models \forall\, \overline{y}_2 \neg Q_2(\overline{a}, \overline{y}_2) \Longleftrightarrow q(N) < \varphi(N, \overline{a}, \overline{\alpha})^+\big].$$

Implication (i) $\Rightarrow$ (ii) is proven.

(ii) $\Rightarrow$ (i). In the proof of this implication, we will need the first part of the following

Lemma 36. *Let $q \in S_1(A)$, where A is a set in a model N, and let $\phi(x, \overline{y})$ be a parameter-free formula.*

(a) *If* $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$ *holds then*

$$\mathrm{LC}\big(H(x, \bar{y}), \overline{y} = \overline{y}, q\big) \ \textit{ and/or } \ \mathrm{RC}\big(H(x, \overline{y}), \overline{y}, = \overline{y}, q\big),$$

where $H(x, \overline{y})$ is either $\phi(x, \overline{y})$ or $\phi(x, \overline{y})^- := \neg\phi(x, \overline{y}) \wedge x < \phi(N, \overline{y})$.

(b) *If* $\mathrm{STC}\big(\phi(x, \overline{y}), q\big)$ *holds then*

$$\mathrm{LC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big) \ \textit{ and } \ \mathrm{RC}\big(\phi(x, \overline{y}), \overline{y}, = \overline{y}, q\big).$$

Proof. (a) For every L-formula $\phi(x, \overline{y})$, we define formulas ϕ_i, $i < \omega$, by induction as follows:

$$\phi_0(x, \overline{y}) := \phi(x, \overline{y}) \wedge \forall\, x_1 \Big[\big(\phi(x_1, \overline{y}) \wedge x_1 < x\big) \to \forall\, z\big(x_1 < z < x \to \phi(z, \overline{y})\big)\Big];$$

$$\phi_{i+1}(x, \overline{y}) := \phi(x, \overline{y}) \wedge \phi_i(N, \overline{y}) < x \wedge \forall\, x_1 \Big[\big(\phi(x_1, \overline{y}) \wedge \phi_i(N, \overline{y}) < x_1 < x\big) \to \forall\, z\big(x_1 < z < x \to \phi(z, \overline{y})\big)\Big].$$

By this definition, for every $i < \omega$ and $\bar{b} \in N$, the set $\phi(N, \overline{b})$ is convex (notice that the empty set is convex by definition).

In view of Assertion 16, we have

$$\phi(x, \overline{y}) \equiv \bigvee_{i<n_\phi} \phi_i(x, \overline{y})$$

for some $n_\phi < \omega$. We show that validity of $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$ yields existence of $i < n_\phi$ such that $\mathrm{WEC}\big(\phi_i(x, \overline{y}), q\big)$ holds. Assume the contrary, i.e., let $\neg\mathrm{WEC}\big(\phi_i(x, \overline{y}), q\big)$ hold for every $i < n_\phi$. Then there exist $L(A)$-formulas $\mu_i(x) \in q$, $i < n_\phi$, such that, for each $\overline{b} \in A$ and each $i < n_\phi$, we have either $\phi_i(N, \overline{b}) \cap \mu_i(N) = \varnothing$ or $\mu_i(N) \subseteq \neg\phi_i(N, \overline{b})$. Put $\mu(x) := \bigwedge \mu_i(x)$. It is obvious that $\mu(x) \in q$. Fix an arbitrary tuple $\overline{b} \in A$. Assume that $\phi_i(N, \overline{b}) \cap \mu_i(N) = \varnothing$ for all $i < n_\phi$. Since $\mu(N) \subseteq \mu_i(N)$, we have $\phi_i(N, \overline{b}) \cap \mu(N) = \varnothing$. Hence,

$$\phi(N, \overline{b}) \cap \mu(N) = \bigcup_i \big(\phi_i(N, \overline{b}) \cap \mu(N)\big) = \varnothing$$

and, consequently, $\phi(x, \overline{b})$ does not divide $\mu(N)$.

If $\mu_i(N) \subseteq \phi_i(N, \overline{b})$ for some $i < n_\phi$ then

$$\mu(N) \subseteq \mu_i(N) \subseteq \phi_i(N, \overline{b}) \subseteq \bigcup_i \phi_i(N, \overline{b}) = \phi(N, \overline{b})$$

and, consequently, $\phi(x, \overline{b})$ does not divide $\mu(N)$. In both cases, $\phi(x, \overline{b})$ does not divide $\mu(N)$, which contradicts validity of $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$ because $\overline{b} \in A$ is arbitrary.

Thus, we assume in the sequel that the formula $\phi(x, \overline{y})$ in the condition $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$ possesses the following additional property: $\phi(x, \overline{b})$ is convex for every $\overline{b} \in N$.

Consider a convex $L(A)$-formula $\mu(x)$ such that $\mu(x) \in q$. The latter means that $q(N) \subset \mu(N)$. Since q is an irrational 1-type, there exist $\gamma_1, \gamma_2 \in \mu(N)$ such that

$$\gamma_1 < q(N) < \gamma_2.$$

Hence, $\mu(N)$ falls into three undefinable (by formulas) convex subsets, $\mu(N) = X_1(\mu) \cup q(N) \cup X_2(\mu)$; moreover,

$$X_1(\mu) < q(N) < X_2(\mu).$$

Since $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$ holds, there exists $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $\mu(N)$. Since q is irrational, neither left nor right bound of the convex set $q(N)$ is definable by formulas. Hence, the following assertion holds.

(*) If a convex formula $\phi(x, \overline{b})$ divides $\mu(N)$ then $\phi(x, \overline{b})$ cannot pass along the bounds of $q(N)$ and must divide at least one of the convex sets $\mu(N)$, $X_1(\mu)$, $X_2(\mu)$.

Assume that $\phi(x, \overline{b})$ divides $X_1(\mu)$. Since the definable set $\phi(N, \overline{b})$ and undefinable set $X_1(\mu)$ are convex, we have either $\phi(N, \overline{b}) < q(N)$ or $q(N) \subset \phi(N, \overline{b})$. If, for every convex $L(A)$-formula $\mu(x) \in q$, there exists $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_1(\mu)$ and $\phi(N, \overline{b}) < q(N)$ then we obtain $\mathrm{LC}\big(\phi(x, \overline{y}), \overline{y} = \overline{y}, q\big)$. If, for every convex $L(A)$-formula $\mu(x) \in q$, there exists $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_1(\mu)$ and $q(N) \subset \phi(N, \overline{b})$ then we obtain $\mathrm{LC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big)$, where $\phi(x, \overline{y})^- := \neg\phi(x, \overline{y}) \wedge x < \phi(N, \overline{y})$.

If, for some convex $L(A)$-formula $\mu(x) \in q$ and every $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_1(\mu)$, we have $\neg\big(q(N) \subset \phi(N, \overline{b})\big)$ and, consequently, $\phi(N, \overline{b}) < q(N)$ then, for every convex $L(A)$-formula $\mu'(x) \in q$ with $\mu'(N) \subset \mu(N)$ and every $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_1(\mu')$, we have $\neg\big(q(N) \subset \phi(N, \overline{b})\big)$ and, consequently, $\phi(N, \overline{b}) < q(N)$. In other words,

$(**)$ if, for every convex $L(A)$-formula $\mu(x) \in q$, there exists $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_1(\mu)$ then $\neg\mathrm{LC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big)$ implies $\mathrm{LC}\big(\phi(x, \overline{y}), \overline{y} = \overline{y}, q\big)$.

In a similar way, we obtain the following assertion.

$(**)'$ If, for every convex $L(A)$-formula $\mu(x) \in q$, there exists $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $X_2(\mu)$ then $\neg\mathrm{RC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big)$ implies $\mathrm{RC}\big(\phi(x, \overline{y}), \overline{y} = \overline{y}, q\big)$.

Assume that there exists a convex $L(A)$-formula $\mu(x) \in q$ such that $\phi(x, \overline{b})$ does not divide $X_2(\mu)$ for any $\overline{b} \in A$. Then, for every $\overline{b} \in A$ such that $\phi(x, \overline{b})$ divides $\mu(N)$, the formula $\phi(x, \overline{b})$ divides $X_1(\mu)$.

Moreover, let $\mu'(x) \in q$ be a convex $L(A)$-formula such that $\mu'(N) \subset \mu(N)$ and let $\overline{b} \in A$ be such that $\phi(x, \overline{b})$ divides $\mu'(N)$. Then the formula $\phi(x, \overline{b})$ does not divide $X_2(\mu')$ but divides $X_1(\mu')$.

Let both conditions $\neg\mathrm{RC}\big(\phi(x, \overline{y}), \overline{y} = \overline{y}, q\big)$ and $\neg\mathrm{RC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big)$ hold. Then there exists a convex formula $\mu(x) \in q$ such that, for any $\overline{b} \in A$, if $\phi(x, \overline{b})$ divides $\mu(N)$ then $\phi(x, \overline{b})$ does not divide $X_2(\mu)$ but divides $X_1(\mu)$. The latter, in view of $\mathrm{WEC}\big(\phi(x, \overline{y}), q\big)$, $(*)$, and $(**)$, means that the following hold:

$(***)$ $\mathrm{LC}\big(\phi(x, \overline{y}), \overline{y} = \overline{y}, q\big)$ and/or $\mathrm{LC}\big(\phi(x, \overline{y})^-, \overline{y} = \overline{y}, q\big)$.

From $(*)$, $(**)$, and $(***)$ we readily obtain assertion (a) of Lemma 36.

The *proof* of assertion (b) is similar. $\square$

Since the irrational type q is not strictly definable, from Lemma 36 (a) it follows that $\mathrm{LC}\big(H(x, \overline{y}), \overline{y} = \overline{y}, q\big)$ and/or $\mathrm{RC}\big(H(x, \overline{y}), \overline{y} = \overline{y}, q\big)$ hold. If the formula $\overline{y} = \overline{y}$ does not satisfy requirements on $\Theta(\overline{y})$ in condition (i) then $\mathrm{C}\big(H(x, \overline{y}), \overline{y} = \overline{y}, q\big)$ must hold. Since q is a definable type, we apply

Theorem 31 and find the required $L(M \cup \overline{\alpha})$-formula $\Theta(\overline{y})$ for the formula $H(x, \overline{y})$. $\square$

4. An example of a D-1-pair of models in a weakly o-minimal theory which is not a D-pair

Theorem 37. *There exists a weakly o-minimal theory T with two models $M_b \prec M$ such that the type $\mathrm{tp}(\overline{\alpha}/M_b)$ is not definable for some tuple $\overline{\alpha} \in M$, while the model M_b is quasi-Dedekind complete in M.*

Proof. We simultaneously construct a model (M, L_0) and prove that the elementary theory $\mathrm{Th}(M, L_0)$ of (M, L_0) is weakly o-minimal. We describe this theory in terms of a finite set of axioms T_0 and prove that T_0 gives rise to a consistent, complete, and weakly o-minimal theory. Our proof falls into the following stages:

4.1. The set T_0 of axioms. If T_0 is a consistent set then the theory derived from T_0 is ω-categorical.

4.2. Construction of a model (M, L), $L \supset L_0$, with $T_0 \vdash_L T := \mathrm{Th}(M, L)$.

4.3. A proof of the facts that T admits quantifier elimination and is weakly o-minimal.

4.4. An example of a pair of models $(M_b, L) \prec (M, L)$ such that (M_b, M) is a D-1-pair but is not a D-pair.

4.1. Set T_0 of axioms. *If T_0 is a consistent set then the theory derived from T_0 is ω_0-categorical.*

Let T_0 stand for the set of axioms Ax (I)–Ax (IX) of language $L_0 := \{=, P^1, <^2, E^3\}$.

Ax (I). *The relation $<$ is a dense linear order without endpoints.*

Ax (II). *The following sentence holds:*

$$\forall x \forall y \Big(\big((P(x) \wedge \neg P(y)) \to y < x \big) \wedge \forall z \forall x \forall y \big(P(z) \to \neg E(x, y, z) \big) \Big).$$

Ax (III). *For every z, if $\neg P(z)$ then $E(x, y, z)$ is an equivalence relation (with respect to x and y) on P, each coset modulo E_z is a nonempty convex set without endpoints, the induced order on the set of cosets modulo E_z is dense, there exists a minimal coset modulo E_z but there is no maximal coset modulo E_z.*

Ax (IV). *For all z, t, and x, if $P(x) \wedge \neg P(z) \wedge \neg P(t) \wedge z < t$ holds then each coset modulo E_z from x is contained in some coset modulo E_t from x, each coset modulo E_t (except for the minimal coset) contains neither minimal nor maximal cosets modulo E_z, and the minimal coset modulo E_t contains the minimal coset modulo E_z but contains no maximal cosets modulo E_z.*

We denote

$$H^2(x,z) := P(x) \wedge \neg P(z) \wedge \forall\, y\Big(\big(P(y) \wedge y < x\big) \to E(x,y,z)\Big),$$
$$\varepsilon^2(x,y) := \exists\, z\big(\neg P(z) \wedge E(x,y,z) \wedge \neg H(x,z)\big).$$

The formula $H^2(x,z)$ says that x is contained in the minimal coset modulo E_z. The formula $\varepsilon^2(x,y)$ says that x and y do not belong to the minimal coset modulo E_z for some $z \in \neg P$.

Ax (V). *The following sentence holds*:

$$\forall\, x\, \forall\, y\Big(\varepsilon^2(x,y) \leftrightarrow \forall z\big(H(x,z) \leftrightarrow H(y,z)\big)\Big).$$

Axioms Ax (V) and Ax (IV) mean that ε^2 is an equivalence relation on P. Moreover, each coset modulo ε^2 is an infinite convex set and the induced order on the set of cosets modulo ε^2 is a dense linear order without endpoints. These assertions can be written as first-order formulas and derived from Ax (I)–Ax (V).

Ax (VI). *The following sentences hold*:

(i) $\forall\, x\, \forall\, y\Big(\big(\varepsilon^2(x,y) \wedge x \neq y\big) \to \exists\, z\big(\neg P(z) \wedge \neg E(x,y,z)\big)\Big)$;

(ii) $\forall\, x\, \forall\, y\, \forall\, z\Big(\big(E(x,y,z) \wedge \neg H(x,z)\big) \to \exists\, t\big(t < z \wedge E(x,y,t)\big)\Big)$.

We denote

$$\varepsilon^4(x,y;u,v) := x < y \wedge u < v \wedge \varepsilon^2(x,y) \wedge \varepsilon^2(y,u) \wedge \varepsilon^2(u,v)$$
$$\wedge\, \forall\, z\Big(\neg P(z) \to \big(E(x,y,z) \leftrightarrow E(u,v,z)\big)\Big).$$

Observe that ε^4 is an equivalence relation on the set of ordered pairs of cosets modulo ε^2; moreover, each coset modulo ε^4 gives rise to a partition of $\neg P$ into two convex sets without endpoints (splits $\neg P$). For every ordered pair (a_1, a_2) whose components belong to the same coset modulo ε^2, the convex set of elements $b \in \neg P$ such that a_1 and a_2 belong to the same coset modulo E_b can serve as an "indicator of nearness" of a_1 and a_2: the larger such a convex set, the "closer" a_1 to a_2. We also notice that, for all $a_1, a_2, a_3 \in P$, the formulas $\varepsilon^4(a_1,a_2,x,a_3)$, $\varepsilon^4(a_1,a_2,a_3,x)$, $\varepsilon^4(a_1,a_2,x,a_2)$, and $\varepsilon^4(a_1,a_2,a_1,x)$ are convex.

Ax (VII). *The following sentences hold*:

(i) $$\forall u_2 \forall u_4 \Bigg(\big(\varepsilon^2(u_2,u_4) \wedge u_2 < u_4\big) \\ \rightarrow \exists u_1 \exists u_3 \exists u_5 \Bigg(\bigwedge_{i<j} u_i < u_j \wedge \varepsilon^2(u_1,u_5) \\ \wedge \bigwedge_{1 \le i<j<k<r \le 5} E(u_i,u_j,u_k,u_r) \Bigg) \Bigg);$$

(ii) $$\forall u_2 \forall u_3 \forall u_4 \Bigg(\big(\varepsilon^2(u_2,u_4) \wedge u_2 < u_3 < u_4\big) \\ \rightarrow \exists u_1 \exists u_5 \big(u_1 < u_3 < u_5 \wedge \varepsilon^4(u_2,u_4,u_1,u_3) \\ \wedge \varepsilon^4(u_2,u_4,u_3,u_5)\big) \Bigg);$$

(iii) $$\forall u_2 \forall u_4 \Bigg(\big(\varepsilon^2(u_2,u_4) \wedge u_2 < u_4\big) \\ \rightarrow \exists u_1 \exists u_3 \exists u_3' \exists u_5 (u_1 < u_2 < u_3 < u_{3'} < u_4 < u_5) \\ \wedge \forall u \forall u' \Big((u_1 \le u \le u_3 \wedge u_{3'} < u' < u_5) \\ \rightarrow \big(\varepsilon^4(u_2,u_4,u,u_4) \wedge \varepsilon^4(u_2,u_4,u_2,u')\big) \Big) \Bigg).$$

From Axiom Ax (VII) it follows that each coset modulo ε^4 is infinite and dense with respect to the induced order; moreover, if elements $a_1, a_2, a_3 \in P$ belong to the same coset modulo ε^2 and satisfy the condition $a_1 < a_3 < a_2$ then the formulas

$$\varepsilon^4(a_1,a_2,x,a_2), \quad \varepsilon^4(a_1,a_2,a_1,x), \quad \varepsilon^4(a_1,a_2,x,a_3), \quad \varepsilon^4(a_1,a_2,a_3,x)$$

define nonempty convex sets without endpoints.

Ax (VIII). *The following sentence holds*:

$$\forall x \forall y \forall u \forall v \Bigg(\big(\varepsilon^2(x,y) \wedge \varepsilon^2(u,v) \wedge x < y \wedge u < v \wedge \neg\varepsilon^2(x,u)\big) \\ \rightarrow \exists z \Big(\neg P(z) \wedge \big(\big(E(x,y,z) \wedge \neg E(u,v,z)\big) \\ \vee \big(E(u,v,z) \wedge \neg E(x,y,z)\big)\big)\Big)\Bigg).$$

Axiom Ax (VIII) says that two distinct cosets modulo ε^2 are “separated” by the relation E_z for some $z \in \neg P$.

Ax (IX). *The following sentence holds*:

$$\forall x \forall y \forall u \Big(P(x) \wedge P(u) \wedge \varepsilon^2(x,y) \wedge \neg\varepsilon^2(x,u) \\ \to \exists z \big(\big(H(u,z) \wedge \neg E(x,y,z)\big) \vee \big(\neg H(u,z) \wedge E(x,y,z)\big)\big)\Big).$$

Axiom Ax (IX) says that each element of P that does not belong to a given coset modulo ε^2 is "separated" from this coset by the relations H_z and E_z for some $z \in \neg P$.

Proposition 38. *If T_0 is a consistent set then the theory derived from T_0 is ω-categorical.*

Proof. Let M be a model of T_0. Consider the set N consisting of cosets modulo ε^2, cosets modulo ε^4, and elements satisfying $\neg P$. On this set, a linear order $\lhd$ can be defined. The resultant linearly ordered set is called a model of language $L' = \{=, \lhd\}$. This model is denoted by (N, L'). For arbitrary countable models (M_1, L_0) and (M_2, L_0) of T_0, we construct an isomorphism g using an isomorphism t between (N_1, L') and (N_2, L'). These two isomorphisms will be constructed simultaneously by induction on $n < \omega$. Each of them is the union of an increasing chain of partial isomorphisms.

Let (M, L_0) be an arbitrary model of T_0. Put $A := P(M)$, $B := \neg P(M)$, and $C := \big\{(a,a') \mid a, a' \in A, M \models \varepsilon^2(a,a') \wedge a < a'\big\}$. We introduce the following notations for elements of C. Given $c = (a_1, a_2) \in C$, let $c = c(a_1, a_2)$, $l(c) = a_1$, and $r(c) = a_2$. We have $M \models \varepsilon^2\big(l(c), r(c)\big) \wedge l(c) < r(c)$ for every $c \in C$.

Let $a \in A$ and let $c \in C$. Put

$$\widehat{c} := \varepsilon^4(M^2, c) = \big\{c' \in C \mid M \models \varepsilon^4(c', c)\big\}, \quad \widehat{C} := \{\,\widehat{c} \mid c \in C\},$$
$$\widehat{a} := \varepsilon^2(M, a) \ = \big\{a' \in A \mid M \models \varepsilon^2(a', a)\big\}, \quad \widehat{A} := \{\,\widehat{a} \mid a \in A\}.$$

From the definitions of ε^2 and ε^4 it follows that

$$\forall c_1 \in C \ \forall c_2 \in C \ \Big[\big(\widehat{c}_1 = \widehat{c}_2 \to \hat{l}(c_1) = \hat{l}(c_2) = \widehat{r}(c_1) = \widehat{r}(c_2)\big)\Big].$$

We define a model as follows:

$$(N, L') := \big(\,\widehat{A} \cup \widehat{C} \cup B, =, \lhd\big).$$

Let $a, a_1 \in A$, $b, b_1 \in B$, and $c, c_1 \in C$. Then

$$
\begin{aligned}
&N \models b \triangleleft b_1 && \Leftrightarrow M \models b < b_1,\\
&N \models \widehat{a} \triangleleft \widehat{a}_1 && \Leftrightarrow M \models \neg\varepsilon^2(a, a_1) \wedge a < a_1,\\
&N \models \widehat{c} \triangleleft b && \Leftrightarrow M \models E\big(l(c), r(c), b\big),\\
&N \models b \triangleleft \widehat{c} && \Leftrightarrow M \models \neg E\big(l(c), r(c), b\big),\\
&N \models \widehat{a} \triangleleft b && \Leftrightarrow M \models H(a, b),\\
&N \models b \triangleleft \widehat{a} && \Leftrightarrow M \models \neg H(a, b),\\
&N \models \widehat{c} \triangleleft \widehat{a} && \Leftrightarrow M \models \exists\, x\Big(\neg P(x) \wedge E\big(l(c), r(c), x\big) \wedge \neg H(a, x)\Big),\\
&N \models \widehat{a} \triangleleft \widehat{c} && \Leftrightarrow M \models \exists\, x\Big(\neg P(x) \wedge \neg E\big(l(c), r(c), x\big) \wedge H(a, x)\Big),\\
&N \models \widehat{c}_1 \triangleleft \widehat{c}_2 && \Leftrightarrow M \models \exists\, x\Big(\neg P(x) \wedge E\big(l(c_1), r(c_1), x\big) \wedge \neg E\big(l(c_2), r(c_2), x\big)\Big).
\end{aligned}
$$

The definitions of ε^2, ε^4, and H^2 together with axioms Ax (I)–Ax (IX) imply that $\triangleleft$ is well defined.

From the definition of $\triangleleft$ we immediately obtain

$$
\begin{aligned}
&N \models \widehat{c} \triangleleft \widehat{a} && \Leftrightarrow \exists\, b \in B\ N \models \widehat{c} \triangleleft b \wedge b \triangleleft \widehat{a},\\
&N \models \widehat{a} \triangleleft \widehat{c} && \Leftrightarrow \exists\, b \in B\ N \models \widehat{a} \triangleleft b \wedge b \triangleleft \widehat{c},\\
&N \models \widehat{c}_1 \triangleleft \widehat{c}_2 && \Leftrightarrow \exists\, b \in B\ N \models \widehat{c}_1 \triangleleft b \wedge b \triangleleft \widehat{c}_2.
\end{aligned}
$$

We put $C_l(a) := \big\{c \in C \mid l(c) = a\big\}$ and $C_r(a) := \big\{c \in C \mid r(c) = a\big\}$.

Assertion 39.

(i)$'$ *For all $a \in A$ and $c_1, c_2 \in C_l(a)$, if $N \models \widehat{c}_1 \triangleleft \widehat{c}_2$ then $r(c_1) < r(c_2)$ and $\widehat{c}\big(r(c_1), r(c_2)\big) = \widehat{c}_2$.*

(i)$''$ *For all $a \in A$ and $c_1, c_2 \in C_r(a)$, if $N \models \widehat{c}_1 \triangleleft \widehat{c}_2$ then $l(c_1) < l(c_2)$ and $\widehat{c}\big(l(c_1), l(c_2)\big) = \widehat{c}_2$.*

(ii) *The structures $\big(\widehat{A}, =, \triangleleft\big)$, $(B, =, \triangleleft)$, and $\big(\widehat{C}, =, \triangleleft\big)$ are dense linearly ordered sets without endpoints.*

(iii) *The relation $\triangleleft$ is a dense linear order on $\widehat{A} \cup \widehat{C} \cup B$; moreover, each of the sets $\widehat{A}$, $\widehat{C}$, and B is $\triangleleft$-dense in $\widehat{A} \cup \widehat{C} \cup B$.*

(iv)$'$ *For every $a \in A$, we have $\widehat{C}_l(a) = \widehat{C}_r(a)$.*

(iv)$''$ *For every $a \in A$, the set $\widehat{C}_l(a)$ is dense in $(-\infty, \widehat{a})_N$.*

Proof. (i)$'$ Let $N \models \widehat{c}_1 \triangleleft \widehat{c}_2$. Then, for some $b_0 \in B$, we have

$$
M \models E\big(a, r(c_1), b\big) \wedge \neg E\big(a, r(c_2), b_0\big) \ \text{ and } \ a < r(c_1), \quad a < r(c_2).
$$

Since $E(a, M, b_0)$ is convex, we find that $r(c_1) < r(c_2)$.

Show that $\widehat{c}\big(r(c_1), r(c_2)\big) = \widehat{c}_2$. Let $b \in B$ and let $M \models E\big(a, r(c_2), b\big)$. Since each coset modulo E_b is a convex set and $a < r(c_1) < r(c_2)$, we obtain $M \models E\big(r(c_1), r(c_2), b\big)$.

Assume that there exists $b_1 \in B$ such that

$$M \models E\big(r(c_1), r(c_2), b_1\big) \wedge \neg E\big(a, r(c_2), b_1\big).$$

Then $E(M, a, b_1) \wedge E\big(M, r(c_1), b_1\big) = \varnothing$ and, consequently, $r(c_1) \notin E(M, a, b_1)$. In view of Ax (IV), the latter means that $E(M, a, b_1) \subset E(M, a, b_0)$ and $b_1 < b_0$. Since $r(c_1) \in E(M, a, b_0)$, from Ax (IV) it follows that $E\big(M, r(c_1), b_1\big) \subset E(M, a, b_0)$. If $r(c_2) \in E\big(M, r(c_1), b_1\big)$ then $r(c_2) \in E\big(M, r(c_1), b_0\big)$, which contradicts the choice of b_0. Hence, for every $b \in B$, we have

$$\Big[M \models E\big(a, r(c_2), b\big) \Longleftrightarrow M \models E\big(r(c_1), r(c_2), b\big)\Big].$$

The proof of assertion (i)$''$ is similar.

(ii) From Ax (I) and Ax (II) it follows that $(B, =, \triangleleft)$ is a dense linearly ordered set without endpoints. From Ax (I)–Ax (V) it follows that $(\widehat{A}, =, \triangleleft)$ is a dense linearly ordered set without endpoints. We show that $(\widehat{C}, =, \triangleleft)$ possesses the required properties. The relation $\triangleleft$ on $\widehat{C}$ is antireflexive in view of the definitions of ε^4 and $\triangleleft$. We show that this relation is transitive. Let $c_1, c_2, c_3 \in C$ and let $N \models \widehat{c}_1 \triangleleft \widehat{c}_2 \wedge \widehat{c}_2 \triangleleft \widehat{c}_3$. Then there exist $b, b_1 \in B$ such that

$$M \models E\big(l(c_1), r(c_1), b\big) \wedge \neg E\big(l(c_2), r(c_2), b\big) \wedge E\big(l(c_2), r(c_2), b_1\big) \\ \wedge \neg E\big(l(c_3), r(c_3), b_1\big).$$

Since $M \models \neg E\big(l(c_2), r(c_2), b\big) \wedge E\big(l(c_2), r(c_2), b_1\big)$, from Ax (IV) we obtain $b < b_1$. Since $M \models \neg E\big(l(c_3), r(c_3), b_1\big) \wedge b < b_1$, from Ax (IV) we obtain $M \models \neg E\big(l(c_3), r(c_3), b\big)$. From $M \models E\big(l(c_1), r(c_1), b\big)$ it follows that $N \models \widehat{c}_1 \triangleleft \widehat{c}_3$. The relation $\triangleleft$ cannot be symmetric because it is transitive and antireflexive.

We prove that this order relation is dense. Let $N \models \widehat{c}_1 \triangleleft \widehat{c}_2$. Then there exists $b \in B$ such that

$$M \models E\big(l(c_1), r(c_1), b\big) \wedge \neg E\big(l(c_2), r(c_2), b\big).$$

By Ax (VI) (ii), there exists b_1 such that $b_1 < b$ and $M \models E\big(l(c_1), r(c_1), b_1\big)$.

In view of Ax (IV), there exists $a \in E\big(l(c_1), M, b\big) \setminus E\big(l(c_1), M, b_1\big)$ such that $a > E\big(l(c_1), M, b_1\big)$. Let c denote the pair $\big(l(c_1), a\big)$. Then $N \models \widehat{c}_1 \triangleleft \widehat{c}$. Since $M \models E\big(l(c_1), a, b\big) \wedge \neg E\big(l(c_2), r(c_2), b\big)$, we obtain $N \models \widehat{c} \triangleleft \widehat{c}_2$.

(iii) Let $b, b_1 \in B$ and let $M \models b < b_1$. From Ax (IV) it follows that $H(M, b_1) \setminus H(M, b) \neq \varnothing$. Take an element $a \in H(M, b_1) \setminus H(M, b)$. Then

$N \models b \lhd \widehat{a} \wedge \widehat{a} \lhd b_1$. In view of Ax (VI) (i), there exist $a_1 \in \varepsilon^2(M, a)$ and $b_2 \in B$ such that $M \models \neg E(a, a_1, b) \wedge E(a, a_1, b_2)$. This means that

$$N \models b \lhd \widehat{c}(a, a_1) \wedge \widehat{c}(a, a_1) \lhd b_2,$$

$E(M, a, b_2) \subset \varepsilon^2(M, a) \subset H(M, b_1)$, and $b_2 \lhd b_1$. Since $\lhd$ is transitive, we obtain $N \models b \lhd \widehat{c} \lhd b_1$.

In a similar way, we can prove that

$$(d, d') \cap \widehat{A} \neq \varnothing, \quad (d, d') \cap B \neq \varnothing, \quad (d, d') \cap \widehat{C} \neq \varnothing$$

for all d, $d' \in \widehat{A} \cup \widehat{C} \cup B$ with $d \lhd d'$.

(iv)$'$ Let $c \in C_l(a)$. By Ax (VII), there exist a_1 and a such that $a_1 < a$ and $\widehat{c} = \widehat{c}(a_1, a)$. Since $c(a_1, a) \in C_r(a)$, we have $\widehat{C}_l(a) \subseteq \widehat{C}_r(a)$. In a similar way, we conclude that $\widehat{C}_r(a) \subseteq \widehat{C}_l(a)$.

(iv)$''$ Let $b_1, b_2 \in (-\infty, \widehat{a})$. Then $N \models b_1 \lhd \widehat{a} \wedge b_2 \lhd \widehat{a}$. Hence, $M \models \neg H(a, b_1) \wedge \neg H(a, b_2)$. Assume that $b_1 < b_2$. Then $E(M, a, b_1) \subset E(M, a, b_2)$.

Let $a_1, a_2 \in E(M, a, b_2) \setminus E(M, a, b_1)$ and let $a_2 < E(M, a, b_1) < a_1$. Then

$$N \models b_1 \lhd \widehat{c}_1 \wedge b_1 \lhd \widehat{c}_2 \wedge \widehat{c}_1 \lhd b_2 \wedge \widehat{c}_2 \lhd b_2,$$

where $c_1 = (a, a_1) \in C_l(a)$ and $c_2 = (a_2, a) \in C_r(a)$. It remains to choose arbitrary $b_1, b_2 \in B$. $\square$

Let (M_1, L_0) and (M_2, L_0) be arbitrary countable models of T_0. Let (N_1, L') and (N_2, L') be models of language L'. Recall that the universe of such a model is the union of the corresponding sets B_i and the sets of cosets modulo ε^2 and ε^4. Using induction on n, we define a sequence of partial isomorphisms

$$\begin{aligned} g_n &\colon \big(M_1^{(n)}, L_0\big) \to \big(M_2^{(n)}, L_0\big), \\ t_n &: \big(N_1^{(n)}, L'_0\big) \to \big(N_2^{(n)}, L'\big), \end{aligned} \qquad n < \omega,$$

so that the finite sets $M_i^{(n)} \subset M_i$ and $N_i^{(n)} \subset N_i$, constructed at Step n remain unchanged in the sequel and the following conditions hold:

(U$_1$) $g_{n-1} \subset g_n$, $t_{n-1} \subset t_n$, g_n is a bijective map preserving $<$ and P^1, and t_n is an L'-isomorphism;

(U$_2$) $t_n(\widehat{a}) = \widehat{g}_n(a)$ for every $a \in A_1^{(n)} \subset A_1$;

(U$_3$) $t_n(\widehat{c}) = \widehat{\gamma}\big(g\big(l(c)\big), g\big(r(c)\big)\big)$ for every $c \in C \cap A_1^{(n)} \times A_1^{(n)}$.

Here γ is an element of $\Gamma = \big\{(\alpha_1, \alpha_2) \bigm| \alpha_1, \alpha_2 \in A_2,\ M \models \alpha_1 < \alpha_2 \wedge \varepsilon^2(\alpha_1, \alpha_2)\big\}$.

At Step n, we define g_n if n is even and g_n^{-1} if n is odd. Fix arbitrary numberings m and μ of M_1 and M_2, i.e., let

$$M_1 = \{m_i \mid i < \omega\}, \quad M_2 = \{\mu_i \mid i < \omega\}.$$

S t e p n. Let n be even. The case of an odd n is similar. In $M_1 \setminus (A_1^{(n)} \cup B_1^{(n)})$, we choose the element with the least m-number. Consider two cases.

C a s e 1. Let $a_n \in A_1 \setminus A_1^{(n)}$ be the element with the least m-number. Consider the set $\varepsilon^2(M_1, a_n) \cap A_1^{(n-1)}$. The following three subcases are possible:

1.1. $\varepsilon^2(M_1, a) \cap A_1^{(n-1)} = \varnothing$;

1.2. $\left|\varepsilon^2(M_1, a) \cap A_1^{(n-1)}\right| = 1$;

1.3. $\left|\varepsilon^2(M_1, a) \cap A_1^{(n-1)}\right| \geq 2$.

1.1. Consider $d, d' \in N_1^{(n-1)}$ such that $\widehat{a}_n \in (d, d')_\triangleleft$ and $(d, d')_\triangleleft \cap N_1^{(n-1)} = \varnothing$, where $(\cdot\,,\cdot)_\triangleleft$ denotes an interval with respect to $\triangleleft$. Let $t_n(\widehat{a}_{2n}) \in \big(t_{n-1}(d), t_{n-1}(d')\big)_\triangleleft \cap \widehat{A}_2$ be arbitrary. Since t_{n-1} is an L'-isomorphism, we have $\big(t_{n-1}(d), t_{n-1}(d')\big)_\triangleleft \cap N_2^{(n-1)} = \varnothing$.

Let $t_n(\widehat{a}_n) = \widehat{\alpha}$ for some $\alpha \in A_2$. Choose an arbitrary $\alpha_n \in \varepsilon^2(M_2, \alpha)$ and put $g_n(a_n) = \alpha_n$.

1.2. Let $a_i \in A_1^{(n-1)}$ and let $\varepsilon^2(M, a_n) \cap A_1^{(n-1)} = \{a_i\}$, $i < n$.

Assume that $a_i < a_n$. Let $d, d' \in N_1^{(n-1)}$, let $\widehat{c}(a_i, a_n) \in (d, d')_\triangleleft$, and let $(d, d')_\triangleleft \cap N_1^{(n-1)} = \varnothing$. In view of Ax (VIII), Ax (IX), and Assertion 39 (iii), such elements d and d' exist. From Assertions 39 (iv)$'$ and 39 (iv)$''$ it follows that $\widehat{c}(a_i, a_n) \triangleleft \widehat{a}_n = \widehat{a}_i$. Hence, we obtain either $d' \triangleleft \widehat{a}_i = \widehat{a}_n$ or $d' = \widehat{a}_i$. By (U_2), we have either $t_{n-1}(d') \triangleleft t_{n-1}(\widehat{a}_i) = \widehat{g}_{n-1}(a_i) = \widehat{\alpha}_i$ or $t_{n-1}(d') = \widehat{a}_i$. By Assertion 39 (iv)$''$, the set $\Gamma_l(\alpha_i)$ is dense in $\big(t_{n-1}(d), t_{n-1}(d')\big)_\triangleleft$. Let $\gamma \in \big(t(d), t(d')\big)_\triangleleft \cap \Gamma_l(\alpha_i)$. It is clear that $l(\gamma) = \alpha_i$. Put $t_n(\widehat{a}_n) := \gamma$, $g_n(a_n) := r(\gamma)$, and $\alpha_n := r(\gamma)$.

The case in which $a_n < a_i$ is similar.

1.3. Put

$$\widehat{c}_n = \min_\triangleleft \Big\{ \widehat{c} \mid c = (a_n, a_i) \text{ or } c = (a_i, a_n),\ a_i \in A_1^{(n-1)},\ M \models \varepsilon^2(a_n, a_i) \Big\}.$$

Consider the following two possible subcases:

1.3 (a) there exist $a_k, a_j \in A_1^{(n-1)}$ such that $\widehat{c}(a_k, a_j) = \widehat{c}_n$;

1.3 (b) $\widehat{c}_n \neq \widehat{c}(a_k, a_j)$ for all $a_k, a_j \in A_1^{(n-1)}$.

1.3 (a) Let $a_k < a_j$ and let $c_n = (a_n, a_i)$. Assume that $a_k < a_n < a_j < a_i$. Since $\widehat{c}_n$ is minimal, from Ax (VII) we obtain

$$\widehat{c}(a_k, a_n) = \widehat{c}(a_n, a_j) = \widehat{c}(a_n, a_i) = \widehat{c}(a_k, a_j).$$

Therefore, we may assume that $a_n \in (a_k, a_j)_\triangleleft$ and $(a_k, a_j) \cap A_1^{(n-1)} = \varnothing$.

In a similar way, using Ax (VII) and the fact that $\widehat{c}_n$ is minimal, we conclude that the following three subcases are possible:

(1) $\alpha_n \in (a_i, a_j)$, $(a_i, a_j) \cap A_1^{(n-1)} = \varnothing$;

(2) $a_n < a_i$, $(a_n, a_i) \cap A_1^{(n-1)} = \varnothing$;

(3) $a_j < a_n$, $(a_j, a_n) \cap A_1^{(n-1)} = \varnothing$.

(1) Let $\alpha_i = g_{n-1}(a_i)$ and let $\alpha_j = g_{n-1}(a_j)$. Find an external $\alpha_n \in (\alpha_i, \alpha_j)$ such that $\widehat{\gamma}(\alpha_i, \alpha_n) = \widehat{\gamma}(\alpha_n, \alpha_j) = \widehat{\gamma}(\alpha_i, \alpha_j)$. Such an element exists in view of Ax (VII). Put $g_n(a_n) := \alpha_n$.

(2) Find an external $\alpha_n < \alpha_i$ such that $\widehat{\gamma}(\alpha_n, \alpha_i) = \widehat{\gamma}(\alpha_i, \alpha_j) = \widehat{\gamma}(\alpha_n, \alpha_j)$. Such an element exists in view of Ax (VII). Put $g_n(a_n) := \alpha_n$.

(3) The reasoning is similar to (2).

In each of the above cases, we have $N_1^{(n-1)} = N_1^{(n)}$ because $\widehat{A}_n = \widehat{a}_i$, $\widehat{c}(a_n, a_j) = \widehat{c}(a_i, a_j)$, and, for every $a_s \in A_1^{(n-1)}$, the fact that $\widehat{c}_n$ is minimal implies either $\widehat{c}(a_n, a_s) = \widehat{c}(a_i, a_s)$ or $\widehat{c}(a_s, a_n) = \widehat{c}(a_s, a_j)$ (see Assertions 39 (i)$'$ and 39 (i)$''$).

1.3 (b) We consider the case in which $a_n < a_i$, $\widehat{c}_n = \widehat{c}(a_n, a_i)$, $i < n$, and $a_i \in A_1^{(n-1)}$.

Since $\widehat{c}_n$ is minimal, from Assertions 39 (i)$'$ and 39 (i)$''$ it follows that

$$\forall j < n\Big[\big(\widehat{c}(a_j, a_n) = \widehat{c}(a_j, a_i) \vee \widehat{c}(a_n, a_j) = \widehat{c}(a_n, a_i)\big) \\ \wedge\ (a_n, a_i) \cap A_1^{(n-1)} = \varnothing\Big].$$

Let $d, d' \in N_1^{(n-1)}$, let $\widehat{c}(a_n, a_i) \in (d, d')_{\triangleleft}$, and let $(d, d')_{\triangleleft} \cap N_1^{(n-1)} = \varnothing$. Such elements d and d' exist because $N_1^{(n-1)}$ is bounded. Observe that $d' \triangleleft \widehat{a}_i$. Hence, $t_{n-1}(d') \leq t_{n-1}(\widehat{a}_i) = \widehat{g}_{n-1}(a_i) = \widehat{\alpha}_i$. Therefore, $\Gamma_r(\alpha_i)$ is dense in $\big(t(d), t(d')\big)_{\triangleleft}$.

Put $\alpha_n := l(\gamma)$, $t\big(\widehat{c}(a_n, a_j)\big) = \widehat{\gamma} = \widehat{\gamma}(\alpha_n, \alpha_i)$, and $g(a_n) = \alpha_n$.

Case 2. Let $b_n \in B_1 \setminus B_1^{(n-1)}$ be the element with the least m-number. Take elements $d, d' \in N_1^{(n-1)}$ such that $b_n \in (d, d')_{\triangleleft}$ and $(d, d') \cap N_1^{(n-1)} = \varnothing$. Consider the interval $\big(t_{n-1}(d), t_{n-1}(d')\big)_{\triangleleft}$. Since t_{n-1} is a partial L'-isomorphism, we have $\big(t_{n-1}(d), t_{n-1}(d')\big)_{\triangleleft} \cap N_1^{(n-1)} = \varnothing$. Let β_n be an arbitrary element of $\big(t(d), t(d')\big)_{\triangleleft} \cap B_2$. Put $t_n(b_n) = g_n(b_n) = \beta_n$.

We now return to the proof of Proposition 38. For each $d \in N_1^{(n-1)}$ and $e \in M_1^{(n-1)}$, put $t_n(d) = t_{n-1}(d)$ and $g_n(e) = g_{n-1}(e)$. From the definition and the choice of the elements $g_n(a_n)$, $g_n(b_n)$, $t_n\big(\widehat{c}(a_n, a_i)\big)$, $t_n(\widehat{a}_n)$, and $t_n(b_n)$

it follows that g_n and t_n satisfy (U$_1$)–(U$_3$). By (U$_1$), we conclude that

$$g = \bigcup_{n<\omega} g_n \colon \big(M_1, =, <, P^1\big) \to \big(M_2, =, <, P^1\big),$$
$$t = \bigcup_{n<\omega} t_n \colon \big(N_1, =, \lhd\big) \to \big(N_2, =, \lhd\big),$$

g is an L_0'-isomorphism, and t is an L'-isomorphism, where $L_0' = \{=, <, P_1\}$.

We prove that g is an L_0-isomorphism. Let $a, a' \in A_1$ and let $b \in B_1$. Then

$$\begin{aligned} M_1 \models E(a, a', b) \wedge a < a' & \\ &\iff N_1 \models \widehat{c}(a, a') \lhd b \\ &\iff N_2 \models t\big(\widehat{c}(a, a')\big) \lhd g(b) \\ &\iff N_2 \models \widehat{\gamma}\big(g(a), g(a')\big) \lhd g(b) \text{ by } (\mathrm{U}_3) \\ &\iff M_2 \models E\big(g(a), g(a'), g(b)\big) \wedge g(a) < g(a'). \end{aligned}$$

Therefore, $M_1 \models E(a, a, b) \iff M_2 \models E\big(g(a), g(a), g(b)\big)$. $\square$

From the proof of Proposition 38, we immediately obtain the following

Assertion 40. *Assume that T_0 is consistent. Let M be a model of T_0. Then every irrational cut (B_1, B_2) in $(B, <)$ is M-definable in (M, L_0) if and only if either $\exists a \in P(M)(B_1 \lhd \widehat{a} \lhd B_2)$ or $\exists a_1, a_2 \in P(M)\big(B_1 \lhd \widehat{c}(a_1, a_2) \lhd B_2\big)$.*

4.2. We construct a model (M, L) such that $L \supset L_0$ and $T_0 \vdash_L T := \mathrm{Th}(M, L)$. Fix the language $L := \{=, P^1, <^2, E^3, H^2, \varepsilon^2, \varepsilon^4, S^4, S^3\}$. Put $M := K \cup \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers and K is defined by induction.

Construction of K. Let $\mathbb{R}$ be the set of real numbers, put $I := \big\{C \mid C \subset \mathbb{R},\ C \cap \mathbb{Q} = \varnothing,\ C \text{ is countable and dense in } \mathbb{R}\big\}$, and let J be a subset of I such that $|J| > \omega$ and, for all $C_1, C_2 \in J$, we have $C_1 \cap C_2 = \varnothing$ provided $C_1 \neq C_2$. Let

$$S := \Big\{a \mid a = (\dots, a_b, \dots)_{b \in \mathbb{Q}'},\ \big(\forall\, b \in \mathbb{Q}',\ a_b \in \mathbb{Q}\big),\ \mathbb{Q}' \subseteq \mathbb{Q}, \\ \text{if } \mathbb{Q}' \subset \mathbb{Q} \text{ then } \exists\, e \in \mathbb{R} \setminus \mathbb{Q},\ \forall\, b \in \mathbb{Q}\big(b < e \to b \in \mathbb{Q}'\big)\Big\}$$

be the set of all $\mathbb{Q}$-sequences of rational numbers and let

$$2^{<\omega} := \Big\{\tau \;\Big|\; \exists\, n < \omega,\ \tau = \big(\tau(1), \dots, \tau(n)\big),\ \forall\, i\, (1 \leq i \leq n),\ \tau(i) \in \{0, 1\}\Big\}.$$

We construct K and functions g and $\mathbf{C}$ such that the following conditions are satisfied:

(Z$_1$) We have $K = \bigcup_{n<\omega} K_n \subset S$, $K_n \cap K_{n+1} = \varnothing$, and $|K_n| = \omega$.

(Z$_2$) The function g maps K_n into $(\mathbb{R}\setminus\mathbb{Q})^{n+1}$. For every $d\in K_n$, let $g(d)=\big(g_0(d),\dots,g_n(d)\big)$. Then $g_0(d)>g_1(d)>\dots>g_n(d)$.

(Z$_3$) The function $\mathbf{C}$ maps K into J. Let $\mathbf{C}(\mathrm{d}):=C_d$. Then $C_{d_1}\neq C_{d_2}$ for all $d_1,d_2\in K$ with $d_1\neq d_2$.

Fix an arbitrary element $a=(\dots,a_b,\dots)_{b\in\mathbb{Q}}$ of S.

Step 0. Fix an arbitrary element $C_0\in J$. For every $\gamma\in C_0$, let $a^\gamma=(\dots,a^\gamma_b,\dots)_{b\in\mathbb{Q},\,b<\gamma}$ be an element of S such that $a^\gamma_b=a_b$ for all $b\in\mathbb{Q}$ with $b<\gamma$. Put $K_0:=\{a^\gamma\mid\gamma\in C_0\}$, $g_0(a^\gamma):=\gamma$, and $g(a^\gamma):=(\gamma)$.

Step $n+1$. For all $d\in K_n$, $\gamma\in C_d$ with $\gamma<g_n(d)$, and $\tau\in 2^{<\omega}$, we define $d^{\gamma\tau}:=(\dots,d^{\gamma\tau}_b\dots)_{b\in\mathbb{Q},\,b<g_0(d)}$ as follows:

$$\forall\, b\in\mathbb{Q}\left[\big(\gamma<b<g_0(d)\Rightarrow d^{\gamma\tau}_b=d_b\big)\wedge\left(b<\gamma\Rightarrow d^{\gamma\tau}_b=d_b+\sum_{i=1}^{l(\tau)}\frac{(-1)^{\tau(i)}}{(n+2)^i}\right)\right].$$

We put

$$K_{n+1}:=\big\{d^{\gamma\tau}\;\big|\;d\in K_n,\ \gamma\in C_d,\ \gamma<g_n(d),\ \tau\in 2^{<\omega}\big\},\quad g(d^{\gamma\tau}):=\big(g(d),\gamma\big).$$

Since $|K_n|=\big|\bigcup_{d\in K_n}C_d\big|=|2^{<\omega}|=\omega$, we have $|K_{n+1}|=\omega$.

Since $|J|>\omega$, we can define a map $\mathbf{C}\colon K_{n+1}\to J$. Observe that

$$\forall\, c,d\in K,\ \forall\, b\in\mathbb{Q}\Big[\big(b<g_0(d)=g_0(c)\wedge d_b=c_b\big)\Rightarrow\forall\, b'\in\mathbb{Q}\big(b<b'<g_0(d)\Rightarrow d_{b'}=c_{b'}\big)\Big].$$

Definition of (M,L). Let $d,c,e,f,b\in M$. The following relations hold:

$$\Big[M\models P^1(d)\iff d\in K\Big];$$

$$\Big[M\models d<^2 c\iff\big(\{d,c\}\subset\mathbb{Q}\wedge d<c\big)\vee(d\in\mathbb{Q}\wedge c\in K)\vee\Big(\{d,c\}\subset K\wedge\Big(g_0(d)<g_0(c)\vee\Big(g_0(d)=g_0(c)\wedge\exists\, x\in\mathbb{R}\setminus\mathbb{Q}\wedge\exists\, b'\in\mathbb{Q}\Big[b'<x<g_0(d)\wedge\forall\, b\in\mathbb{Q}\Big(\big[x<b<g_0(d)\Rightarrow c_b=d_b\big]\wedge\big[b'<b<x\Rightarrow d_b<c_b\big]\Big)\Big]\Big)\Big)\Big)\Big];$$

$$\Big[M \models E^3(c,d,b) \iff \{c,d\} \subset K \wedge b \in \mathbb{Q} \wedge \Big(b > \max\big\{g_0(d), g_0(c)\big\} \vee \big(b < g_0(d) = g_0(c) \wedge c_b = d_b\big)\Big)\Big];$$

$$\Big[M \models H^2(d,b) \iff d \in K \wedge b \in \mathbb{Q} \wedge g_0(d) < b\Big];$$

$$\Big[M \models \varepsilon^2(c,d) \iff \{c,d\} \subset K \wedge g_0(c) = g_0(d)\Big];$$

$$\Big[M \models \varepsilon^4(d,c,e,f) \iff \{d,e,c,f\} \subset K \wedge \exists\, n < \omega$$
$$g_n(d) = g_n(c) = g_n(e) = g_n(f) \wedge g_{n+1}(d) \neq g_{n+1}(c) \wedge g_{n+1}(e) \neq g_{n+1}(f),$$
$$\exists\, x \in \mathbb{R} \setminus \mathbb{Q}, \exists\, b' \in \mathbb{Q}\Big[b' < x < g_n(d) \wedge \forall\, b \in \mathbb{Q}$$
$$\Big(\big[x < b < g_n(d) \Rightarrow (c_b = d_b \wedge e_b = f_b)\big] \wedge \big[b' < b < x \Rightarrow (d_b < c_b \wedge e_b < f_b)\big]\Big)\Big]\Big];$$

$$\Big[M \models S_1^4(d,c,e,f) \iff \{d,c,e,f\} \subset K$$
$$\wedge\, g_0(d) = g_0(c) \wedge g_0(e) = g_0(f) \wedge \exists\, b \in \mathbb{Q}(d_b = c_b \wedge e_b \neq f_b) \wedge \exists\, x \in \mathbb{R} \setminus \mathbb{Q}$$
$$\wedge\, \exists\, b' \in \mathbb{Q}\Big[b' < x < g_0(d) \wedge \forall\, b \in \mathbb{Q}\Big(\big[x < b < g_0(d) \Rightarrow c_b = d_b\big] \wedge \big[b' {<} b {<} x \Rightarrow d_b {<} c_b\big]\Big)\Big] \wedge \exists\, x {\in} \mathbb{R} \setminus \mathbb{Q}$$
$$\wedge\, \exists\, b' \in \mathbb{Q}\Big[b' < x < g_0(e) \wedge \forall\, b \in \mathbb{Q}\Big(\big[x < b < g_0(e) \Rightarrow e_b = f_b\big] \wedge \big[b < b < x \Rightarrow e_b < f_b\big]\Big)\Big]\Big];$$

$$\Big[M \models S_2^3(d,c,f) \iff \{d,c,f\} \subset K \wedge g_0(d)$$
$$= g_0(c) \wedge \Big(g_0(d) > g_0(f) \to \exists\, b \in \mathbb{Q}\big(b < g_0(f) \wedge d_b = c_b\big)\Big) \wedge \exists\, x \in \mathbb{R} \setminus \mathbb{Q}$$
$$\wedge\, \exists\, b' \in \mathbb{Q}\Big[b' < x < g_0(d) \wedge \forall\, b \in \mathbb{Q}\Big(\big[x < b < g_0(d) \Rightarrow c_b = d_b\big] \wedge \big[b' < b < x \Rightarrow d_b < c_b\big]\Big)\Big]\Big].$$

Observe that the relations ε^2, ε^4, S_1^4, S_2^3, and H^2 are defined in $\big(M, =, P^1, <^2, E^3\big)$; moreover, (M, L_0) satisfies Ax (I)–Ax (IX).

Let $a, a_1, a_2, a_3 \in P(M)$ and let N be the model constructed in Proposition 38. Then the following conditions hold:

(F1) $[M \models S^3(a_1, a_2, a) \iff N \models \widehat{c}(a_1, a_2) \lhd \widehat{a}]$;

(F2) $[M \models S^4(a, a_1, a_2, a_3) \iff N \models \widehat{c}(a, a_1) \lhd \widehat{c}(a_2, a_3)]$;

(F3) $[M \models \neg\varepsilon^2(a_1, a_2) \wedge a_1 < a_2 \iff N \models \widehat{a}_1 < \widehat{a}_2]$.

4.3. We prove that T admits quantifier elimination and is weakly o-minimal.

Proposition 41. *The theory* $T = \mathrm{Th}(M, L)$ *is* ω*-categorical and finitely axiomatizable, admits quantifier elimination, and is weakly* o*-minimal.*

Proof. Let M be a countable model of T. Consider subsets $A_i \subset P(M)$ and $B_i \subset \neg P(M)$, $i = 1, 2$, such that the set $(A_1 \cup B_1)$ is finite and $(A_1 \cup B_1, L) \cong (A_2 \cup B_2, L)$. Introduction of H^2, ε^2, ε^4, S^3, and S^4 allows us to define finite L'-structures $\big(N(A_i \cup B_i), =, \lhd\big)$ and an L'-isomorphism

$$t\colon \big(N(A_1 \cup B_1), =, \lhd\big) \to \big(N(A_2 \cup B_2), =, \lhd\big)$$

in such a way that t satisfies (U_2) of Proposition 38.

Indeed, let $a, a_1, a_2, a_3 \in A$ and let $b \in B$. Then $\widehat{a}$ is defined via ε^2 and $\widehat{c}(a, a_1)$ is defined via ε^4, ε^2, and $<^2$. Moreover, $\widehat{a}$ and b are $\lhd$-comparable on using H^2; $\widehat{a}$ and $\widehat{c}(a_1, a_2)$ are $\lhd$-comparable on using S^3 (see (F1)); b and $\widehat{c}(a_1, a_2)$ are $\lhd$-comparable on using S^3; $\widehat{c}(a, a_1)$ and $\widehat{c}(a_2, a_3)$ are $\lhd$-comparable on using S^4 (see (F2)); $\widehat{a}_1$ and $\widehat{a}_2$ are $\lhd$-comparable on using ε^2 and $<^2$ (see (F3)).

Employing the method of the proof of Proposition 38, we can extend an isomorphism between $(A_1 \cup B_1, L)$ and $(A_2 \cup B_2, L)$ to an automorphism of (M, L). This means that T admits quantifier elimination [10]. We prove that T is weakly o-minimal. Since T is ω-categorical (see Proposition 38) and admits quantifier elimination, it suffices to show that every atomic 1-formula with parameters is convex. Observe that every 1-formula definable by a finite set of parameters from $(A \cup B)$ is a Boolean combination of $(A \cup B)$-definable atomic 1-formulas. Atomic 1-formulas of the form

$$x < a(b),\quad a(b) < x,\quad P(x),\quad H(x, b),\quad H(a, y),\quad E(x, a, b),\quad E(a, x, b),$$
$$E(a_1, a_2, y),\quad \varepsilon^2(x, a),\quad \varepsilon^2(a, x),\quad \varepsilon^4(x, a_1, a_2, a_3),\quad \varepsilon^4(a_1, x, a_2, a_3),$$
$$\varepsilon^4(a_1, a_2, x, a_3),\quad \varepsilon^4(a_1, a_2, a_3, x),\quad S^3(x, a_1, a_2),\quad S^3(a_1, x, a_2),\quad S^3(a_1, a_2, x),$$
$$S^4(x, a_1, a_2, a_3),\quad S^4(a_1, x, a_2, a_3),\quad S^4(a_1, a_2, x, a_3),\quad S^4(a_1, a_2, a_3, x)$$

are convex in view of Ax (I)–Ax (IX) and the properties of the linear order $\lhd$ on N. Indeed, let a_1', a_1, a_1'', a_2, and a_3 be elements of $P(M)$ such that

$$M \models S^3(a_1', a_2, a_3) \wedge S^3(a_1'', a_2, a_3) \wedge a_1' < a_1 < a_1''.$$

By (F1), we have $N \models \widehat{c}(a_1', a_2, a_3) \lhd \widehat{a}_3 \wedge \widehat{c}(a_1'', a_2) \lhd \widehat{a}_3$. Since $a_1' < a_1 < a_1''$, from the definitions of ε^4 and $\lhd$ it follows that

$$N \models \Big(\widehat{c}(a_1', a_2) \lhd \widehat{c}(a_1, a_2) \vee \widehat{c}(a_1', a_2) = \widehat{c}(a_1, a_2)\Big) \wedge \Big(\widehat{c}(a_1, a_2) \lhd \widehat{c}(a_1'', a_2) \vee \widehat{c}(a_1, a_2) = \widehat{c}(a_1'', a_2)\Big).$$

Since $\lhd$ is transitive, we find that $N \models \widehat{c}(a_1, a_2) \lhd \widehat{a}_3$. By (F1), we have $M \models S^3(a_1, a_2, a_3)$ which means that $S^3(x, a_2, a_3)$ is convex. In a similar way, we can verify that all atomic 1-formulas of the form S^3 and S^4 are convex. For the remaining atomic 1-formulas, the required assertion is immediate from definitions and axioms. $\square$

4.4. An example of a pair of models $(M_b, L) \prec (M, L)$ such that (M_b, M) is a D-1-pair but is not a D-pair.

Let (M, L) be the model constructed at Stage 4.2 and let b be an arbitrary element of $\mathbb{Q}$ $\big($i.e., $\neg P(M)\big)$. Let Q_b denote the set $\{x \in \mathbb{Q} \mid x < b\}$ and let K_b denote the set $\bigcup_{b'<b} H(M, b')$. Put $M_b = Q_b \cup K_b$. It is easy to see that (M_b, L) is a submodel of (M, L). Since $\mathrm{Th}(M)$ admits quantifier elimination, we have $(M_b, L) \prec (M, L)$.

Assertion 42. *Let $d_1, d_2 \in P(M) \setminus M_b$ and let $d_1 \neq d_2$. If there exists $n < \omega$ such that $n > 0$ and $g_n(d_1) = g_n(d_2) < b$ then the type $\mathrm{tp}(d_1 d_2/M_b)$ is not definable.*

Proof. In view of axiom Ax (VII), the formula $E(d_1, d_2, x)$ is defined by some irrational cut (B_1, B_2) in $(\mathbb{Q}, <)$ and the conditions $g_n(d_1) = g_n(d_2) < b$ and $B_1 < b$. Hence, the irrational cut $\Big(B_1, \big(B_2 \cap \{b\}^-_M\big) \cup P(M_b)\Big)$ is not definable in (M_b, L). This is a consequence of Assertion 40 and condition (Z_3) from the construction of K (cf. Stage 4.2). There exists a unique 1-type $p \in S_1(M_b)$ extending the cut $\Big(B_1, \big(B_2 \cap \{b\}^-_M\big) \cup P(M_b)\Big)$. In view of Proposition 26, we have $p \perp^w \mathrm{tp}(d_1/M_b)$ because p is irrational while $\mathrm{tp}(d_1/M_b)$ is quasirational. Put $q := \mathrm{tp}(d_2/M_b \cup d_1)$ and take $p' \in S_1(M_b \cup d_1)$ with $p \subseteq p'$. We have $d_1 d_2 \not\perp^w p$. Hence, $q \not\perp^w p'$ because $p(M) = p'(M)$. Observe that p' is an irrational type defined by a cut. Therefore, p' is not definable. By Proposition 22, q is not definable either. Thus, $\mathrm{tp}(d_1 d_2/M_b)$ is not definable. This means that (M_b, M) is not a D-pair. $\square$

Observe that (M_b, L) is a quasi-Dedekind complete model in (M, L). Indeed, we have $P(M) \setminus P(M_b) > P(M_b)$ and $\neg P(M) \setminus \neg P(M_b) > \neg P(M_b)$. Thus, Assertion 42 implies Theorem 37. $\square$

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